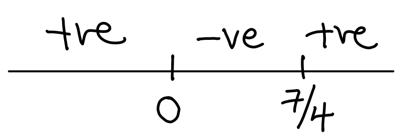
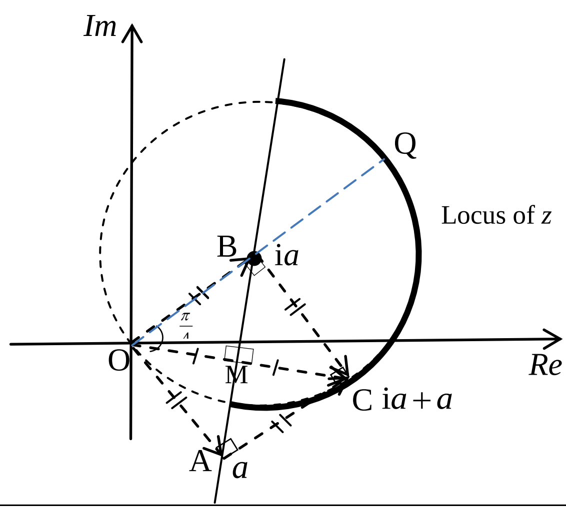
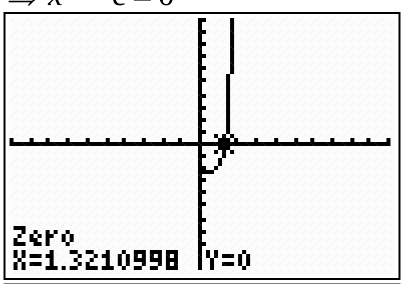
2012 HCI H2 Math Prelim Paper 1 Marking Scheme

Qn.	Solutions
1	$\mathrm{f}'(x) = \dfrac{(x-2)(2kx-5)-(kx^2-5x+3)(1)}{(x-2)^2}$ $= \dfrac{2kx^2-5x-4kx+10-kx^2+5x-3}{(x-2)^2}$ $= \dfrac{kx^2-4kx+7}{(x-2)^2}$ Since f is an increasing function, $\mathrm{f}'(x) = \dfrac{kx^2-4kx+7}{(x-2)^2} \geq 0$ $\Rightarrow kx^2-4kx+7 \geq 0$ for all values of x Hence we have $k=0$ (linear) or for $k>0$, $(-4k)^2-4k(7)\leq 0$ (quadratic) $16k^2-28k\leq 0$ $4k(4k-7)\leq 0$  Hence $0 \leq k \leq \dfrac{7}{4}$.
2 (i) and (ii)	
	$z = \overrightarrow{OQ} = 2ia$
3	$\ln y = xy\ln x$ Differentiate implicitly w.r.t x,

	$\frac{1}{y}\frac{dy}{dx} = xy\left(\frac{1}{x}\right) + \ln x\left(x\frac{dy}{dx} + y\right)$ $\frac{1}{y}\frac{dy}{dx} = y + x\ln x\frac{dy}{dx} + y\ln x$ $\left(\frac{1}{y} - x\ln x\right)\frac{dy}{dx} = y + y\ln x$ $\frac{dy}{dx} = \frac{y^2(1+\ln x)}{1-xy\ln x} = \frac{y^2(1+\ln x)}{1-\ln y}$ For the tangent to be parallel to the y–axis, $\frac{dx}{dy} = 0$ $\frac{1-\ln y}{y^2(1+\ln x)} = 0$ $1-\ln y = 0$ $y = e$ Hence from $y = x^{xy}$, we have $e = x^{ex}$ $\Rightarrow x^{ex} - e = 0$  Hence x = 1.32, and since the tangent parallel to the y–axis takes the form $x = c$, hence equation of tangent is $x = 1.32$
4(a)	$3iz = -i\left(z^* - 1 + 2i\right)$ $3i\left(x + iy\right) = -i\left(x - iy - 1 + 2i\right)$ $3ix - 3y = -ix - y + i + 2$ $3ix - 3y = (1-x)i + (2-y)$ Comparing Re and Im parts, $-3y = 2 - y \Rightarrow y = -1$ $3x = 1 - x \Rightarrow x = \frac{1}{4}$ $\therefore z = \frac{1}{4} - i$

(b)	$z=\frac{-1\pm\sqrt{1-4p}}{2}=-\frac{1}{2}\pm\frac{\sqrt{4p-1}}{2}\mathrm{i}$ $\lvert z_1-z_2\rvert=\sqrt{4p-1}=\sqrt{3}$ $\therefore\ p=1$
5(i)	$y=\left[1+\ln(1+2x)\right]^{-\frac{1}{2}}$ $=\left[1+(2x)-\frac{(2x)^2}{2}+\ldots\right]^{-\frac{1}{2}}$ $\approx(1+2x-2x^2)^{-\frac{1}{2}}$ $=1+\left(-\frac{1}{2}\right)(2x-2x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(2x-2x^2)^2+\ldots$ $=1-x+x^2+\frac{3}{8}(4x^2)+\ldots$ $=1-x+\frac{5}{2}x^2+\ldots$
(ii)	$-1<2x\le 1$ and $\lvert\ln(1+2x)\rvert<1$ $\Rightarrow\frac{-1}{2}<x\le\frac{1}{2}$ and $\frac{\mathrm{e}^{-1}-1}{2}<x<\frac{\mathrm{e}-1}{2}$ $\Rightarrow\frac{\mathrm{e}^{-1}-1}{2}<x\le\frac{1}{2}\quad(\text{OR }-0.316<x\le 0.5)$
(iii)	$\int_0^2 y\,\mathrm{d}x\approx\int_0^2\left(1-x+\frac{5}{2}x^2\right)\mathrm{d}x=6.67\quad\left(\text{OR }\frac{20}{3}\right)$ Since the integration in $(0,2)$ does not fall into the valid range of $-0.316<x\le 0.5$, the approximation is not good.
6(a)	Assume the x th day to be the day with maximum amount of goods delivered. For the first x days: (a = 1000, d = 100) $S_x=\frac{x}{2}[2(1000)+(x-1)(100)]$ $=\frac{x}{2}(1900+100x)$ For the remaining (15- x) days: (a = 1000+ (x - 1)100 – 100 = 800+100x, d = -100)

	$S_{15-x}=\frac{(15-x)}{2}[2(800+100x)+(15-x-1)(-100)]$ $=\frac{15-x}{2}(200+300x)$ Since total goods delivered is 21300 tons, $\frac{x}{2}(1900+100x)+\frac{15-x}{2}(200+300x)=21300$ $x^2-31x+198=0$ $(x-9)(x-22)=0$ $x=9$ or $x=22$ (NA, since $x\le 15$) Therefore, 9th June was the day with max goods delivered. Goods delivered = 1000 + (9 – 1)(100) = 1800 tons
(b) (i)	Series H is a GP with common ratio r. $H=\frac{a(1-r^n)}{1-r}$ or $H=\frac{a(r^n-1)}{r-1}$ C is a GP with common ratio 1/r. $C=\frac{\frac{1}{a}(1-\frac{1}{r^n})}{1-\frac{1}{r}}=\frac{1}{ar^{n-1}}\frac{r^n-1}{r-1}$ $\frac{H}{C}=\frac{a(r^n-1)}{r-1}\times\frac{ar^{n-1}(r-1)}{r^n-1}$ $=a\times ar^{n-1}$ $=u_1u_n$
(b) (ii)	$u_1\cdot u_2\cdot\ldots\cdot u_n=a(ar)(ar^2)(ar^3)\ldots(ar^{n-1})$ $=a^n(r^{1+2+3+\ldots+(n-1)})$ $=a^nr^{\frac{(n-1)n}{2}}$ Since $\frac{H}{C}=u_1u_n=a^2r^{n-1}$, $u_1\cdot u_2\cdot\ldots\cdot u_n=a^nr^{\frac{(n-1)n}{2}}$ $=(a^2r^{n-1})^{\frac{n}{2}}=(\frac{H}{C})^{\frac{n}{2}}$
7(a)	$u_2-u_1=2(3^{-1})+a$ $u_3-u_2=2(3^{-2})+a$ $u_4-u_3=2(3^{-3})+a$ ... $u_n-u_{n-1}=2(3^{-(n-1)})+a$

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Sum of all equations:

$$u_n - u_1 = 2(3^{-1} + 3^{-2} + 3^{-3} + ... + 3^{-(n-1)}) + (n-1)a$$

$$u_n = u_1 + 2\left(\frac{\frac{1}{3}(1-(\frac{1}{3})^{n-1})}{1-\frac{1}{3}}\right) + (n-1)a$$

$$u_n = -1 + (1 - \frac{1}{3^{n-1}}) + (n-1)a$$

$$= (n-1)a - \frac{1}{3^{n-1}}$$
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<tr><td>(b)</td><td>
Let P_n be the statement denoting

$1^3 + 2^3 + ... + n^3 + 3(1^5 + 2^5 + ... + n^5) = \frac{1}{2}n^3(n+1)^3$ for $n \in \mathbb{Z}^+$,

When n = 1,

LHS = $1^3 + 3(1^5) = 4$

RHS = $\frac{1}{2}(1^3)(2^3)$ = 4 =LHS

Therefore, P_1 is true.

Assume P_k is true for some values of $k \in \mathbb{Z}^+$, i.e.

$1^3 + 2^3 + ... + k^3 + 3(1^5 + 2^5 + ... + k^5) = \frac{1}{2}k^3(k+1)^3$

To prove P_{k+1}, i. e.

$1^3 + 2^3 + ... + k^3 + (k+1)^3 + 3(1^5 + 2^5 + ... + k^5 + (k+1)^5)$

$= \frac{1}{2}(k+1)^3(k+2)^3$

LHS =

$1^3 + 2^3 + ... + k^3 + (k+1)^3 + 3(1^5 + 2^5 + ... + k^5 + (k+1)^5)$

$= \frac{1}{2}k^3(k+1)^3 + (k+1)^3 + 3(k+1)^5$

$= \frac{1}{2}(k+1)^3[k^3 + 2 + 6(k+1)^2]$

$= \frac{1}{2}(k+1)^3[k^3 + 6k^2 + 12k + 8]$

$= \frac{1}{2}(k+1)^3(k+2)^3$

=RHS

Therefore P_{k+1} is true.
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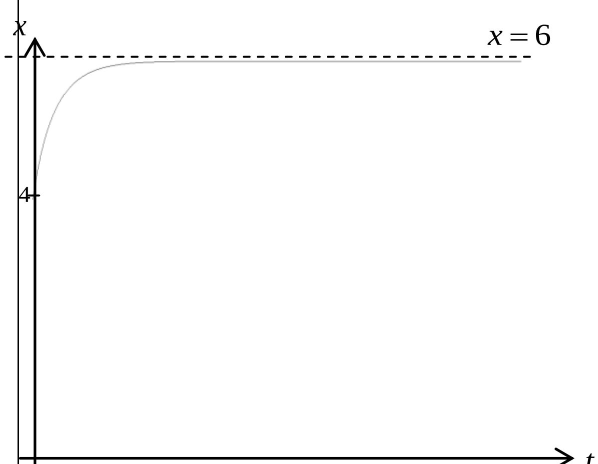
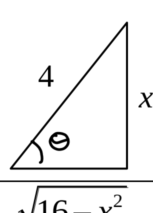
	Since P_1 is true, P_k is true $\Rightarrow P_{k+1}$ is true. By mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$. $\sum_{r=1}^{2n} r^5 = \frac{1}{3}[\frac{1}{2}(2n)^3(2n+1)^3 - \frac{1}{4}(2n)^2(2n+1)^2]$ $= \frac{1}{3}[4n^3(2n+1)^3 - n^2(2n+1)^2]$ $= \frac{1}{3}n^2(2n+1)^2[4n(2n+1)-1]$ $= \frac{1}{3}n^2(2n+1)^2(8n^2+4n-1)$
8(i)	$x^2 + y^2 = 36$
(ii)	$\frac{dy}{dt} = -ky$ Method 1 Implicit Differentiation w.r.t. time $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$ $= -\frac{y}{x}(-ky)$ $= k\frac{y^2}{x} = \frac{k(36-x^2)}{x}$ (shown) Method 2 Implicit Differentiation w.r.t x then using chain rule (rate of change equation) $2x + 2y\frac{dy}{dx} = 0$ $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ $\left(\text{or } \frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}\right)$ $\Rightarrow \frac{dx}{dt} = \frac{-ky}{-\frac{x}{y}}$ $= k\frac{y^2}{x} = \frac{k(36-x^2)}{x}$ (shown)

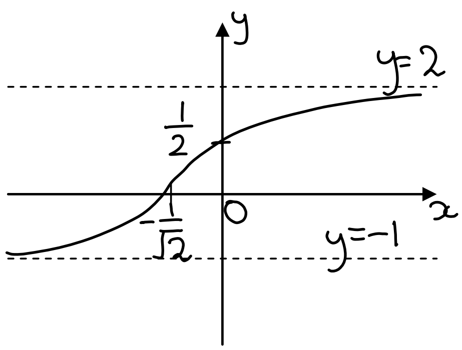
(iii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2(36-x^2)}{x}$ (given) $\int \frac{x}{36-x^2}\,\mathrm{d}x = \int 2\,\mathrm{d}t$ $\frac{1}{-2}\int \frac{-2x}{36-x^2}\,\mathrm{d}x = \int 2\,\mathrm{d}t$ $-\frac{1}{2}\ln\left\|36-x^2\right\| = 2t + C$ $\ln\left\|36-x^2\right\| = -4t + C'$ $36 - x^2 = A\mathrm{e}^{-4t}$ $x^2 = 36 - A\mathrm{e}^{-4t} \Rightarrow x = \sqrt{36 - A\mathrm{e}^{-4t}}$ $(\because x > 0)$ Using initial conditions, when $t = 0,\ x = 4$ $4 = \sqrt{36 - A}$ $\Rightarrow A = 20$ $\Rightarrow x = \sqrt{36 - 20\mathrm{e}^{-4t}}$ For OY to be 3, $OX = \sqrt{36-9} = \sqrt{27}$ $\Rightarrow 27 = 36 - 20\mathrm{e}^{-4t}$ $\mathrm{e}^{-4t} = \frac{9}{20}$ $\therefore t = -\frac{1}{4}\ln\frac{9}{20} = 0.2$ s
(iv)	

	Jesse's model is **not appropriate**. Based on Jesse's model, the rod would never fall flat on the ground.
9 (i)	$P(4,0,0), Q(6,4,6), R(6,2,0)$ $\overrightarrow{PR}\times\overrightarrow{RQ}=\begin{pmatrix}2\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\2\\6\end{pmatrix}$ $=\begin{pmatrix}12\\-12\\4\end{pmatrix}=4\begin{pmatrix}3\\-3\\1\end{pmatrix}$ So the equation of the plane is $\mathbf{r}\cdot\begin{pmatrix}3\\-3\\1\end{pmatrix}=\begin{pmatrix}4\\0\\0\end{pmatrix}\cdot\begin{pmatrix}3\\-3\\1\end{pmatrix}=12$ i.e. $3x-3y+z=12$
(ii)	<u>Method 1</u> the length of projection of $\overrightarrow{CP}$ onto the normal of the plane PQR $=\left\|\begin{pmatrix}4\\-4\\0\end{pmatrix}\cdot\frac{1}{\sqrt{3^2+3^2+1^2}}\begin{pmatrix}3\\-3\\1\end{pmatrix}\right\|$ $=\frac{24}{19}\sqrt{19}$ cm = 5.51 cm (correct to 3 sig figs) <u>Method 2</u> Denote the foot of perpendicular of C to the plane PQR as N. Then $\overrightarrow{ON}=\begin{pmatrix}0\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\-3\\1\end{pmatrix}$ Sub into the equation of plane PQR, $\left[\begin{pmatrix}0\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\-3\\1\end{pmatrix}\right]\cdot\begin{pmatrix}3\\-3\\1\end{pmatrix}=12\Rightarrow\lambda=\frac{24}{19}$ $\Rightarrow N\left(\frac{72}{19},\frac{4}{19},\frac{24}{19}\right), CN=\frac{24}{19}\sqrt{19}$ or 5.51 cm

(iii)	$\cos\theta = \dfrac{\left\lvert\begin{pmatrix}3\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}3\\-3\\1\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert} = \dfrac{1}{\sqrt{19}}$ $\Rightarrow \theta = 76.7°$
(iv)	Line PQ: $\mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + s\begin{pmatrix}1\\2\\3\end{pmatrix}$, Plane *OCGD:* $x = 0$ At point *M*, $4 + s = 0 \Rightarrow s = -4 \therefore \overrightarrow{OM} = \begin{pmatrix}0\\-8\\-12\end{pmatrix}$ The distance from *M* to the plane *OABC* is 12 cm
(v)	The point of reflection, Q', of Q about the plane $OABC$ is $(6,4,-6)$ $\Rightarrow$ equation of the reflection plane is $\mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\0\end{pmatrix} + \mu\begin{pmatrix}2\\4\\-6\end{pmatrix}$ *Note: the plane contains three points, $Q'(6,4,-6), P(4,0,0), R(6,2,0)$ and three vectors parallel to the plane, $\overrightarrow{PR} = \begin{pmatrix}2\\2\\0\end{pmatrix}, \overrightarrow{PQ'} = \begin{pmatrix}2\\4\\-6\end{pmatrix}, \overrightarrow{RQ'} = \begin{pmatrix}0\\2\\-6\end{pmatrix}$ All possible answers using one point and two vectors will be correct.
10 (a) (i)	$\int(\cos^4 x - \sin^4 x)\ \mathrm{d}x$ $= \int(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)\ \mathrm{d}x$ $= \int(\cos 2x)(1)\ \mathrm{d}x$ $= \frac{1}{2}\sin 2x + C$
(ii)	$\int \log_3(3x-1)\ \mathrm{d}x$

	$= \int \frac{\ln(3x-1)}{\ln 3}\ \mathrm{d}x$ Let $u = \ln(3x-1), \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3}{3x-1}, \qquad v = x$ $\int \log_3(3x-1)\ \mathrm{d}x$ $= \frac{1}{\ln 3}\left[\left(\ln(3x-1)\right)(x) - \int\left(\frac{3}{3x-1}\right)(x)\ \mathrm{d}x\right]$ $= \frac{1}{\ln 3}\left[\ x\ln(3x-1)\ - \int\left(1+\frac{1}{3x-1}\right)\ \mathrm{d}x\right]$ $= \frac{1}{\ln 3}\left[\ x\ln(3x-1)\ - x - \frac{1}{3}\ln(3x-1)\right] + C$ $= \frac{1}{\ln 3}\left[\left(x-\frac{1}{3}\right)\ln(3x-1) - x\right] + C$
(b)	Let $x = 4\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 4\cos\theta$ $\int \frac{\sqrt{16-x^2}}{x^2}\ \mathrm{d}x$ $= \int \frac{\sqrt{16-(4\sin\theta)^2}}{(4\sin\theta)^2}\ (4\cos\theta)\mathrm{d}\theta$ $= \int \frac{\sqrt{16\cos^2\theta}}{16\sin^2\theta}\ (4\cos\theta)\mathrm{d}\theta$ $= \int \frac{\cos^2\theta}{\sin^2\theta}\ \mathrm{d}\theta$ $= \int \cot^2\theta\ \mathrm{d}\theta$ $= \int\left(\operatorname{cosec}^2\theta - 1\right)\ \mathrm{d}\theta$ $= -\cot\theta - \theta + C$ Now, $x = 4\sin\theta$, $\therefore\ \theta = \sin^{-1}\left(\frac{x}{4}\right)$ $\cot\theta = \frac{\sqrt{16-x^2}}{x}$ 

	$\therefore \int \frac{\sqrt{16-x^2}}{x^2}\,\mathrm{d}x$ $= -\frac{\sqrt{16-x^2}}{x} - \sin^{-1}\left(\frac{x}{4}\right) + C$ (shown)
11 (a) (i)	$x = t - \frac{1}{t}$ $\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}$ $y = 2t + \frac{1}{t}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2 - \frac{1}{t^2} = \frac{2t^2-1}{t^2}$ Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$ $= \frac{2t^2-1}{t^2+1}$ $= 2 - \frac{3}{t^2+1}$ Hence we have $0 < \frac{3}{t^2+1} < \frac{3}{0+1} = 3$ Hence $2-3 < 2 - \frac{3}{t^2+1} < 2-0$ $-1 < 2 - \frac{3}{t^2+1} < 2$ (shown)
(a) (ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{2t^2-1}{t^2+1} = 0$ $\Rightarrow 2t^2 - 1 = 0$ $\Rightarrow t^2 = \frac{1}{2}$ $\Rightarrow t = \frac{1}{\sqrt{2}}$ (since t is +ve)

	At $t = \frac{1}{\sqrt{2}}$, $x = t - \frac{1}{t} = \frac{1}{\sqrt{2}} - \frac{1}{\frac{1}{\sqrt{2}}}$ $= \frac{1}{\sqrt{2}} - \sqrt{2} = \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}$ $= -\frac{1}{\sqrt{2}}$
(iii)	At $x = 0$, $t = 1 \Rightarrow \frac{dy}{dx} = \frac{2(1)^2 - 1}{(1)^2 + 1} = \frac{1}{2}$ 
(b)	$A = xy = \left(t - \frac{1}{t}\right)\left(2t + \frac{1}{t}\right)$ $= 2t^2 + 1 - 2 - \frac{1}{t^2}$ $= 2t^2 - 1 - t^{-2}$ We have $\frac{dA}{dt} = 4t + 2t^{-3}$ When $t = 5$ $\frac{dA}{dx} = \frac{dA}{dt} \times \frac{dt}{dx}$ $= (4t + 2t^{-3}) \times \frac{t^2}{t^2 + 1}$ = 19.2