2023 SAJC H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**

Attempt all questions.

1 The sum of the first n terms of a sequence, S_n, is given by $S_n = an^3 + bn^2 + cn + d$, where a, b, c and d are constants . It is given that $S_1 = 5,\ \ S_2 = 20,\ \ S_3 = 57$ and $S_4 = 128$. Find S_n. [4]

2 **(a)** Differentiate $\dfrac{\sin^{-1}(2x)}{1-4x^2}$, for $-\dfrac{1}{2} < x < \dfrac{1}{2}$, with respect to x, simplifying your answer as much as possible as a single fraction. [3]

(b) It is given that $y^2 = 3\mathrm{e}^{4x} + 4$. By repeated differentiation, show that $y\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 8y^2 = k$, where k is a constant to be determined. [4]

3 **(a)** Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, find $\sum_{r=1}^{n} 6r(r+1)$, in terms of n. [2]

(b) **(i)** Express $\dfrac{1}{r(r+1)(r+2)}$ in partial fractions. [1]

(ii) Hence find S_N in terms of N, where $S_N = \sum_{r=1}^{N} \dfrac{1}{r(r+1)(r+2)}$, and deduce that $S_N < \dfrac{1}{4}$. [6]

4 An arithmetic sequence $u_1, u_2, u_3, \cdots, u_n, \cdots$ is known to have a common difference of $\ln 3$.

The rth term, w_r, of another sequence $w_1, w_2, w_3, \cdots$, is given by $w_r = \mathrm{e}^{-u_r}$.

(i) Show that the sequence $w_1, w_2, w_3, \cdots, w_n, \cdots$, is a geometric progression with common ratio $\dfrac{1}{3}$. [2]

(ii) Explain why $\sum_{r=1}^{\infty} w_r$ converges. [1]

(iii) Given that $u_1 = \ln 3$, find the smallest possible value of n such that the sum of the first n terms of the geometric progression given by $w_1, w_2, w_3, \cdots$ is within 0.5% of the sum to infinity of the geometric series. [3]

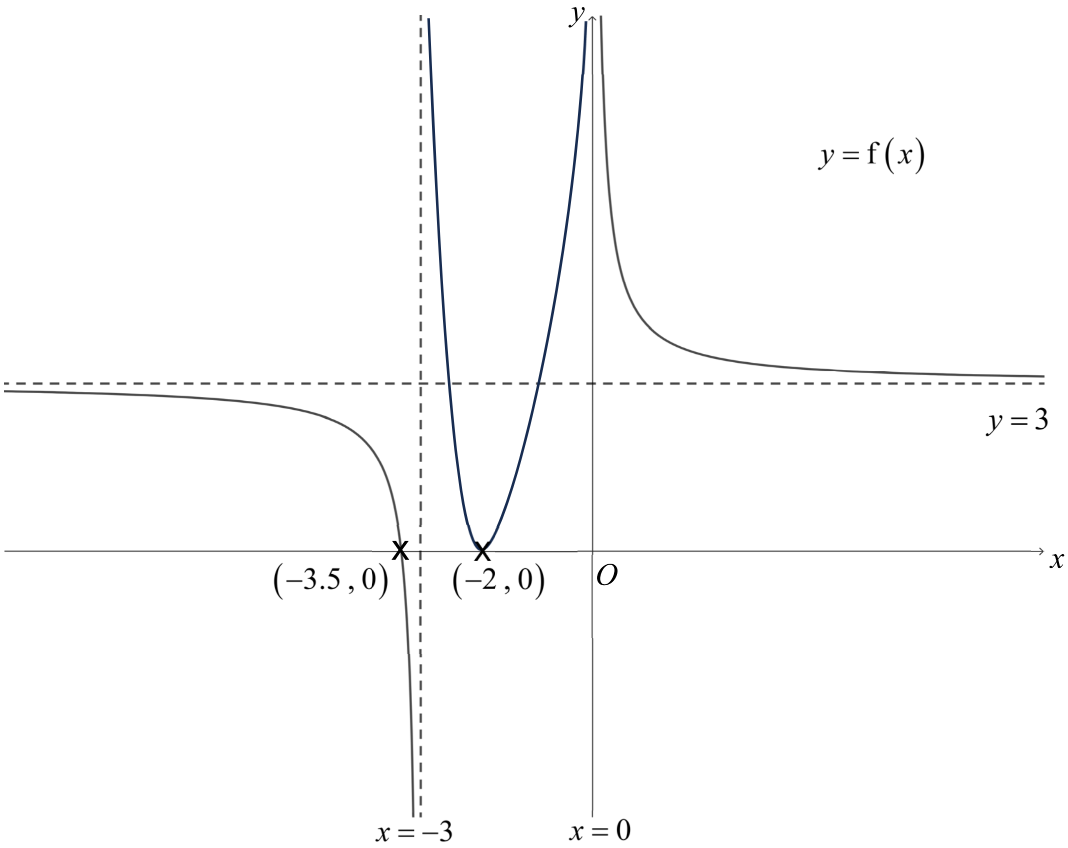
5 The position vectors of the points A and B relative to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively.

(i) Point M on AB is such that $AM : MB = 1:2$. Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. If the area of triangle OBM is 4 units^2, find $|\mathbf{a}\times\mathbf{b}|$. [5]

(ii) Another point P has position vector $\mathbf{p}$ and $\mathbf{p}\neq\mathbf{0}$. Given $(\mathbf{p}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$, what can you deduce about the relationship between the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$? Hence find the vector equation of line l that passes through points A and P in terms of $\mathbf{b}$ and $\mathbf{a}$. [2]

(iii) Given instead that $\mathbf{a}=(4\mathbf{a}\cdot\mathbf{b})\mathbf{b}$, describe in words, the relation between $\mathbf{a}$ and $\mathbf{b}$ and find $|\mathbf{b}|$. [3]

6 **(a)**



The diagram above shows the graph of $y = \mathrm{f}\left(x\right)$. The curve crosses the x-axis at $\left(-3.5, 0\right)$ and has a minimum point at $\left(-2, 0\right)$. The lines $x = -3$, $x = 0$ and $y = 3$ are asymptotes of the curve.

On separate diagrams, sketch the graphs in **(i)** and **(ii)**, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes and of any turning points in exact form, if possible.

(i) $y = \mathrm{f}\left(|x|\right)$, and [2]

(ii) $y = \mathrm{f}'(x)$. [4]

(b) The transformations A, B and C are given as follows:

A: Reflection about the x-axis;

B: Translation of 4 units in the positive x-direction;

C: Scaling parallel to the x-axis by a factor of 2.

A curve undergoes in succession, the transformations A, B and C, and the equation of the resulting curve is $y = \frac{1}{3}\left(x+1\right)^2$.

Determine the original equation of the curve. [3]

7 **(i)** Sketch, on the same diagram, the graphs $y = \frac{2x-1}{x-3}$ and $y = \left|\ln\left(1-x\right)\right|$, stating clearly the coordinates of axial intercept(s) and equation of asymptote(s).

Hence solve $\frac{2x-1}{x-3} = \left|\ln\left(1-x\right)\right|$. [5]

(ii) Solve $\frac{2x-1}{x-3} \le \left|\ln\left(1-x\right)\right|$. [1]

(iii) Hence, solve $\frac{2x+1}{x-2} \le \left|\ln\left(-x\right)\right|$. [3]

8 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{-x-3}{x+1}, \text{ for } x \in \mathbb{R}, x \neq k\ .$$

(i) State the value of k and explain why this value has to be excluded from the domain of f. [2]

(ii) Find $\mathrm{f}^{-1}(x)$. Hence find $\mathrm{f}^{2}(x)$. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto \sqrt{x+1}\,, x \in \mathbb{R}\,, x > -1.$$

(iii) Find the range of fg . [2]

9 A curve C is defined by the parametric equations

$$x = t^2, \quad y = t^3 .$$

(i) Prove that the equation of the tangent at the point $(t^2\ , t^3)$ on the curve is $2y - 3tx + t^3 = 0$. [3]

(ii) This tangent passes through a fixed point (a,b).

Explain why there cannot be more than 3 tangents through (a,b). [1]

(iii) The tangent at the point P when $t = 2$ meets the curve again at the point Q where $t = k$. Find the value of k. [3]

(iv) Find the equation of the normal at the point P. [2]

(v) Sketch the curve C, indicating the intersection point(s) with the axes. [2]

(vi) By sketching the tangent and normal at the point P on the sketch in **(v)**, explain why $\tan^{-1}(3) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$. [2]

10 The line l_1 and the plane π_1 have equations $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -5 \\ p \end{pmatrix} = 4$

respectively, where p is a real constant.

(i) Given that the line l_1 and the plane π_1 intersect at the point $A(-5, -7, 7)$, show that $p = -3$. [2]

(ii) Find the acute angle between the line l_1 and the plane π_1. [2]

(iii) B is the point on l_1 where $\lambda = 1$. Find the position vector of the foot of perpendicular, F, from the point B to π_1. [4]

(iv) Find the equation of the line of reflection of l_1 in the plane π_1. [3]

Another line l_2 has the following properties.

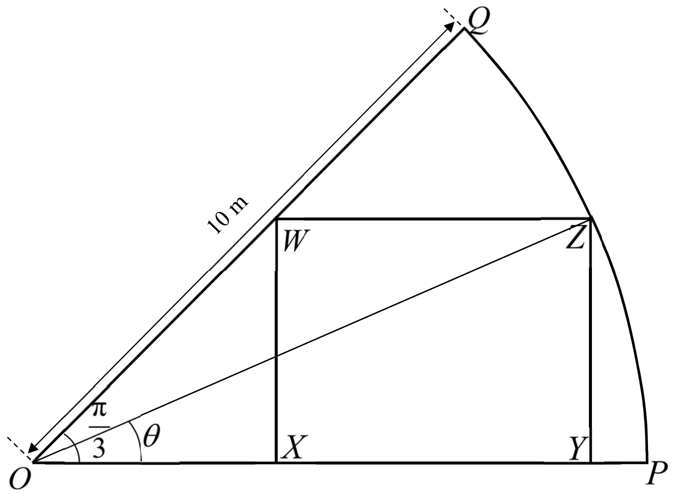
- l_2 passes through point B,
- l_2 is perpendicular to l_1, and
- l_2 is parallel to π_1.

(v) Find, in vector form, the equation of l_2. [2]

11 [It is given that the area of a sector of a circle with radius r and angle θ is given by $\frac{1}{2}r^2\theta$.]

A florist owns a greenhouse in the shape of a circular sector OPQ, with center O, and intends to enclose a rectangular area to grow roses.

The garden is represented in the diagram below by a fixed sector OPQ where $OP = OQ = 10\,\text{m}$ and $\angle POQ = \frac{\pi}{3}$ radians. The rectangular area $WXYZ$ is inscribed in sector OPQ such that $\angle ZOP = \theta$ radians.



(i) By considering triangles OWX and OZY, or otherwise, show that the area of rectangle $WXYZ$, $A = 50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right)$. [3]

(ii) Using differentiation, find the maximum value of A in exact form. [6]

The florist subsequently decides to use remaining regions in the sector to grow marigolds.

The cost of maintaining the plants each month, $\$C$, can be broken down into $\$5/\text{m}^2$ for roses and $\$4/\text{m}^2$ for marigolds.

(iii) Sketch the graph of C as θ varies, indicating any turning point(s) and end point(s). [3]

(iv) Hence determine the range of values of θ for the florist to have a maintenance cost of less than \$220. [1]