**1** Express $\dfrac{12}{x+1}-(7-x)$ as a single simplified fraction. [1]

Without using a calculator, solve $\dfrac{12}{x+1} \le 7-x$. [3]

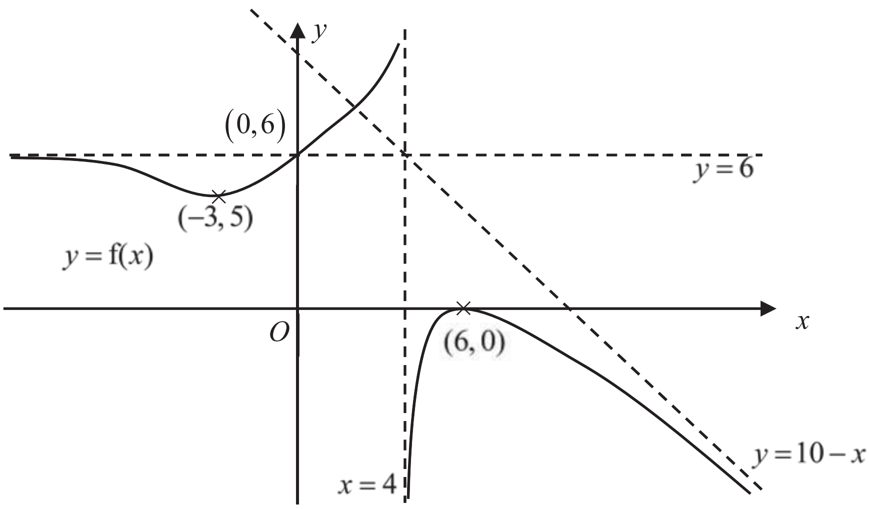
**2** **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}\left(x^2\right)$. [1]

**(ii)** Hence, or otherwise, evaluate $\int_0^1 x\tan^{-1}\left(x^2\right)\mathrm{d}x$ exactly. [3]

**3** **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(3x^2 2^x\right)$. [2]

**(ii)** Find the equation of the tangent to the curve $y = 3x^2 2^x$ at the point where $x = 1$, giving your answer in exact form. [3]

**4** The graph for $y = \mathrm{f}(x)$ is given below, where $y = 10 - x$, $y = 6$ and $x = 4$ are asymptotes. The turning points are $(-3, 5)$ and $(6, 0)$, and the graph intersects the $y$-axis at $(0,6)$.




On separate diagrams, sketch the graphs of

**(i)** $y = \mathrm{f}(|x|)$, [3]

**(ii)** $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

**5** Referred to the origin $O$, points $P$ and $Q$ have position vectors $3\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$ respectively. Point $M$ is a point on $QP$ extended such that $PM$:$QM$ is 2:3.

**(i)** Find the position vector of point $M$ in terms of **a** and **b**. [2]

**(ii)** Find $\overrightarrow{PQ} \times \overrightarrow{OM}$ in terms of **a** and **b**. [3]

**(iii)** State the geometrical meaning of $\dfrac{\left|\overrightarrow{PQ} \times \overrightarrow{OM}\right|}{\left|\overrightarrow{PQ}\right|}$. [1]

**6** A curve $C$ has equation $y = \mathrm{f}(x)$, where the function f is defined by

$$\mathrm{f} : x \mapsto \frac{12 - 3x}{x^2 + 4x - 5}, \quad x \in \mathbb{R}, x \neq -5, x \neq 1.$$

**(i)** Find algebraically the range of f. [3]

**(ii)** Sketch $C$, indicating all essential features. [4]

**(iii)** Describe a pair of transformations which transforms the graph of $C$ on to the graph of $y = \dfrac{9 - x}{x^2 - 6x}$. [2]

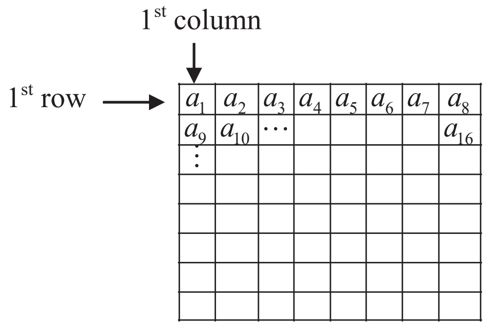
**7** Given that $\sin^{-1} y = \ln(1 + x)$, where $0 < x < 1$, show that $(1 + x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 - y^2}$. [2]

**(i)** By further differentiation, find the Maclaurin expansion of $y$ in ascending powers of $x$, up to and including the term in $x^2$. [4]

**(ii)** Use your expansion from **(i)** and integration to find an approximate expression for $\displaystyle\int \frac{\sin(\ln(1 + x))}{x}\,\mathrm{d}x$. Hence find an approximate value for $\displaystyle\int_0^{0.5} \frac{\sin(\ln(1 + x))}{x}\,\mathrm{d}x$. [3]

[Turn over

**8** **(a)** A sequence of numbers $a_1, a_2, a_3, \ldots, a_{64}$ is such that $a_{n+1} = a_n + d$, where $1 \le n \le 63$ and $d$ is a constant. The 64 numbers fill the 64 squares in the $8 \times 8$ grid in such a way that $a_1$ to $a_8$ fills the first row of boxes from left to right in that order. Similarly, $a_9$ to $a_{16}$ fills the second row of boxes from left to right in that order.




Given that the sums of the numbers in the **first row** and in the **third column** are 58 and 376 respectively, find the values of $a_1$ and $d$. [4]

**(b)** A geometric series has first term $a$ and common ratio $r$, where $a$ and $r$ are non-zero. The sum to infinity of the series is 2. The sum of the six terms of this series from the 4th term to the 9th term is $-\frac{63}{256}$. Show that $512r^9 - 512r^3 - 63 = 0$.

Find the two possible values of $r$, justifying the choice of your answers. [5]

**9** One of the roots of the equation $z^3 - az - 66 = 0$, where $a$ is real, is $w$.

**(i)** Given that $w = b - \sqrt{2}\mathrm{i}$, where $b$ is real, find the exact values of $a$ and $\frac{w}{w^*}$. [6]

**(ii)** Given instead that $w = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, $-\pi < \theta < -\frac{3\pi}{4}$, find $\left|aw^2 + 66w\right|$ and $\arg\left(aw^2 + 66w\right)$ in terms of $r$ and $\theta$. [4]

**10** The point $M$ has position vector relative to the origin $O$, given by $6\mathbf{i} - 5\mathbf{j} + 11\mathbf{k}$. The line $l_1$ has equation $x - 7 = \frac{y}{3} = \frac{z+2}{-2}$, and the plane $\pi$ has equation $4x - 2y - z = 30$.

**(i)** Show that $l_1$ lies in $\pi$. [2]

**(ii)** Find a cartesian equation of the plane containing $l_1$ and $M$. [3]

The point $N$ is the foot of perpendicular from $M$ to $l_1$. The line $l_2$ is the line passing through $M$ and $N$.

**(iii)** Find the position vector of $N$ and the area of triangle $OMN$. [5]

**(iv)** Find the acute angle between $l_2$ and $\pi$, giving your answer correct to the nearest $0.1°$. [3]

**11** [It is given that the volume of a cylinder with base radius $r$ and height $h$ is $\pi r^2 h$ and the volume of a cone with the same base radius and height is a third of a cylinder.]

A manufacturer makes double-ended coloured pencils that allow users to have two different colours in one pencil. The manufacturer determines that the shape of each coloured pencil is formed by rotating a trapezium $PQRS$ completely about the $x$-axis, such that it is a solid made up of a cylinder and two cones. The volume, $V$ cm$^3$, of the coloured pencil should be as large as possible.

It is given that the points $P$, $Q$, $R$ and $S$ lie on the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a$ and $b$ are positive constants. The points $R$ and $S$ are $(-a,0)$ and $(a,0)$ respectively, and the line $PQ$ is parallel to the $x$-axis.

**(i)** Verify that $P(a\cos\theta,\ b\sin\theta)$, where $0<\theta<\frac{\pi}{2}$, lies on the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Write down the coordinates of the point $Q$. [2]

**(ii)** Show that $V$ can be expressed as $V=k\pi\sin^2\theta(2\cos\theta+1)$, where $k$ is a constant in terms of $a$ and $b$. [3]

**(iii)** Given that $\theta=\theta_1$ is the value of $\theta$ which gives the maximum value of $V$, show that $\theta_1$ satisfies the equation $3\cos^2\theta+\cos\theta-1=0$. Hence, find the value of $\theta_1$. [4]

At $\theta=\frac{\pi}{6}$, the manufacturer wants to change one end of the coloured pencil to a rounded-end eraser. The eraser is formed by rotating the arc $PS$ completely about the $x$-axis.

**(iv)** Find the volume of the eraser in terms of $a$ and $b$. [3]

**12** A ball-bearing is dropped from a point $O$ and falls vertically through the atmosphere. Its speed at $O$ is zero, and $t$ seconds later, its velocity is $v$ m s$^{-1}$ and its displacement from $O$ is $x$ m. The rate of change of $v$ with respect to $t$ is given by $10 - 0.001v^2$.

**(i)** Show that $v=100\left(\frac{e^{\frac{t}{5}}-1}{e^{\frac{t}{5}}+1}\right)$. [4]

**(ii)** Find the value of $v_0$, where $v_0$ is the value approached by $v$ for large values of $t$. [1]

**(iii)** By using chain rule, form an equation relating $\frac{dx}{dt}$, $\frac{dv}{dt}$ and $\frac{dv}{dx}$. Given that $v=\frac{dx}{dt}$, form a differential equation relating $v$ and $x$. Show that

$$v=100\sqrt{1-e^{-\frac{x}{500}}}.$$ [5]

**(iv)** Find the distance of the ball-bearing from $O$ after 5 seconds, giving your answer correct to 2 decimal places. [3]

[Turn over

**Section A: Pure Mathematics [40 Marks]**

**1** Express $2\cos\theta\sin\dfrac{\theta}{2}$ in the form $\sin a\theta - \sin b\theta$, where $a$ and $b$ are constants to be found. [2]

Hence, find the exact value of $\alpha$, where $0<\alpha<\pi$, for which

$$\int_{\alpha}^{\pi}\left(3\cos\frac{3\theta}{2}-\cos\frac{\theta}{2}\right)e^{\cos\theta\sin\frac{\theta}{2}}\,d\theta = 4\left(\frac{1}{e}-1\right).$$ [5]

**2** **(i)** Show that $\dfrac{2}{2r+1}-\dfrac{3}{2r+3}+\dfrac{1}{2r+5}=\dfrac{Ar+B}{(2r+1)(2r+3)(2r+5)}$, where $A$ and $B$ are constants to be found. [2]

**(ii)** Hence find $\displaystyle\sum_{r=1}^{n}\frac{2r+9}{(2r+1)(2r+3)(2r+5)}$. (There is no need to express your answer as a single algebraic fraction.) [4]

**(iii)** It is given that $\displaystyle\sum_{r=1}^{n}\frac{2r+9}{(2r+1)(2r+3)(2r+5)}$ is within 0.01 of the sum to infinity.

Write down an inequality in terms of $n$, and hence find the smallest possible value of $n$. [3]

**3** The function f is defined by

$$\mathrm{f}: x \mapsto \frac{-x^2+5x-11}{x-2}, \quad x\in\mathbb{R}, x\neq 2.$$

**(i)** Find the equations of the asymptotes of the curve $y=\mathrm{f}(x)$. [3]

**(ii)** Determine whether f has an inverse, justifying your answer. [2]

Given that the function g is defined by

$$\mathrm{g}: x \mapsto \mathrm{f}(x), \quad x\in\mathbb{R}, 2<x\leqslant 4,$$

find $\mathrm{g}^{-1}(x)$ and state the domain of $\mathrm{g}^{-1}$. [4]

Sketch the graph of $y=\mathrm{g}^{-1}\mathrm{g}(x)$. [2]

**4** A curve $C$ has parametric equations

$$x = -\sqrt{t^2+4}, \quad y = \frac{\ln t}{t}, \quad \text{where } t > 0.$$

**(i)** Show that $\dfrac{dy}{dx} = \dfrac{(\ln t - 1)\sqrt{t^2+4}}{t^3}$. [3]

**(ii)** Find the exact coordinates of the turning point on $C$, and explain why it is a maximum. [4]

**(iii)** Sketch $C$. [3]

**(iv)** Show that the area bounded by $C$ and the lines $x = -\sqrt{13}$ and $x = -\sqrt{5}$ is given by

$$\int_1^3 \frac{\ln t}{\sqrt{t^2+4}}\, dt.$$

Find the area, giving your answer to 4 decimal places. [3]

**Section B: Probability and Statistics [60 Marks]**

**5** Mr and Mrs Lee participate in a game show, together with 3 other men and 5 other women. In the first round, the 10 participants are grouped into 5 pairs.

**(i)** Find the number of ways the pairings can be done if there is only 1 pair of the same gender. [3]

After the first round, Mr and Mrs Lee are both eliminated. The remaining 8 participants are seated around a round table. Find the number of ways this can be done if

**(ii)** there are no restrictions, [1]

**(iii)** the 3 men are not all seated together. [3]

[Turn over

6 As the use of email becomes more prevalent, the number of unsolicited email (also known as spam) received increases. Besides advertisements, spam can now be cleverly disguised as business emails and contain malware. Hence, there is a need to use spam filter.

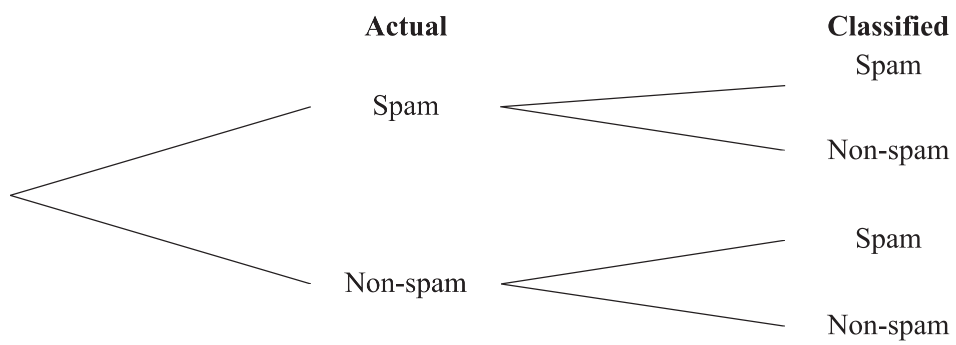
The probability that Yip receives a spam email is $p$. He uses a spam filter, Spam Guard Plus, to filter his emails. He has the following information:

P(an email is classified correctly) = $\dfrac{41}{50}$;

P(an email is classified correctly | it is classified as spam) = $\dfrac{38}{45}$;

P(an email is classified correctly | it is a spam email) = $\dfrac{19}{20}$.

**(i)** Show that $p = \dfrac{4}{5}$. Hence, complete the probability tree below. [5]




**(ii)** Andy and Betty notices that, on average, 30% and 70% of their emails are spam respectively. State whether Spam Guard Plus would be **(a)** more appropriate for Andy, **(b)** more appropriate for Betty, or **(c)** just as appropriate for both Andy and Betty. Justify your answer. [2]

7 Seven members of the school cross country team undergo a new training programme to improve their fitness. During a particular session, each of them has to complete a 200 metre run and to achieve as many push-ups as possible in one minute. The times taken for the 200 metre run, $t$ seconds, together with the number of push-ups each runner achieves, $n$, are shown in the table.

| Student | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| $t$ | 38.3 | 42.1 | 35.1 | 40.1 | 32.0 | 31.6 | 41.0 |
| $n$ | 44 | 35 | 48 | 42 | 49 | 49 | 40 |

**(i)** Draw a scatter diagram to illustrate the data, labelling the axes. [1]

**(ii)** Explain using your scatter diagram why the linear model $n = at + b$ would not be appropriate. [1]

It is thought that the relationship between $n$ and $t$ can be modelled by one of the formulae

$$n = c\left(t-30\right)^2 + d \qquad \text{or} \qquad n = e\left(t-30\right)^3 + f$$

where $c$, $d$, $e$ and $f$ are constants.

**(iii)** The product moment correlation coefficient between $n$ and $\left(t-30\right)^2$ is $-0.980$, correct to 3 decimal places. Determine, with a reason, which of the 2 models is more appropriate. [2]

**(iv)** It is known that student H is able to do 48 push-ups in one minute. It is required to estimate student H's timing for the 200 metre run. Find the equation of a suitable regression line, and use it to find the required estimate. Comment on the reliability of this estimate. [5]

8 A bag contains two balls numbered 3, $n$ balls numbered 2 and three balls numbered 1. A player picks two balls at random from the bag at the same time.

If the difference between the numbers on the two balls is 2, the player receives \$6.

If the difference between the numbers on the two balls is 1, the player does not receive or lose any money.

If the numbers on the 2 balls are the same, the player loses \$1.

**(i)** Show that the largest value of $n$ such that player is expected to receive money from this game is 8. [5]

For the rest of this question, take the value of $n$ to be 8.

**(ii)** Show that the probability that a player loses money in a game is $\frac{16}{39}$. [1]

Victoria plays this game 50 times.

**(iii)** Find the probability that she lost money for at least 20 games. [2]

**(iv)** The probability that Victoria loses money in $r$ games is more than 0.1. Find the set of values of $r$. [3]

[Turn over

**9** Exposure to Volatile Organic Compounds (VOCs), which have been identified in indoor air, is suspected as a cause for headaches and respiratory symptoms. Indoor plants have not only a positive psychological effect on humans, but may also improve the air quality. Certain species of indoor plants were found to be effective removers of VOCs.

A commonly known VOC is Benzene. The following data gives the benzene levels, $x$ (in ppm) in 40 test chambers containing the indoor plant *Epipremnum aureum*.

$$n = 40, \qquad \sum(x - 26.0) = -30.1, \qquad \sum(x - 26.0)^2 = 214.61.$$

The initial mean Benzene level (in ppm) without *Epipremnum aureum* was found to be 26.0.

**(i)** Test, at the 5% level of significance, the claim that the mean Benzene level, $\mu$ (in ppm), has decreased as a result of the indoor plant *Epipremnum aureum*. You should state your hypotheses clearly. [5]

**(ii)** State, giving a reason, whether there is a need to make any assumptions about the population distribution of the Benzene level in order for the test to be valid. [2]

The Benzene levels of another 50 test chambers containing the indoor plant *Epipremnum aureum* were recorded, The sample mean is $\overline{x}$ ppm and the sample variance is 8.33 ppm$^2$.

**(iii)** The acceptance region of a test of the null hypothesis $\mu = 26.0$ is $\overline{x} > 25.1$. State the alternative hypothesis and find the level of significance of the test. [4]

**(iv)** If the null hypothesis is $\mu = \mu_0$, where $\mu_0 > 26.0$, would the significance level of a test with the same acceptance region in part **(iii)** be larger or smaller than that found in part **(iii)**? Give a reason for your answer. [2]

**10 In this question, you should state clearly the values of the parameters of any distribution you use.**

A bus service plies from a point $A$ in the city, through a point $B$, and then to its terminal station at point $C$.

Journey times in minutes from $A$ to $B$ have the distribution $\mathrm{N}\left(28, 4^2\right)$.

**(i)** Find the probability that a randomly selected bus journey from $A$ to $B$ is completed within 35 minutes. [1]

The journey times in minutes from $A$ to $C$, have the distribution $\mathrm{N}\left(46.2, 4.8^2\right)$.

**(ii)** The journey times in minutes from $B$ to $C$ have the distribution $\mathrm{N}\left(\mu, \sigma^2\right)$. Given that the journey times from $A$ to $B$ are independent of the journey times from $B$ to $C$, find the value of $\mu$ and show that $\sigma^2 = 7.04$. [3]

**(iii)** Find the set of values of $k$ such that at least 90% of all journey times from $A$ to $C$ can be completed within $k$ minutes. [2]

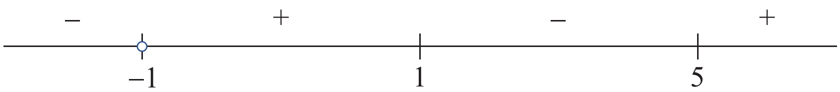
The performance of the bus operation is deemed as "unreliable" if a random sample of 70 journeys from $A$ to $C$ yields a mean journey time exceeding 47 minutes.

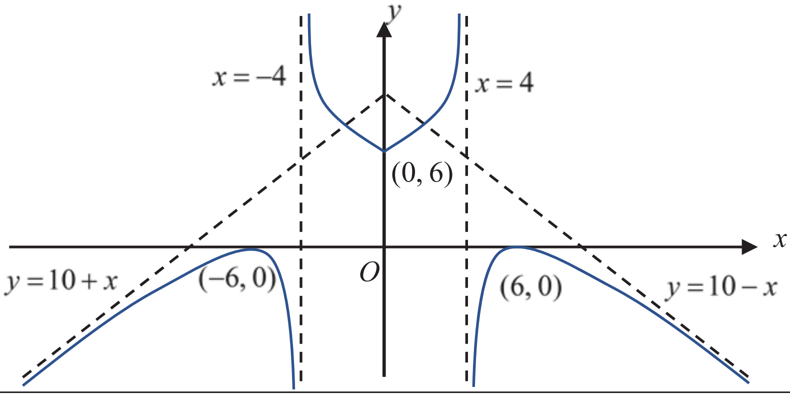
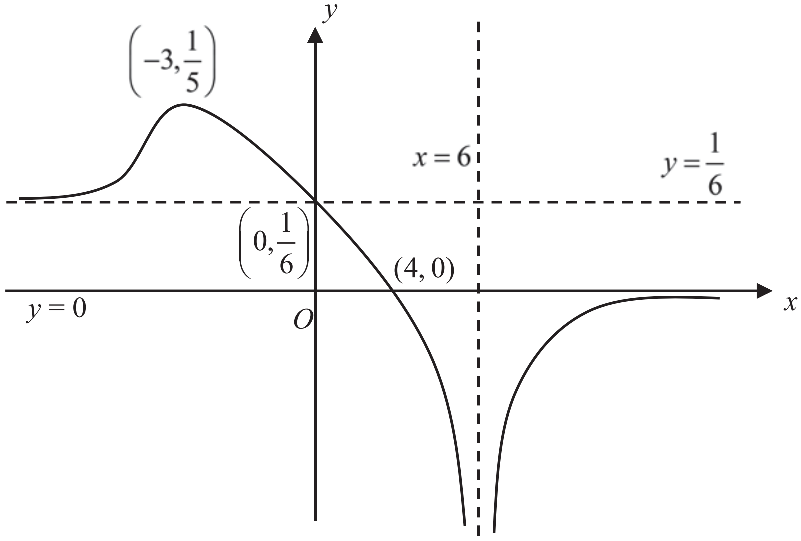
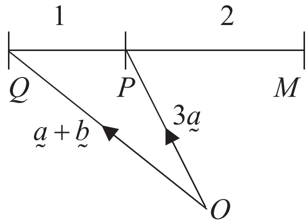
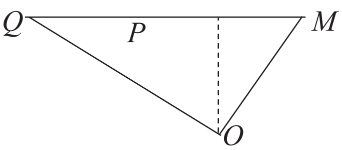
**(iv)** Two independent random samples of 70 journeys from $A$ to $C$ are taken. Find the probability that both samples will result in the performance of the bus operation to be deemed as "unreliable". [3]

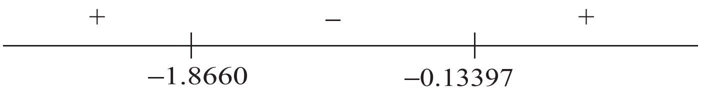
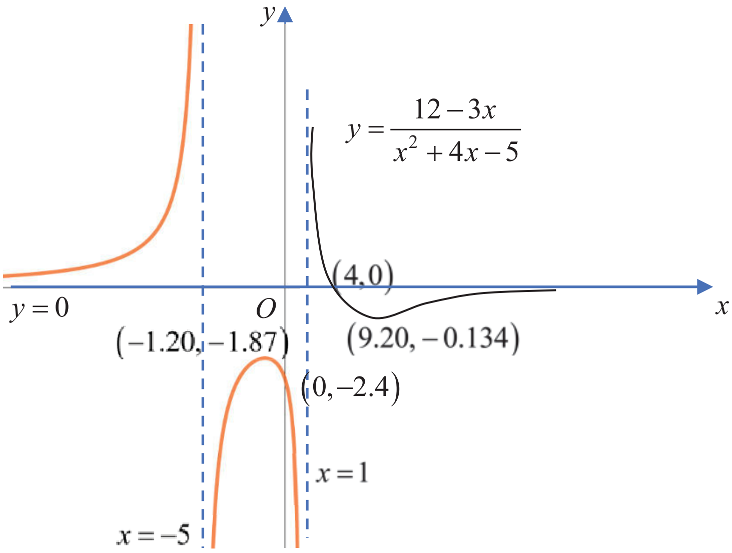
To improve the reliability performance of the bus operation, more bus lanes are introduced and some bus stops along the bus route are removed. The journey times from $A$ to $B$ are now reduced by 10%, and the journey times from $B$ to $C$ now have the distribution $\mathrm{N}(\mu - 1, 8)$.

**(v)** Find the probability that two journeys from $A$ to $C$ are completed within a total of 90 minutes. [4]

[Turn over

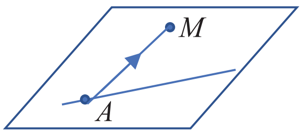
| Qn | Solution | Comments |
|---|---|---|
| 1 | $\frac{12}{x+1}-(7-x)=\frac{12+(x-7)(x+1)}{x+1}$<br>$=\frac{x^2-6x+5}{x+1}$ | |
| | $\frac{12}{x+1}\le 7-x$<br>$\frac{x^2-6x+5}{x+1}\le 0$<br>$\frac{(x-1)(x-5)}{x+1}\le 0$<br><br><br>$x<-1$ or $1\le x\le 5$ | |
| 2i | $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x^2)=\frac{2x}{1+(x^2)^2}$<br>$=\frac{2x}{1+x^4}$ | |
| ey2ii | $\int_0^1 x\tan^{-1}(x^2)\,\mathrm{d}x=\left[\frac{x^2}{2}\tan^{-1}(x^2)\right]_0^1-\int_0^1\frac{x^2}{2}\left(\frac{2x}{1+x^4}\right)\mathrm{d}x$<br>$=\frac{1}{2}\left(\frac{\pi}{4}\right)-\left[\frac{1}{4}\ln(1+x^4)\right]_0^1$<br>$=\frac{\pi}{8}-\frac{1}{4}\ln 2$ | |
| 3i | $\frac{\mathrm{d}}{\mathrm{d}x}(3x^2 2^x)=3(x^2 2^x\ln 2+2^{x+1}x)$<br>$=3x2^x(x\ln 2+2)$ | |
| 3ii | At $x=1$, gradient $=6(\ln 2+2)$<br>$y=6$<br><br>Equation of tangent:<br>$y-6=6(\ln 2+2)(x-1)$<br>$y=6x(\ln 2+2)-6\ln 2-6$ | |

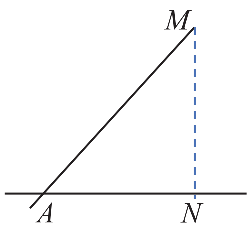
| Qn | Solution | Comments |
|---|---|---|
| 4i |  | |
| 4ii |  | |
| 5i | $\overrightarrow{OP}=\dfrac{2\overrightarrow{OQ}+\overrightarrow{OM}}{3}$<br>$\overrightarrow{OM}=3\overrightarrow{OP}-2\overrightarrow{OQ}$<br>$=3(3\underset{\sim}{a})-2(\underset{\sim}{a}+\underset{\sim}{b})$<br>$=7\underset{\sim}{a}-2\underset{\sim}{b}$<br> | |
| 5ii | $\overrightarrow{PQ}\times\overrightarrow{OM}=(\underset{\sim}{a}+\underset{\sim}{b}-3\underset{\sim}{a})\times(7\underset{\sim}{a}-2\underset{\sim}{b})$<br>$=(\underset{\sim}{b}-2\underset{\sim}{a})\times(7\underset{\sim}{a}-2\underset{\sim}{b})$<br>$=7\underset{\sim}{b}\times\underset{\sim}{a}-2\underset{\sim}{b}\times\underset{\sim}{b}-14\underset{\sim}{a}\times\underset{\sim}{a}+4\underset{\sim}{a}\times\underset{\sim}{b}$<br><br>Since $\underset{\sim}{a}\times\underset{\sim}{a}=\underset{\sim}{b}\times\underset{\sim}{b}=\underset{\sim}{0}$ and $\underset{\sim}{a}\times\underset{\sim}{b}=-\underset{\sim}{b}\times\underset{\sim}{a}$,<br>$\overrightarrow{PQ}\times\overrightarrow{OM}=3\underset{\sim}{b}\times\underset{\sim}{a}$ | |
| 5iii | $\dfrac{\left\lvert\overrightarrow{PQ}\times\overrightarrow{OM}\right\rvert}{\left\lvert\overrightarrow{PQ}\right\rvert}=\left\lvert\overrightarrow{OM}\times\dfrac{\overrightarrow{PQ}}{\left\lvert\overrightarrow{PQ}\right\rvert}\right\rvert$<br><br>$\dfrac{\left\lvert\overrightarrow{PQ}\times\overrightarrow{OM}\right\rvert}{\left\lvert\overrightarrow{PQ}\right\rvert}$ is the shortest distance from $O$ to line $PQ$. | |

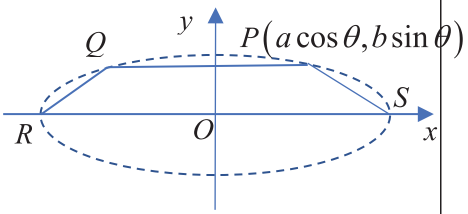
| Qn | Solution | Comments |
|---|---|---|
| 6i | Let $y=\dfrac{12-3x}{x^2+4x-5}$ $\Rightarrow yx^2+(4y+3)x+(-5y-12)=0$<br>When $x$ is real $\Rightarrow$ discriminant $\geqslant 0$<br>$\Rightarrow (4y+3)^2-4y(-5y-12)\geqslant 0$<br>$\Rightarrow 16y^2+24y+9+20y^2+48y\geqslant 0$<br>$\Rightarrow 36y^2+72y+9\geqslant 0$<br>$\Rightarrow 36(y+1.8660)(y+0.13397)\geqslant 0$<br><br>$y\leqslant -1.8660$ or $y\geqslant -0.13397$<br>$R_f=(-\infty,-1.87]\cup[-0.134,\infty)$ | |
| 6ii |  | |
| 6iii | $y=\dfrac{12-3x}{(x-1)(x+5)}$<br>Stretch parallel to the $y$-axis with $x$-axis invariant with factor $\dfrac{1}{3}$:<br>$y=\dfrac{4-x}{(x-1)(x+5)}$<br>Translate in the positive $x$-direction by 5 units:<br>$y=\dfrac{4-(x-5)}{(x-5-1)(x-5+5)}$ ie. $y=\dfrac{9-x}{(x-6)(x)}$ | |

| Qn | Solution | Comments |
|---|---|---|
| 7 | $\sin^{-1} y = \ln(1+x)$<br>$\frac{1}{\sqrt{1-y^2}}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x}$<br>$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-y^2}$ | |
| 7i | $(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-y^2}$<br>$\frac{\mathrm{d}y}{\mathrm{d}x} + (1+x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{y}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$<br>When $x=0, y=0, \frac{\mathrm{d}y}{\mathrm{d}x}=1, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{\mathrm{d}y}{\mathrm{d}x} = -1$<br>$y = 0 + x - \frac{1}{2!}x^2 + \ldots \approx x - \frac{1}{2}x^2$ | |
| 7ii | $\sin^{-1} y = \ln(1+x)$<br>$y = \sin\left(\ln(1+x)\right) \approx x - \frac{1}{2}x^2$<br>$\int \frac{\sin\left(\ln(1+x)\right)}{x}\,\mathrm{d}x \approx \int\left(1-\frac{1}{2}x\right)\mathrm{d}x$<br>$= x - \frac{x^2}{4} + C$<br>$\int_0^{0.5} \frac{\sin\left(\ln(1+x)\right)}{x}\,\mathrm{d}x \approx \left[x - \frac{x^2}{4}\right]_0^{0.5}$<br>$= 0.4375$ | |
| 8i | The sum of the numbers in the first row:<br>$\frac{8}{2}\left[2a_1 + 7d\right] = 58 \Rightarrow 4a_1 + 14d = 29$ ------- (1)<br>The sum of the numbers in the third column:<br>$a_3, a_{11}, a_{19}, \cdots, a_{59}$ are in arithmetic progression with common difference $8d$.<br>$a_3 + a_{11} + a_{19} + \cdots + a_{59} = \frac{8}{2}\left[2(a_1+2d) + 7(8d)\right] = 376$<br>$\Rightarrow a_1 + 30d = 47$ -------- (2)<br>By GC, $a_1 = 2, d = 1.5$ | |
| 8ii | Sum to infinity $= 2 \Rightarrow \frac{a}{1-r} = 2$ -------------- (1)<br>Sum of the terms from the 4th term to 9th term:<br>$\frac{ar^3\left(1-r^6\right)}{1-r} = -\frac{63}{256}$ -------- (2)<br>Sub (1) into (2): | |

| Qn | Solution | Comments |
|---|---|---|
| | $2r^3\left(1-r^6\right)=-\dfrac{63}{256}$<br>$\Rightarrow 512r^3-512r^9=-63$<br>$\Rightarrow 512r^9-512r^3-63=0$<br><br>From the GC, $r=1.02(\text{NA}),\ -0.977$ or $-0.5$ as $-1<r<1$ for the sum to infinity to exist. | |
| 9i | Coefficients of equation are real $\Rightarrow b+\sqrt{2}\mathrm{i}$ is also a root.<br>$z^3-az-66=\left(z-b+\sqrt{2}\mathrm{i}\right)\left(z-b-\sqrt{2}\mathrm{i}\right)(z+k)$<br>$=\left[(z-b)^2-\left(\sqrt{2}\mathrm{i}\right)^2\right](z+k)$<br>$=\left(z^2-2bz+b^2+2\right)(z+k)$<br>$=z^3+(-2b+k)z^2+\left(b^2+2-2bk\right)z+k\left(b^2+2\right)$<br>Comparing coefficients of $z^2$, $z$ and constant:<br>$-2b+k=0$<br>$\left(b^2+2\right)-2kb=-a$<br>$k\left(b^2+2\right)=-66$<br>Solving the equations,<br>$k=2b \quad \Rightarrow 2b\left(b^2+2\right)=-66 \qquad \Rightarrow b=-3$<br><br>**Alternatively**<br>Substitute $z=b-\sqrt{2}\mathrm{i}$<br>$\left(b-\sqrt{2}\mathrm{i}\right)^3-a\left(b-\sqrt{2}\mathrm{i}\right)-66=0$<br>$b^3+3b^2\left(-\sqrt{2}\mathrm{i}\right)+3b\left(-\sqrt{2}\mathrm{i}\right)^2+\left(-\sqrt{2}\mathrm{i}\right)^3-ab+\sqrt{2}a\mathrm{i}-66=0$<br>Comparing real and imaginary parts<br>$b^3-6b-ab-66 = 0 \quad --(1)$<br>$-3\sqrt{2}b^2+2\sqrt{2}+\sqrt{2}a = 0 \quad --(2)$<br>From (2): $a=3b^2-2--(3)$<br>Substitute (3) into (1):<br>$b^3-6b-b\left(3b^2-2\right)-66=0$<br>$2b^3+4b+66=0$<br>$b=-3$<br><br>$b=-3\Rightarrow a=25$<br><br>$\dfrac{w}{w^*}=\dfrac{-3-\sqrt{2}\mathrm{i}}{-3+\sqrt{2}\mathrm{i}}\times\dfrac{-3-\sqrt{2}\mathrm{i}}{-3-\sqrt{2}\mathrm{i}}=\dfrac{7+6\sqrt{2}\mathrm{i}}{11}$ | |

| Qn | Solution | Comments |
|---|---|---|
| 9ii | $w$ is a root $\Rightarrow w^3 - aw - 66 = 0 \quad \Rightarrow aw + 66 = w^3 \Rightarrow aw^2 + 66w = w^4$<br>$w^4 = \left(re^{i\theta}\right)^4 = r^4 e^{i(4\theta)}$<br>$\therefore \left\lvert aw^2 + 66w \right\rvert = \left\lvert w^4 \right\rvert = r^4$<br>$\arg\left(aw^2 + 66w\right) = \arg\left(r^4 e^{i(4\theta)}\right)$<br>$-\pi < \theta < -\dfrac{3\pi}{4} \quad \Rightarrow -4\pi < 4\theta < -3\pi \qquad \Rightarrow 0 < 4\theta + 4\pi < \pi$<br>$\therefore \arg\left(aw^2 + 66w\right) = 4\theta + 4\pi$ | |
| 10i | Equation of $l_1$ is $\underset{\sim}{r} = \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, s \in \mathbb{R}$<br>$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} = 4 - 6 + 2 = 0$<br>$\therefore$ line $l_1$ is perpendicular to normal vector of $\pi$.<br>$\therefore$ line $l_1$ is parallel to $\pi$.<br><br>$\begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} = 28 + 0 + 2 = 30$<br>$\therefore (7,0,-2)$ is in $\pi$.<br>Thus, $l_1$ is in $\pi$. | |
| 10ii | Let $A(7,0,-2)$.<br>$\overrightarrow{AM} = \begin{pmatrix} 6 \\ -5 \\ 11 \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 13 \end{pmatrix}$<br><br><br>normal vector of plane $= \begin{pmatrix} -1 \\ -5 \\ 13 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -29 \\ 11 \\ 2 \end{pmatrix}$<br>$\begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -29 \\ 11 \\ 2 \end{pmatrix} = -207$<br><br>Cartesian equation of the plane is $-29x + 11y + 2z = -207$ | |
| 10iii | Since $N$ is a point on $l_1$,<br>$\overrightarrow{ON} = \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$, for some $s \in \mathbb{R}$ | |

| Qn | Solution | Comments |
|---|---|---|
| | $\overrightarrow{MN}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}=0$<br>$\left[\left(\begin{pmatrix}7\\0\\-2\end{pmatrix}+s\begin{pmatrix}1\\3\\-2\end{pmatrix}\right)-\begin{pmatrix}6\\-5\\11\end{pmatrix}\right]\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}=0$<br>$\left[\begin{pmatrix}1\\5\\-13\end{pmatrix}+s\begin{pmatrix}1\\3\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}=0$<br>$42+14s=0$<br>$s=-3$<br>$\therefore\overrightarrow{ON}=\begin{pmatrix}7-3\\3(-3)\\-2-2(-3)\end{pmatrix}=\begin{pmatrix}4\\-9\\4\end{pmatrix}$<br><br>Alternatively,<br>$\overrightarrow{AN}=\left(\overrightarrow{AM}\cdot\frac{1}{\sqrt{14}}\begin{pmatrix}1\\3\\-2\end{pmatrix}\right)\frac{1}{\sqrt{14}}\begin{pmatrix}1\\3\\-2\end{pmatrix}$<br>$=\frac{1}{14}\left(\left(\begin{pmatrix}6\\-5\\11\end{pmatrix}-\begin{pmatrix}7\\0\\-2\end{pmatrix}\right)\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}\right)\begin{pmatrix}1\\3\\-2\end{pmatrix}$<br>$=\frac{1}{14}\left(\begin{pmatrix}-1\\-5\\13\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}\right)\begin{pmatrix}1\\3\\-2\end{pmatrix}$<br>$=\frac{-1-15-26}{14}\begin{pmatrix}1\\3\\-2\end{pmatrix}$<br>$=\begin{pmatrix}-3\\-9\\6\end{pmatrix}$<br><br><br>$\therefore\overrightarrow{ON}=\begin{pmatrix}7\\0\\-2\end{pmatrix}+\begin{pmatrix}-3\\-9\\6\end{pmatrix}=\begin{pmatrix}4\\-9\\4\end{pmatrix}$<br><br>Area of triangle $=\frac{1}{2}\left\|\overrightarrow{OM}\times\overrightarrow{ON}\right\|$<br>$=\frac{1}{2}\left\|\begin{pmatrix}6\\-5\\11\end{pmatrix}\times\begin{pmatrix}4\\-9\\4\end{pmatrix}\right\|$<br>$=\frac{1}{2}\left\|\begin{pmatrix}79\\20\\-34\end{pmatrix}\right\|$<br>$=44.2\text{ unit}^2$ | |
| \10iv | Let $\theta$ be the angle between the normal of $\pi$ and $l_2$.<br>$\overrightarrow{MN}=\begin{pmatrix}4\\-9\\4\end{pmatrix}-\begin{pmatrix}6\\-5\\11\end{pmatrix}=\begin{pmatrix}-2\\-4\\-7\end{pmatrix}$ | |

| Qn | Solution | Comments |
|---|---|---|
| | $\cos\theta = \dfrac{\begin{pmatrix}-2\\-4\\-7\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\-1\end{pmatrix}}{\sqrt{69}\sqrt{21}} = \dfrac{7}{\sqrt{69}\sqrt{21}}$<br>$\theta = 79.403°$<br><br>Acute angle between $\pi$ and $l_2$ = $90° - 79.403° = 10.6°$ | |
| 11i | Given the curve $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$,<br><br><br>LHS = $\dfrac{(a\cos\theta)^2}{a^2}+\dfrac{(b\sin\theta)^2}{b^2}$<br>$=\cos^2\theta+\sin^2\theta$<br>$=1=\text{RHS}$<br>$P(a\cos\theta,\ b\sin\theta)$ lies on the curve.<br>$Q$ is $(-a\cos\theta,\ b\sin\theta)$. [by symmetry in the $y$-axis] | |
| 11ii | $V$ = cylinder + 2 identical cones<br>$=\pi(b\sin\theta)^2(2a\cos\theta)+2*\dfrac{1}{3}\pi(b\sin\theta)^2(a-a\cos\theta)$<br>$=\dfrac{2}{3}\pi b^2\sin^2\theta(3a\cos\theta+a-a\cos\theta)$<br>$=\dfrac{2}{3}\pi ab^2\sin^2\theta(2\cos\theta+1)$<br><br>$\therefore k=\dfrac{2}{3}ab^2$ | |
| 11iii | At $\theta=\theta_1$,<br>$\dfrac{dV}{d\theta}=0$<br>$\dfrac{dV}{d\theta}=\dfrac{2}{3}\pi ab^2\left[\sin^2\theta(-2\sin\theta)+(2\cos\theta+1)(2\sin\theta\cos\theta)\right]=0$<br>$\sin\theta\left(-2\sin^2\theta+2\cos\theta+4\cos^2\theta\right)=0$<br>$\sin\theta=0$ or $-2(1-\cos^2\theta)+2\cos\theta+4\cos^2\theta=0$<br>$\left(\text{NA since } 0<\theta<\dfrac{\pi}{2}\right)$<br>$6\cos^2\theta+2\cos\theta-2=0$<br>$3\cos^2\theta+\cos\theta-1=0$ (shown)<br>$\cos\theta=0.43425, -0.76759\left(\text{NA} \because 0<\theta<\dfrac{\pi}{2}\right)$<br>$\Rightarrow \theta_1=1.1216\approx1.12$<br>$\dfrac{dV}{d\theta}=\dfrac{2}{3}\pi ab^2\left[-2\sin^3\theta+(2\cos\theta+1)\sin2\theta\right]$<br>$\dfrac{d^2V}{d\theta^2}=\dfrac{2}{3}\pi ab^2\left[-6\sin^2\theta\cos\theta+(-2\sin\theta)\sin2\theta\right.$<br>$\left.+2\cos2\theta(2\cos\theta+1)\right]$ | |

| Qn | Solution | Comments |
| --- | --- | --- |
| | When $\theta_1 = 1.1216$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = -3.90\pi ab^2$<br>Since $a$ and $b$ are positive, $\dfrac{\mathrm{d}^2V}{\mathrm{d}\theta^2} < 0$<br>Hence, $\theta_1$ gives a maximum value of $V$. | |
| 11iv | $\text{Volume} = \pi\int_{a\cos\left(\frac{\pi}{6}\right)}^{a} y^2 \, \mathrm{d}x$<br>$= \pi\int_{\frac{\sqrt{3}}{2}a}^{a} b^2\left(1-\dfrac{x^2}{a^2}\right)\mathrm{d}x$<br>$= \pi b^2\int_{\frac{\sqrt{3}}{2}a}^{a} \left(1-\dfrac{x^2}{a^2}\right)\mathrm{d}x$<br>$= \pi b^2\left[x-\dfrac{x^3}{3a^2}\right]_{\frac{\sqrt{3}}{2}a}^{a}$<br>$= \pi b^2\left\{\left(a-\dfrac{a^3}{3a^2}\right)-\left(\dfrac{\sqrt{3}}{2}a-\dfrac{3\sqrt{3}a^3}{24a^2}\right)\right\}$<br>$= \pi ab^2\left(\dfrac{2}{3}-\dfrac{3\sqrt{3}}{8}\right)$ (**or** $0.0171\pi ab^2$) | |
| 12i | $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 10-0.001v^2 = \dfrac{10000-v^2}{1000}$<br>$\Rightarrow \int \dfrac{1}{10000-v^2}\mathrm{d}v = \int \dfrac{1}{1000}\mathrm{d}t$<br>$\Rightarrow \dfrac{1}{200}\ln\left\|\dfrac{100+v}{100-v}\right\| = \dfrac{t}{1000}+d$<br>$\Rightarrow \ln\left\|\dfrac{100+v}{100-v}\right\| = \dfrac{t}{5}+d'$<br>$\Rightarrow \left\|\dfrac{100+v}{100-v}\right\| = Ce^{\frac{t}{5}}$<br>$\Rightarrow \dfrac{100+v}{100-v} = De^{\frac{t}{5}}$<br>$t=0, v=0 \Rightarrow D=1$<br>$\dfrac{100+v}{100-v} = e^{\frac{t}{5}} \Rightarrow 100+v = e^{\frac{t}{5}}(100-v) \Rightarrow v\left(e^{\frac{t}{5}}+1\right) = 100e^{\frac{t}{5}}-100$<br>$v = 100\left(\dfrac{e^{\frac{t}{5}}-1}{e^{\frac{t}{5}}+1}\right)$ (shown) | |
| 12ii | Method 1:<br>$v = 100\left(\dfrac{e^{\frac{t}{5}}-1}{e^{\frac{t}{5}}+1}\right) = 100\left(1-\dfrac{2}{e^{\frac{t}{5}}+1}\right)$ | |

| Qn | Solution | Comments |
|---|---|---|
| | When $t \to \infty,\ \dfrac{1}{e^{\frac{t}{5}}+1} \to 0,\ v \to 100 \quad \therefore v_0 = 100$<br><br>Method 2:<br>$v = 100\left(\dfrac{e^{\frac{t}{5}}-1}{e^{\frac{t}{5}}+1}\right) = 100\left(\dfrac{e^{\frac{t}{5}}\left(1-e^{-\frac{t}{5}}\right)}{e^{\frac{t}{5}}\left(1+e^{-\frac{t}{5}}\right)}\right)$<br><br>When $t \to \infty,\ e^{-\frac{t}{5}} \to 0,\quad v \to 100 \quad \therefore v_0 = 100$<br><br>Method 3:<br>When $t \to \infty,\ \dfrac{1}{e^{\frac{t}{5}}+1} \to 0,\ v \to 100 \quad \therefore v_0 = 100$<br>$v = 100\left(\dfrac{e^{\frac{t}{5}}-1}{e^{\frac{t}{5}}+1}\right) \to 100\left(\dfrac{e^{\frac{t}{5}}}{e^{\frac{t}{5}}}\right)$<br>$v \to 100 \quad \therefore v_0 = 100$ | |
| 12iii | $\dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt}$<br>$\Rightarrow 10 - 0.001v^2 = \dfrac{dv}{dx}(v)$<br>$\Rightarrow \int 1\,dx = \displaystyle\int \frac{v}{10-0.001v^2}dv = \int \frac{1000v}{10000-v^2}dv$<br>$\Rightarrow -500\displaystyle\int \frac{-2v}{10000-v^2}dv = x + c$<br>$\Rightarrow \ln\lvert 10000 - v^2\rvert = -\dfrac{x}{500} + c'$<br>$\Rightarrow \lvert 10000 - v^2\rvert = Ae^{-\frac{x}{500}}$<br>$\Rightarrow 10000 - v^2 = Be^{-\frac{x}{500}}$<br>$x = 0, v = 0 \Rightarrow B = 10000$<br>$10000 - v^2 = 10000e^{-\frac{x}{500}}$<br>$v^2 = 10000 - 10000e^{-\frac{x}{500}}$<br>$v \geqslant 0 \quad \Rightarrow v = 100\sqrt{1-e^{-\frac{x}{500}}}$ (shown) | |
| 12iv | When $t = 5, v = 100\left(\dfrac{e-1}{e+1}\right) = 46.2117$<br><br>When $v = 46.2117, 46.2117 = 100\sqrt{1-e^{-\frac{x}{500}}}$<br>$\therefore x = 120.11$<br>The required distance is 120.11 m. | |

| Qn | Solution | Comments |
|---|---|---|
| 1 | $2\cos\theta\sin\frac{\theta}{2}=\sin P-\sin Q$<br>where $\frac{P+Q}{2}=\theta,\ \frac{P-Q}{2}=\frac{\theta}{2}$<br>$\Rightarrow P=\frac{3\theta}{2},\ Q=\frac{\theta}{2}$<br>$2\cos\theta\sin\frac{\theta}{2}=\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}$<br>$\therefore a=\frac{3}{2},\ b=\frac{1}{2}.$ | |
| | $\cos\theta\sin\frac{\theta}{2}=\frac{1}{2}\sin\frac{3\theta}{2}-\frac{1}{2}\sin\frac{\theta}{2}=\mathrm{f}(\theta)$<br>$\mathrm{f}'(\theta)=\frac{1}{4}\left(3\cos\frac{3\theta}{2}-\cos\frac{\theta}{2}\right)$<br>$\int_{\alpha}^{\pi}\left(3\cos\frac{3\theta}{2}-\cos\frac{\theta}{2}\right)\mathrm{e}^{\cos\theta\sin\frac{\theta}{2}}\ \mathrm{d}\theta=4\left(\frac{1}{\mathrm{e}}-1\right)$<br>$4\int_{\alpha}^{\pi}\frac{1}{4}\left(3\cos\frac{3\theta}{2}-\cos\frac{\theta}{2}\right)\mathrm{e}^{\cos\theta\sin\frac{\theta}{2}}\ \mathrm{d}\theta=4\left(\frac{1}{\mathrm{e}}-1\right)$<br>$4\int_{\alpha}^{\pi}\mathrm{f}'(\theta)\mathrm{e}^{\mathrm{f}(\theta)}\ \mathrm{d}\theta=4\left(\frac{1}{\mathrm{e}}-1\right)$<br>$\left[\mathrm{e}^{\mathrm{f}(\theta)}\right]_{\alpha}^{\pi}=\frac{1}{\mathrm{e}}-1$<br>$\mathrm{e}^{\cos\pi\sin\frac{\pi}{2}}-\mathrm{e}^{\cos\alpha\sin\frac{\alpha}{2}}=\frac{1}{\mathrm{e}}-1$<br>$\mathrm{e}^{\cos\alpha\sin\frac{\alpha}{2}}=1$<br>$\Rightarrow\cos\alpha\sin\frac{\alpha}{2}=0$<br>$\cos\alpha=0$ or $\sin\frac{\alpha}{2}=0$<br>$\alpha=\ldots,-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\ldots$ or $\frac{\alpha}{2}=\ldots,-\pi,0,\pi,2\pi,\ldots$<br>or $\alpha=\ldots,-2\pi,0,2\pi,4\pi,\ldots$<br>$\therefore$ since $0<\alpha<\pi,\ \alpha=\frac{\pi}{2}$ and reject all other values. | |
| 2i | $\frac{2}{2r+1}-\frac{3}{2r+3}+\frac{1}{2r+5}$<br>$=\frac{2(2r+3)(2r+5)-3(2r+1)(2r+5)+(2r+1)(2r+3)}{(2r+1)(2r+3)(2r+5)}$<br>$=\frac{8r^2+32r+30-(12r^2+36r+15)+4r^2+8r+3}{(2r+1)(2r+3)(2r+5)}$<br>$=\frac{4r+18}{(2r+1)(2r+3)(2r+5)}$ | |

| Qn | Solution | Comments |
|---|---|---|

**2ii**

$$\sum_{r=1}^{n} \frac{2r+9}{(2r+1)(2r+3)(2r+5)}$$

$$= \frac{1}{2}\sum_{r=1}^{n} \frac{4r+18}{(2r+1)(2r+3)(2r+5)}$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left(\frac{2}{2r+1} - \frac{3}{2r+3} + \frac{1}{2r+5}\right)$$

$$= \frac{1}{2}\left[\begin{array}{cccccc} & \frac{2}{3} & - & \frac{3}{5} & + & \frac{1}{7} \\ + & \frac{2}{5} & - & \frac{3}{7} & + & \frac{1}{9} \\ + & \frac{2}{7} & - & \frac{3}{9} & + & \frac{1}{11} \\ & & & \vdots & & \\ + & \frac{2}{2n-3} & - & \frac{3}{2n-1} & + & \frac{1}{2n+1} \\ + & \frac{2}{2n-1} & - & \frac{3}{2n+1} & + & \frac{1}{2n+3} \\ + & \frac{2}{2n+1} & - & \frac{3}{2n+3} & + & \frac{1}{2n+5} \end{array}\right]$$

$$= \frac{1}{2}\left[\frac{2}{3} - \frac{3}{5} + \frac{2}{5} + \frac{1}{2n+3} - \frac{3}{2n+3} + \frac{1}{2n+5}\right]$$

$$= \frac{7}{30} - \frac{1}{2n+3} + \frac{1}{2(2n+5)}$$

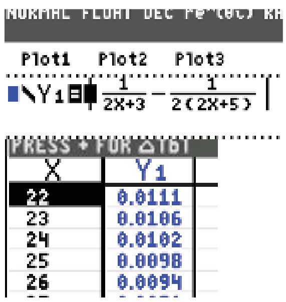
**2iii**

Sum to infinity $= \frac{7}{30}$

$$\left|\frac{7}{30} - \left(\frac{7}{30} - \frac{1}{2n+3} + \frac{1}{2(2n+5)}\right)\right| < 0.01$$

$$\left|\frac{1}{2n+3} - \frac{1}{2(2n+5)}\right| < 0.01$$

| $n$ | $\left\lvert\frac{1}{2n+3} - \frac{1}{2(2n+5)}\right\rvert$ |
|---|---|
| 24 | 0.010174 |
| 25 | 0.009777 |
| 26 | 0.009409 |



From GC, the smallest value of $n$ is 25

**3i**

**Method 1**

$$\frac{-x^2+5x-11}{x-2} = Ax + B + \frac{C}{x-2}$$

$$= \frac{(Ax+B)(x-2)+C}{x-2}$$

Comparing coefficients:

$A = -1$,

$B - 2A = 5 \Rightarrow B = 3$,

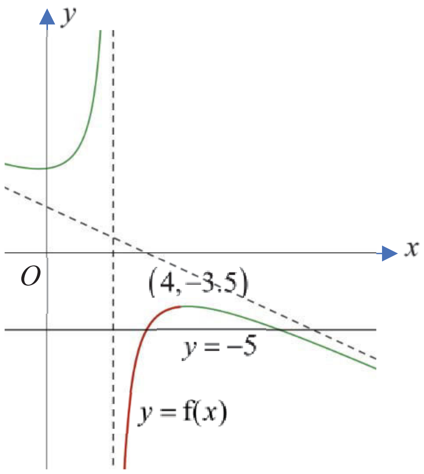
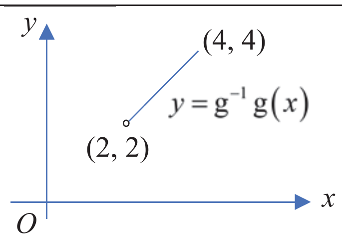
$-2B + C = -11 \Rightarrow C = -5$

**Method 2**

$$\begin{array}{r} -x + 3 \\ x-2\,\overline{)\,-x^2 + 5x - 11} \\ \underline{+\,x^2 - 2x\phantom{- 11}} \\ 3x - 11 \\ \underline{-\,3x + 6} \\ -\,5 \end{array}$$

$$\frac{-x^2+5x-11}{x-2} = -x + 3 - \frac{5}{x-2}$$

Asymptotes have equations $x = 2$ and $y = -x + 3$.

| Qn | Solution | Comments |
|---|---|---|
| 3ii | <br><br>The horizontal line $y=-5$ cuts the graph $y=\mathrm{f}(x)$ more than once, hence f is not one-one, and f does not have an inverse. | |
| 3iii | Let $y=\dfrac{-x^2+5x-11}{x-2}$<br>$\Rightarrow -x^2+(5-y)x+2y-11=0$<br>$\Rightarrow x=\dfrac{y-5\pm\sqrt{(5-y)^2-4(-1)(2y-11)}}{-2}=\dfrac{y-5}{-2}\pm\dfrac{\sqrt{y^2-2y-19}}{2}$<br>$2<x\leqslant 4 \quad \Rightarrow x=\dfrac{y-5}{-2}-\dfrac{\sqrt{y^2-2y-19}}{2}$<br>$\therefore \mathrm{g}^{-1}(x)=\dfrac{5-x}{2}-\dfrac{\sqrt{x^2-2x-19}}{2}$<br>Domain of $\mathrm{g}^{-1}=(-\infty,-3.5]$ | |
| 3iv | <br> | |
| 4i | $x=-\sqrt{t^2+4}\,,\ y=\dfrac{\ln t}{t},\ t>0.$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{t.\frac{1}{t}-\ln t.1}{t^2}=\dfrac{1-\ln t}{t^2}$<br>$\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\dfrac{1}{2}\left(t^2+4\right)^{-\frac{1}{2}}.(2t)=-\dfrac{t}{\sqrt{t^2+4}}$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(\dfrac{1-\ln t}{t^2}\right)\left(-\dfrac{\sqrt{t^2+4}}{t}\right)$<br>$=\dfrac{(\ln t-1)\sqrt{t^2+4}}{t^3}$ (shown) | |
| 4ii | At stationary point, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$. | |

| Qn | Solution | Comments |
|---|---|---|

$$\frac{dy}{dx}=\frac{(\ln t-1)\sqrt{t^2+4}}{t^3}=0$$

$\ln t-1=0$

$\Rightarrow t=e$

Stationary point is $\left(-\sqrt{4+e^2},\frac{1}{e}\right)$.

To determine that it is a max turning point, use first derivative sign test.

| $t$ | $t=e^{+}(2.72)$ | $t=e$ | $t=(2.71)$ |
|---|---|---|---|
| $x$ | $x=-3.376$ | $x=-\sqrt{e^2+4}$ | $x=-3.368$ |
| $\frac{dy}{dx}$ | $1.06\times10^{-4}$ | 0 | $-5.16\times10^{-4}$ |
| sign of $\frac{dy}{dx}$ | Positive | Zero | Negative |

As seen from the table above, stationary point is a maximum.

**Alternatively (not in syllabus)**

$$t^3\frac{dy}{dx}=(\ln t-1)\sqrt{t^2+4}$$

$$3t^2\frac{dy}{dx}+t^3\left(\frac{d^2y}{dx^2}\right)\left(\frac{dx}{dt}\right)=\frac{\sqrt{t^2+4}}{t}+\frac{t(\ln t-1)}{\sqrt{t^2+4}}$$

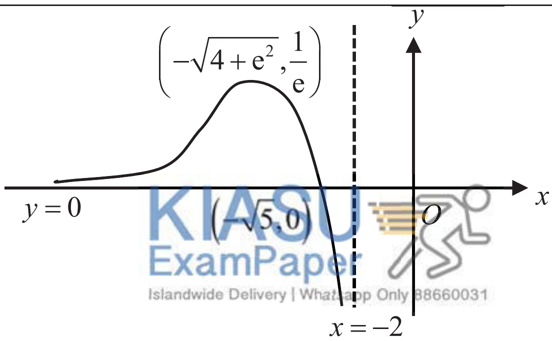
When $t=e$, $\frac{dy}{dx}=0$,

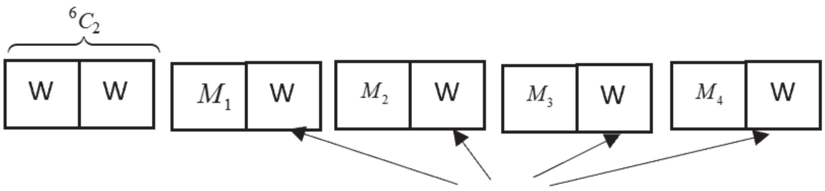
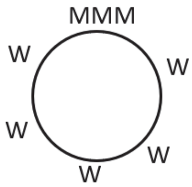
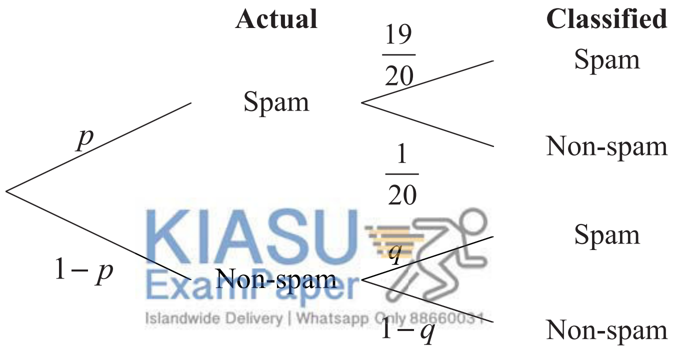
$$e^3\left(\frac{d^2y}{dx^2}\right)\left(-\frac{e}{\sqrt{e^2+4}}\right)=\frac{\sqrt{e^2+4}}{e}$$

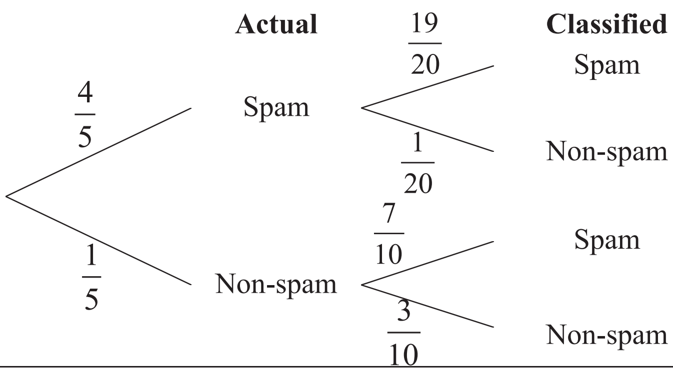
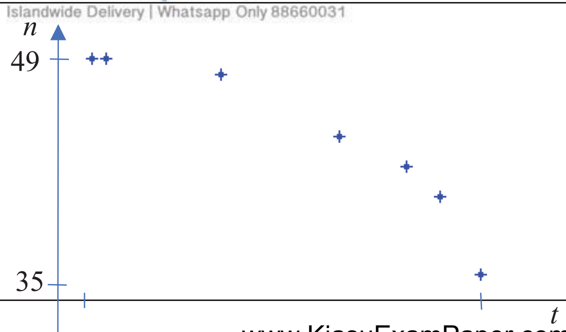
$$\frac{d^2y}{dx^2}=-\frac{e^2+4}{e^5}<0$$

Stationary point is a maximum.

4iii

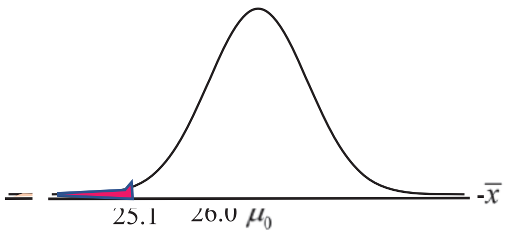
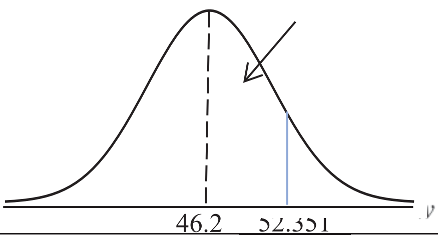


| Qn | Solution | Comments |
|---|---|---|
| 4iv | Area $=\int_{-\sqrt{13}}^{-\sqrt{5}} y \, \mathrm{d}x$<br>$=\int_3^1 \left(\frac{\ln t}{t}\right) \cdot \left(-\frac{t}{\sqrt{t^2+4}}\right) \mathrm{d}t$<br>$=\int_1^3 \frac{\ln t}{\sqrt{t^2+4}} \mathrm{d}t$<br>$=\int_1^3 \frac{\ln x}{\sqrt{x^2+4}} \mathrm{d}x$<br>$=0.4317$<br><br>When $x=-\sqrt{5}, t^2+4=5$<br>$\Rightarrow t=1 (\because t>0)$<br><br>When $x=-\sqrt{13}, t^2+4=13$<br>$\Rightarrow t=3 (\because t>0)$ | |
| 5i | No. of ways $= {}^6\mathrm{C}_2 \times 4! = 360$<br><br><br>4! ways to pair a woman to each of the men | |
| 5ii | No. of ways $= 7! = 5040$ | |
| 5iii | No. of ways 3 men seated together $= 5! \times 3! = 720$<br>No. of ways 3 men not seated together = 5040 – 720 = 4320<br><br><br>Thought process:<br>W,W,W,W,W, ($M_1M_2M_3$) – 6 objects in a circle: $(6-1)!$<br>Arrange $M_1M_2M_3$ : 3!<br><br>Hence, applying the complement method:<br><br>No. of ways 3 men not seated together = 5040 – 720 = 4320 | |
| 6i | Let P(classified as spam \| the email is not spam) $= q$<br><br><br>$\frac{19}{20}p+(1-p)(1-q)=\frac{41}{50}$<br>$\frac{19}{20}p+1-p-(1-p)q=\frac{41}{50}$<br>$\frac{1}{20}p+(1-p)q=\frac{9}{50}$ | |

| Qn | Solution | Comments |
|---|---|---|
| | $\dfrac{\frac{19}{20}p}{\frac{19}{20}p+(1-p)q}=\dfrac{38}{45}$<br>$\dfrac{19}{20}p=\dfrac{361}{450}p+\dfrac{38}{45}(1-p)q$<br>$\dfrac{133}{900}p-\dfrac{38}{45}(1-p)q=0$<br><br>By GC, $p=\dfrac{4}{5},(1-p)q=\dfrac{7}{50}$<br>$q=\dfrac{7}{10}$<br><br> | |
| 6ii | Since there is a high "cost" to classifying non-spam email wrongly in real life, and Spam Guard Plus classifies non-spam email wrongly 70% of the time, it would be more beneficial for someone with higher proportion of spam email to use Spam Guard Plus.<br>Hence, Spam Guard Plus would be more suitable for Betty.<br><br>OR<br>Email classified correctly<br>Andy: $0.3\times 0.95+0.7\times 0.3=0.495$<br>Betty: $0.7\times 0.95+0.3\times 0.3=0.755$<br>More suitable for Betty<br><br>OR<br>Email classified correctly, given that it is spam is $\dfrac{19}{20}$ for both Andy and Betty, hence Spam Guard Plus is just as effective in filtering out spam email for both of them. Therefore, Spam Guard Plus is just as appropriate for both of them. | |
| 7i | <br> | |

| Qn | Solution | Comments |
|---|---|---|
| | 31.6 42.1 | |
| 7ii | The linear model $n = at + b$ is not appropriate because from the scatter diagram, the **<u>points</u>** do not lie close to a straight line.<br>OR<br>The linear model $n = at + b$ is not appropriate because from the scatter diagram, **<u>$n$</u>** decreases at increasing rate as $t$ increases. | |
| 7iii | Product moment correlation coefficient between $n$ and $(t-30)^3$ is $-0.993$.<br>Since $-0.993$ is closer to $-1$ compared to $-0.980$, $n = e(t-30)^3 + f$ is the more appropriate model of the two. | |
| 7iv | Regression equation is $(t-30)^3 = -132.12n + 6487.1$.<br>When $n = 48$, $t = 35.3$<br><br>Since $n - 48$ is within the data range of $35 \le n \le 49$, the product moment correlation coefficient between $n$ and $(t-30)^3$ is close to $-1$, the estimate is likely to be reliable. | |
| 8i | Let \$$X$ be the amount the player receives after a game.<br>$\mathrm{P}(X=6) = \frac{2}{n+5} \times \frac{3}{n+4} \times 2 = \frac{12}{(n+5)(n+4)}$<br>$\mathrm{P}(X=0) = \frac{n}{n+5} \times \frac{5}{n+4} \times 2 = \frac{10n}{(n+5)(n+4)}$<br>$\mathrm{P}(X=-1) = \frac{2}{n+5} \times \frac{1}{n+4} + \frac{3}{n+5} \times \frac{2}{n+4} + \frac{n}{n+5} \times \frac{n-1}{n+4}$<br>$= \frac{n^2-n+8}{(n+5)(n+4)}$<br><br>When the player is expected to receive money in a game, $\mathrm{E}(X) > 0$<br>$\mathrm{E}(X) = 6\left(\frac{12}{(n+5)(n+4)}\right) - \left(\frac{n^2-n+8}{(n+5)(n+4)}\right)$<br>$= \frac{-n^2+n+64}{(n+5)(n+4)}$<br><br>Since $(n+5)(n+4) > 0$,<br>$-n^2+n+64 > 0$<br>$-7.5156 < n < 8.5156$<br>Since $n$ is a positive integer, largest $n = 8$ | |
| 8ii | $\mathrm{P}(X=-1) = \frac{8^2-8+8}{(13)(12)} = \frac{16}{39}$ | |
| 8iii | Let $L$ be the number of games that Victoria lost money, out of 50<br>$L \sim \mathrm{B}\left(50, \frac{16}{39}\right)$ | |

<table>
<tr><th>Qn</th><th>Solution</th><th>Comments</th></tr>
<tr><td></td><td>$\mathrm{P}(L \geq 20) = 1 - \mathrm{P}(L \leq 19) = 0.61132 \approx 0.611$</td><td></td></tr>
<tr><td>8iv</td><td>$\mathrm{P}(L = r) > 0.1$<br>By GC,<br>
<table>
<tr><th colspan="2">NORMAL FLOAT AUTO REAL RADIAN MP<br>PRESS + FOR △Tbl</th></tr>
<tr><th>X</th><th>Y1</th></tr>
<tr><td>14</td><td>0.0199</td></tr>
<tr><td>15</td><td>0.0332</td></tr>
<tr><td>16</td><td>0.0506</td></tr>
<tr><td>17</td><td>0.0703</td></tr>
<tr><td>18</td><td>0.0897</td></tr>
<tr><td>19</td><td>0.1051</td></tr>
<tr><td>20</td><td>0.1133</td></tr>
<tr><td>21</td><td>0.1126</td></tr>
<tr><td>22</td><td>0.1033</td></tr>
<tr><td>23</td><td>0.0875</td></tr>
<tr><td>24</td><td>0.0684</td></tr>
</table>
X=24<br>
the set of values that $r$ can take is $\{19, 20, 21, 22\}$, or<br>$\{r \in \mathbb{Z} : 19 \leq r \leq 22\}$</td><td></td></tr>
<tr><td>9i</td><td>$\mathrm{H}_0 : \mu = 26.0$<br>$\mathrm{H}_1 : \mu < 26.0$<br>Level of significance: 5%<br>Test Statistic: Since $n$ is sufficiently large, by CLT, $\overline{X}$ is approximately normal.<br>When $\mathrm{H}_0$ is true $Z = \dfrac{\overline{X} - 26.0}{S/\sqrt{n}} \sim \mathrm{N}(0,1)$ approx<br>Computation:<br>$n = 40,\ \overline{x} = 26 - \dfrac{30.1}{40} = 25.2475$<br>$s^2 = \dfrac{1}{39}\left[214.61 - \dfrac{(-30.1)^2}{40}\right] = 4.9220$<br>$p - \text{value} = 0.015969 \approx 0.0160$<br><br>Conclusion: Since $p - \text{value} = 0.0160 < 0.05$, $\mathrm{H}_0$ is rejected at the 5% level of significance. Therefore, there is sufficient evidence to conclude that the mean benzene level has decreased as a result of the indoor plant *Epipremnum aureum* at the 5% level of significance.</td><td></td></tr>
<tr><td>9ii</td><td>There is no need to assume the population distribution of the benzene level because $n = 40$ is sufficiently large, so by Central Limit Theorem, the sample mean benzene level, $\overline{X}$, follows a normal distribution approximately.</td><td></td></tr>
<tr><td>9iii</td><td>$\mathrm{H}_0 : \mu = 26.0$<br>$\mathrm{H}_1 : \mu < 26.0$<br><br>Test Statistic: Since $n$ is sufficiently large, by Central Limit Theorem, $\overline{X}$ is approximately normal.<br>When $\mathrm{H}_0$ is true $Z = \dfrac{\overline{X} - 26.0}{S/\sqrt{n}} \sim \mathrm{N}(0,1)$ approx.</td><td></td></tr>
</table>

| Qn | Solution | Comments |
|---|---|---|
| | Computation: $s^2 = \frac{50}{49}(8.33) = 8.5$<br><br>Rejection region is $\overline{x} \le 25.1$<br>Level of significance<br>$= \mathrm{P}\left(\overline{X} < 25.1 \text{ when } \mathrm{H}_0 \text{ is true}\right)$ OR $\mathrm{P}\left(Z < -2.1828 \text{ when } \mathrm{H}_0 \text{ is true}\right)$<br>$= 0.014524$<br>Level of significance of this test is 1.45%. | |
| 9iv | $\mathrm{H}_0 : \mu = \mu_0, \quad$ where $\mu_0 > 26.0$<br>$\mathrm{H}_1 : \mu < \mu_0$<br><br>Level of significance $= \mathrm{P}\left(\overline{X} < 25.1 \text{ when } \mu = \mu_0\right)$<br>$< \mathrm{P}\left(\overline{X} < 25.1 \text{ when } \mu = 26.0\right)$<br>(refer to diagram above)<br>Level of significance is smaller than that in **(iii)**. | |
| 10i | Let $X$ min be the journey times from A to B.<br>$X \sim \mathrm{N}(28, 4^2)$<br>$\mathrm{P}(X \le 35) = 0.959945 \approx 0.960$ | |
| 10ii | Let $Y$ min and $W$ min be the journey times from B to C and from A to C respectively.<br>$Y \sim \mathrm{N}\left(\mu, \sigma^2\right)$<br>Since $X$ and $Y$ are independent,<br>$W = X + Y$<br><br>$\mathrm{E}(W) = \mathrm{E}(X) + \mathrm{E}(Y)$<br>$46.2 = 28 + \mu$<br>$\mu = 18.2$<br><br>$\mathrm{Var}(W) = \mathrm{Var}(X) + \mathrm{Var}(Y)$<br>$4.8^2 = 4^2 + \sigma^2$<br>$\sigma^2 = 23.04 - 16$<br>$= 7.04$ | |
| 10iii | $W \sim \mathrm{N}(46.2, 4.8^2)$<br>$\mathrm{P}(W \le k) \ge 0.9$<br>By GC, $\mathrm{P}(W \le 52.351) = 0.9$<br> | |

| Qn | Solution | Comments |
| --- | --- | --- |
| | $\{k \in \mathbb{R} : k \geq 52.4\}$ | |
| 10iv | Let $\overline{W}$ min be the mean bus journey time from A to C in 70 such journeys.<br>$\overline{W} \sim N\left(46.2, \frac{4.8^2}{70}\right)$<br>$\left[\mathrm{P}\left(\overline{W} > 47\right)\right]^2 = (0.081593)^2 \approx 0.0066574 \approx 0.00666$ | |
| 10v | Let $T$ min be the new journey times from A to C.<br>$\mathrm{E}(T) = 0.9 \times 28 + 17.2 = 42.4$<br>$\mathrm{Var}(T) = 0.9^2 \times 4^2 + 8 = 20.96$<br>$T \sim \mathrm{N}(42.4, 20.96)$<br>$T_1 + T_2 \sim \mathrm{N}(84.8, 41.92)$<br>$\mathrm{P}(T_1 + T_2 \leq 90) = 0.78905 \approx 0.789$ | |