Qn	Solution
1	**Transformation of Curves**
(i)	$y=\frac{2x+1}{x-4}=\frac{2(x-4)+9}{x-4}=2+\frac{9}{x-4}$ where $a=2$ and $b=9$.
(ii)	See below

Note: For sequence of transformation questions, you MUST describe the transformations (using the keywords) and not just write the replacements

Template {delete as appropriate}:

Translate ___ units in the **{positive / negative} {x-direction / y-direction}**	
x-direction [Replace x by $x-(k)$]	y-direction [Replace y by $y-(k)$]
• **Sign** of (k) determines **positive / negative** • **Magnitude** of (k) determines **no. of units of translation**	
Stretch by a **factor** of **k parallel to the {x-axis / y-axis}**	
Parallel to the x-axis $\left[\text{Replace } x \text{ by } \frac{x}{k}\right]$	Parallel to the y-axis $\left[\text{Replace } y \text{ by } \frac{y}{k}\right]$
• k is the stretch factor	
Reflection in the **{x-axis / y-axis}**	
In the y-axis [Replace x by $-x$]	In the x-axis [Replace y by $-y$]

Method 1: Stretch parallel to y-axis

$$y=\frac{1}{x}\xrightarrow{(1)}y=\frac{1}{(x-4)}\xrightarrow{(2)}\left(\frac{y}{9}\right)=\frac{1}{x-4}\Rightarrow y=\frac{9}{x-4}\xrightarrow{(3)}y=2+\frac{9}{x-4}$$

(1) Translate 4 units in the positive x-direction
(2) Stretch by a factor of 9 parallel to the y-axis
(3) Translate 2 units in the positive y-direction

Alternative: In the sequence (2), (3), (1) **or** (2), (1), (3)

Method 2: Stretch parallel to x-axis

$$y=\frac{1}{x}\xrightarrow{(1)}y=\frac{1}{\left(\frac{x}{9}\right)}=\frac{9}{x}\xrightarrow{(2)}y=\frac{9}{(x-4)}\xrightarrow{(3)}(y-2)=\frac{9}{x-4}\Rightarrow y=2+\frac{9}{x-4}$$

(1) Stretch by a factor of 9 parallel to the x-axis
(2) Translate 4 units in the positive x-direction
(3) Translate 2 units in the positive y-direction

Alternative: In the sequence (1), (3), (2) **or** (3), (1), (2)

Qn	Solution
2	**Application of Differentiation (Maxima/Minima)**

2

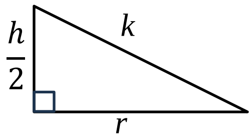
Let V be the volume of the cylinder (in cm^3).

<u>Steps to solve Maxima/Minima Problems</u>

1. Draw a clear diagram and define all variables, where necessary.
2. Form equation(s) relating the variables.
3. Express quantity to be maximized/minimized in terms of a **<u>single</u>** variable, say x (if there are 2 variables, express one in terms of another).
4. Differentiate w.r.t. x and equate to 0 to find the stationary point(s).
5. Use 1st or 2nd derivative test to determine/prove nature of the stationary point.

By Pythagoras Theorem,

$$\left(\frac{h}{2}\right)^2 + r^2 = k^2$$



$$r^2 = k^2 - \frac{h^2}{4} \quad \text{--- (1)}$$

Question asks for **'volume of the cylinder is maximum'**
Hence the formula you need is the volume of cylinder

$$V = \pi r^2 h$$

$$= \pi\left(k^2 - \frac{h^2}{4}\right)h \quad \text{,from (1)}$$

$$= \pi k^2 h - \frac{\pi}{4}h^3$$

Question asks for **'exact value of *h*'**
You should **keep *h*** and **get rid of *r*** so that you can directly solve and answer the question
Note: k is a constant (radius of sphere)

Differentiate w.r.t h,

$$\frac{dV}{dh} = \pi k^2 - \frac{3\pi}{4}h^2$$

For maximum V, $\frac{dV}{dh} = 0$

$$\therefore \frac{dV}{dh} = \pi k^2 - \frac{3\pi}{4}h^2 = 0$$

$$k^2 - \frac{3}{4}h^2 = 0$$

$$h^2 = \frac{4k^2}{3}$$

$$h = \frac{2k}{\sqrt{3}} \quad (\because h > 0)$$

Reminder: When you square root both sides of the equation, you will have ±, i.e.

$$h^2 = \frac{4k^2}{3} \Rightarrow h = \pm\frac{2k}{\sqrt{3}}$$

So you should state why you only want $h = \frac{2k}{\sqrt{3}}$ from context of the question

Remember to **verify using the 1st or 2nd derivative test** that the value of h gives the maximum volume

<u>Testing *V* is maximum</u>

For 1st derivative test, you need to state clearly the value of h

<u>Method 1:</u> 1st derivative test

h	$\left(\frac{2k}{\sqrt{3}}\right)^-$	$\frac{2k}{\sqrt{3}}$	$\left(\frac{2k}{\sqrt{3}}\right)^+$
$\frac{dV}{dh}$	+	0	–
Slope	/	—	\

For 1st derivative test, you need to state the signs of $\frac{dV}{dh}$ and corresponding slope

$\therefore V$ is maximum when $h = \frac{2k}{\sqrt{3}}$.

	Method 2: 2^{nd} derivative test Differentiate w.r.t h, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-\frac{3\pi}{2}h$ When $h=\frac{2k}{\sqrt{3}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-\frac{3\pi}{2}\left(\frac{2k}{\sqrt{3}}\right)=-\sqrt{3}\pi k<0 \quad (\because k>0)$ $\therefore V$ is maximum when $h=\frac{2k}{\sqrt{3}}$.

For 2^{nd} derivative test, you need to

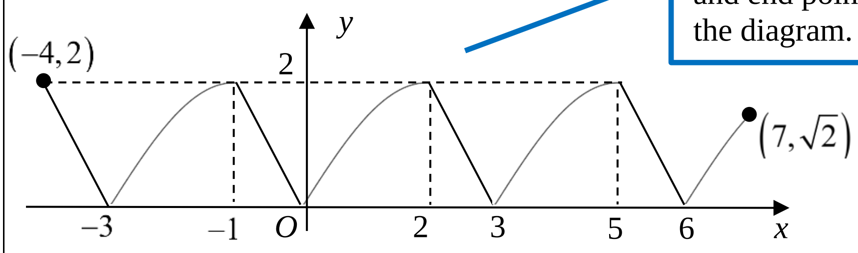
(1) find the second derivative, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}$

(2) explain clearly why it is < 0 for the value of h

(3) conclude that the volume is maximum

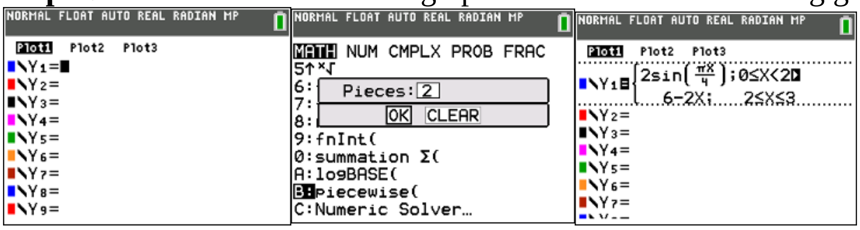
Qn	Solution
3	**Techniques of Integration**
(i)	Using $\mathrm{f}(x)=\mathrm{f}(x+3)$, we have $\mathrm{f}(7)=\mathrm{f}(4)=\mathrm{f}(1)=2\sin\left(\frac{\pi}{4}\right)=\sqrt{2}$ or 1.41 (3s.f.)

(ii)

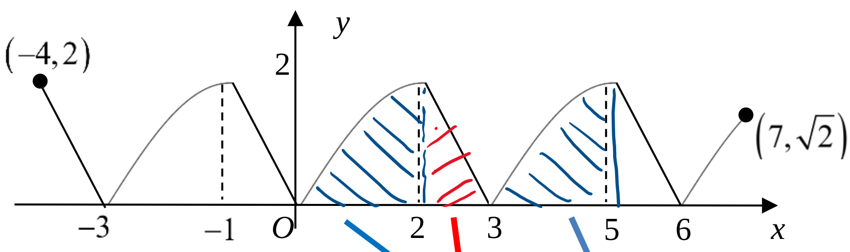


Label the axial intercepts and end points clearly on the diagram.

Step 1: Use of GC to sketch the graph from $x=0$ to $x=3$ using given equation.



Step 2: Since $\mathrm{f}(x)=\mathrm{f}(x+3)$ for all real values of x, the curve is a periodic curve with a period of 3, that is, the graph repeats itself every 3 units. Thus, sketch the curve from $x=-4$ to $x=7$ using this property.



$$\text{Volume}=\pi\int_0^5 y^2\,\mathrm{d}x$$

$$=2\pi\int_0^2 4\sin^2\left(\frac{\pi x}{4}\right)\mathrm{d}x+\frac{1}{3}\pi(2)^2(1)$$

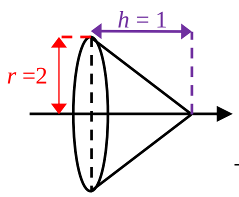
$$=4\pi\int_0^2 1-\cos\left(\frac{\pi x}{2}\right)\mathrm{d}x+\frac{4}{3}\pi$$

$$=4\pi\left[x-\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_0^2+\frac{4}{3}\pi$$

$$=4\pi\left[(2-0)-0\right]+\frac{4}{3}\pi$$

$$=\frac{28}{3}\pi$$

Rotating the triangle about the x-axis gives a cone. Thus volume is found using $\frac{1}{3}\pi r^2 h$.



Note:
The equation of the curve from $x=0$ to $x=2$ is given as $y=2\sin\left(\frac{\pi x}{4}\right)$.

The equation of the curve from $x=3$ to $x=5$ is **NOT** $y=2\sin\left(\frac{\pi x}{4}\right)$.

Observe that the volume generated between $x=3$ to $x=5$ is identical as the volume generated between $x=0$ to $x=2$.

Thus, the volume generated between $x=0$ to $x=2$ and $x=3$ to $x=5$ is given by $2\pi\int_0^2 4\sin^2\left(\frac{\pi x}{4}\right)\mathrm{d}x$.

Qn	Solution
4	**Maclaurin Series**
(i)	Using cosine rule, $2^2 = r^2 + r^2 - 2r^2\cos\left(\frac{\pi}{3}-\theta\right)$ $4 = 2r^2 - 2r^2\cos\left(\frac{\pi}{3}-\theta\right)$ $r^2 = \dfrac{4}{2\left(1-\cos\left(\frac{\pi}{3}-\theta\right)\right)}$ $= \dfrac{4}{2\left(1-\left(\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta\right)\right)}$ $= \dfrac{4}{2\left(1-\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)\right)}$ $= \dfrac{4}{2-\sqrt{3}\sin\theta-\cos\theta}$ (shown) **Formula NOT given in MF26**: 1) Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos A$ 2) Sine Rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ Use MF26: $\cos(A-B) = \cos A\cos B + \sin A\sin B$
(ii)	$r^2 = \dfrac{4}{2-\sqrt{3}\sin\theta-\cos\theta}$ $\approx \dfrac{4}{2-\sqrt{3}\theta-\left(1-\frac{1}{2}\theta^2\right)}$ $= \dfrac{4}{1-\sqrt{3}\theta+\frac{1}{2}\theta^2}$ "θ is a sufficiently small angle" means use small angle approximation (θ^3 and above can be neglected) Use MF26: $\sin\theta \approx \theta$ $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$ $r \approx 2\left(1-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)^{-\frac{1}{2}} = 2\left(1+\left(-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)\right)^{-\frac{1}{2}}$ Use MF26: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2}x^2 + \ldots$ $= 2\left(1+\left(-\frac{1}{2}\right)\left(-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)+\dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\sqrt{3}\theta+\frac{1}{2}\theta^2\right)^2+\ldots\right)$ $= 2\left(1+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2+\frac{9}{8}\theta^2+\ldots\right)$ $= 2\left(1+\frac{\sqrt{3}}{2}\theta+\frac{7}{8}\theta^2+\ldots\right)$ $\approx 2+\sqrt{3}\theta+\frac{7}{4}\theta^2$ $a = \sqrt{3}$ and $b = \frac{7}{4}$ Answer question

Qn	Solution
5	**AP and GP**
(i)	<u>**Method 1**</u> $br = a+10d$ ----- (1) $br^2 = a+2d$ ----- (2) $br^3 = a$ ------ (3) $r = \frac{a}{a+2d} = \frac{a+2d}{a+10d}$ $r = \frac{\text{fourth term of } G}{\text{third term of } G} = \frac{a}{a+2d}$ $r = \frac{\text{third term of } G}{\text{second term of } G} = \frac{a+2d}{a+10d}$ $a(a+10d) = (a+2d)^2$ $a^2 + 10ad = a^2 + 4ad + 4d^2$ $6ad = 4d^2$ $a = \frac{2}{3}d$ since $d \neq 0$ Hence, $r = \frac{a}{a+2d} = \frac{\frac{2}{3}d}{\frac{2}{3}d+2d} = \frac{1}{4}$ Since $\lvert r \rvert = \frac{1}{4} < 1$, hence the series is convergent. To prove geometric series G is convergent, show $\lvert r \rvert < 1$. <u>**Method 2**</u> $br = a+10d$ ----- (1) $br^2 = a+2d$ ----- (2) $br^3 = a$ ------ (3) Eqn (2) – (1) $br^2 - br = -8d$ -----(4) Eqn (3) – (2) $br^3 - br^2 = -2d$ -----(5) $\frac{(5)}{(4)}$ $\frac{br^3 - br^2}{br^2 - br} = \frac{-2d}{-8d}$ $\frac{r^3 - r^2}{r^2 - r} = \frac{1}{4}$ $\frac{r^2(1-r)}{r(1-r)} = \frac{1}{4}$ Need to reject $r = 1$. Since $d \neq 0 \Rightarrow r \neq 1$, $r = \frac{1}{4}$ Since $\lvert r \rvert = \frac{1}{4} < 1$, hence the series is convergent.

(ii)

$H: b^2, b^2r^2, b^2r^4, ...$

Sum to infinity of $H = \dfrac{b^2}{1-r^2} = \dfrac{b^2}{1-\frac{1}{9}} = \dfrac{9}{8}b^2$

First term of $H = b^2$
Common ratio of $H = r^2$
Use the formula:

$$S_\infty = \frac{\text{first term}}{1-\text{common ratio}}$$

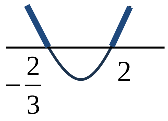
Sum to infinity of $G = \dfrac{b}{1-r} = \dfrac{b}{1-\frac{1}{3}} = \dfrac{3}{2}b$

$$\frac{9}{8}b^2 - \frac{3}{2}b > \frac{3}{2}$$

$$3b^2 - 4b - 4 > 0$$

$$(3b+2)(b-2) > 0$$

Sketch the graph to solve the quadratic inequality



$$b < -\frac{2}{3} \text{ or } b > 2$$

Important: Use ‘or’ (DO NOT use ‘and’)

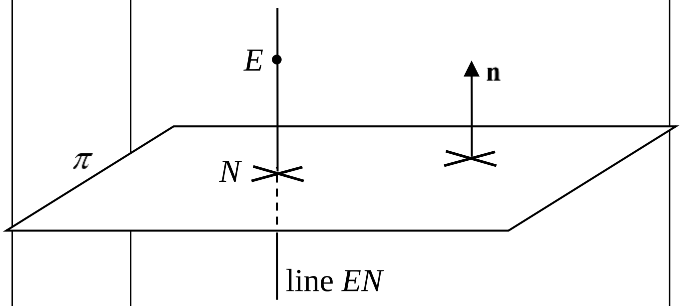
Qn	Solution
6	**APGP + Seq & Series**
(i)	$\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{1}{2^r+1}-\frac{1}{2^{r+1}+1}$ $=\frac{2^{r+1}+1-(2^r+1)}{(2^r+1)(2^{r+1}+1)}$ $=\frac{2(2^r)+1-2^r-1}{(2^r+1)(2^{r+1}+1)}$ $=\frac{2^r}{(2^r+1)(2^{r+1}+1)}$ Need to indicate Step 3 leading to show the result stated in question.
(ii)	$\sum_{r=1}^{n}\frac{2^r}{(2^r+1)(2^{r+1}+1)}$ $=\sum_{r=1}^{n}[\mathrm{f}(r)-\mathrm{f}(r+1)]$ $=\left[\begin{array}{l}\mathrm{f}(1)-\cancel{\mathrm{f}(2)}\\+\cancel{\mathrm{f}(2)}-\cancel{\mathrm{f}(3)}\\+\cancel{\mathrm{f}(3)}-\cancel{\mathrm{f}(4)}\\+\ldots\\+\cancel{\mathrm{f}(n-1)}-\cancel{\mathrm{f}(n)}\\+\cancel{\mathrm{f}(n)}-\mathrm{f}(n+1)\end{array}\right]$ $=\mathrm{f}(1)-\mathrm{f}(n+1)$ $=\frac{1}{3}-\frac{1}{2^{n+1}+1}$ This part of the question was quite well done as most students were able to write out the MOD steps. Misconception of replacing f(r) = $\frac{2^r}{(2^r+1)(2^{r+1}+1)}$ instead of $\frac{1}{2^r+1}$.
(iii)	$\sum_{r=1}^{n}\left[\frac{2^{r+3}}{(2^{r+2}+1)(2^{r+3}+1)}-(2r+5)\right]$ Replace r with $r-2$ $=\sum_{r=3}^{n+2}\frac{2^{r+1}}{(2^r+1)(2^{r+1}+1)}-\sum_{r=1}^{n}(2r+5)$ $=2\sum_{r=1}^{n+2}\frac{2^r}{(2^r+1)(2^{r+1}+1)}-2\sum_{r=1}^{2}\frac{2^r}{(2^r+1)(2^{r+1}+1)}-\frac{n}{2}(7+2n+5)$ $=2\left[\frac{1}{3}-\frac{1}{2^{n+3}+1}\right]-2\left[\frac{1}{3}-\frac{1}{2^3+1}\right]-\frac{n(2n+12)}{2}$ $=\frac{2}{9}-\frac{2}{2^{n+3}+1}-n(n+6)$ **Note:** - Need to replace the r by (r-2) in the expression in (iii) instead of (ii). - Not recommended to replace r in (iii) by (r+3) or r in (ii) by (r+2). Steps will be tedious. - Observed and factorised the number '2' to obtain the same expression in (ii). Note: - Need to split the summation in order to **use the result in (ii)** i.e. $\sum_{r=1}^{n}\frac{2^r}{(2^r+1)(2^{r+1}+1)}=\frac{1}{3}-\frac{1}{2^{n+1}+1}$. - Do not apply MOD again as it is a "Hence" question. - Some applied sum of AP formula wrongly with incorrect first term or missed out $\frac{n}{2}$ or wrote $\frac{n}{2}$ (1st term – last term).

Qn	Solution
7	**Techniques of Differentiation**
(i)	$y^2 = 3x^3 - x + 1$ Differentiate w.r.t. x, $2y\frac{dy}{dx} = 9x^2 - 1$ Use implicit differentiation At stationary point, $\frac{dy}{dx} = 0$ $9x^2 - 1 = 0$ $x = \frac{1}{3}$ or $-\frac{1}{3}$ Always give your answer in the simplest form! i.e $\sqrt{9} = 3$ e.g. $y = \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3}$ or $-\sqrt{\frac{7}{9}} = -\frac{\sqrt{7}}{3}$ When $x = \frac{1}{3}$, $y = \frac{\sqrt{7}}{3}$ or $-\frac{\sqrt{7}}{3}$ When $x = -\frac{1}{3}$, $y = \frac{\sqrt{11}}{3}$ or $-\frac{\sqrt{11}}{3}$ Answer the question. Give the **COORDINATES** of the stationary points. You do NOT need to determine the nature of the stationary points. Stationary points are $\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right), \left(-\frac{1}{3}, \frac{\sqrt{11}}{3}\right), \left(-\frac{1}{3}, -\frac{\sqrt{11}}{3}\right), \left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$,
(ii)	The stationary point in the first quadrant is $S\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$. Since the x-coordinate is increasing, $\frac{dx}{dt} = \frac{1}{9}$ $\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{d\left(\frac{dy}{dx}\right)}{dx} \times \frac{dx}{dt} \Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{d^2y}{dx^2} \times \frac{dx}{dt}$ Important step: Use chain rule and note that $\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$ $2y\frac{dy}{dx} = 9x^2 - 1$ Differentiate w.r.t. x, $2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 18x$ Use implicit differentiation to find the 2nd derivative $\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 9x$

	At point $S\left(\frac{1}{3},\frac{\sqrt{7}}{3}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{9}$ $\frac{\sqrt{7}}{3}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=9\left(\frac{1}{3}\right)\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{9}{\sqrt{7}}$ $\therefore\frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}t}=\frac{9}{\sqrt{7}}\times\left(\frac{1}{9}\right)=\frac{1}{\sqrt{7}}$ Hence the rate of change of its gradient at point S is $\frac{1}{\sqrt{7}}$ units per second.

Qn	Solution
8	**Vectors**
(i)	$\overrightarrow{OE}=\overrightarrow{OA}+\overrightarrow{AE}=\begin{pmatrix}4\\3\\0\end{pmatrix}+\begin{pmatrix}0\\0\\20\end{pmatrix}=\begin{pmatrix}4\\3\\20\end{pmatrix}$ Using Ratio Theorem, $\overrightarrow{OF}=\dfrac{k\overrightarrow{OA}+(1-k)\overrightarrow{OE}}{k+(1-k)}=k\begin{pmatrix}4\\3\\0\end{pmatrix}+(1-k)\begin{pmatrix}4\\3\\20\end{pmatrix}=\begin{pmatrix}4\\3\\20(1-k)\end{pmatrix}$ Since it is given that $AF:FE=1-k:k$, we can apply Ratio Theorem to find $\overrightarrow{OF}$.
(ii)	$\overrightarrow{DC}=\overrightarrow{OB}=\begin{pmatrix}0\\6\\0\end{pmatrix}=6\begin{pmatrix}0\\1\\0\end{pmatrix}$ Find the two vectors parallel to the plane. Can use $\overrightarrow{DC}$, $\overrightarrow{DF}$ or $\overrightarrow{CF}$ $\overrightarrow{DF}=\overrightarrow{OF}-\overrightarrow{OD}$ $=\begin{pmatrix}4\\3\\20(1-k)\end{pmatrix}-\begin{pmatrix}0\\0\\20\end{pmatrix}$ $=\begin{pmatrix}4\\3\\-20k\end{pmatrix}$ $n=\overrightarrow{DC}\times\overrightarrow{DF}=\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}4\\3\\-20k\end{pmatrix}=\begin{pmatrix}-20k\\0\\-4\end{pmatrix}=-4\begin{pmatrix}5k\\0\\1\end{pmatrix}$ Find the cross product of the two vectors parallel to the plane to find the normal vector Equation of the plane *CDF* $\mathbf{r}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\20\end{pmatrix}\cdot\begin{pmatrix}5k\\0\\1\end{pmatrix}=20$ (Shown) Use $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$ to find and show the equation of the plane. Any one of the points *C*, *D*, or *F* can be used.

(iii)	Let N be the foot of perpendicular from point E to the plane CDF.



Form the equation of the line passing through points E and N. Since $\overrightarrow{EN}$ is perpendicular to the plane, we can use the normal vector of the plane to be the direction vector of the line.

$$l_{EN} : \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Since N lies on the line, $\overrightarrow{ON} = \begin{pmatrix} 4 \\ 3 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

Remark: Do not label the foot of perpendicular as point F since it is a point given in the question.

$$\overrightarrow{ON} = \begin{pmatrix} 4 \\ 3 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$\begin{pmatrix} 4+5k\lambda \\ 3 \\ 20+\lambda \end{pmatrix} \cdot \begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix} = 20$$

$$5k\left(4+5k\lambda\right)+20+\lambda = 20$$

$$20k + 25k^2\lambda + \lambda = 0$$

$$\left(25k^2+1\right)\lambda = -20k$$

$$\lambda = \frac{-20k}{\left(25k^2+1\right)}$$

Since N lies on the plane, it satisfies the equation of the plane. Hence $\overrightarrow{ON} \cdot \begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix} = 20$. Students should solve for the value of λ to find point N, not k.

The x-coordinate of N is $4+\lambda\left(5k\right) = 4 + \frac{-20k}{\left(25k^2+1\right)}\left(5k\right) = 4 - \frac{100k^2}{\left(25k^2+1\right)} = \frac{4}{\left(25k^2+1\right)}$

The question only requires the x-coordinate of the foot of perpendicular.

Alternatively,

$$\overrightarrow{ON} = \begin{pmatrix} 4 \\ 3 \\ 20 \end{pmatrix} + \left[\frac{-20k}{\left(25k^2+1\right)}\right]\begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 - \frac{100k^2}{\left(25k^2+1\right)} \\ 3 \\ 20 - \frac{20k}{\left(25k^2+1\right)} \end{pmatrix}$$

The x-coordinate of N = $4 - \frac{100k^2}{\left(25k^2+1\right)} = \frac{4}{\left(25k^2+1\right)}$

Alternative Method:
Let N be the foot of perpendicular from point E to the plane CDF.

$$\overrightarrow{ED} = \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix}$$

$$\overrightarrow{EN} = \left(\overrightarrow{ED} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}$$

$$= \frac{\left[\begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix}\right]}{25k^2+1}\begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix}$$

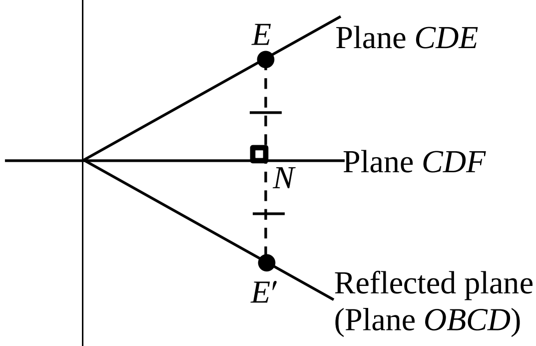
$$= \frac{-20k}{25k^2+1}\begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix}$$

Students using the projection method needs to be careful when applying the projection formula. It must start or end at the same point.

$$\overrightarrow{ON} = \begin{pmatrix} 4 \\ 3 \\ 20 \end{pmatrix} + \frac{-20k}{25k^2+1}\begin{pmatrix} 5k \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 - \dfrac{100k^2}{\left(25k^2+1\right)} \\ 3 \\ 20 - \dfrac{20k}{\left(25k^2+1\right)} \end{pmatrix}$$

The x-coordinate of $N = 4 - \dfrac{100k^2}{\left(25k^2+1\right)} = \dfrac{4}{\left(25k^2+1\right)}$

(iv) Let E' be the reflection of point E in the plane CDF.



Thought process:
Point E lies on plane CDE.
We can use the foot of perpendicular from point E to plane CDF found in (iii) to find the reflected point.
Since the reflected plane lies on the plane $OBCD$, the reflected point will lie on plane $OBCD$.
From the diagram, we observe that plane $OBCD$ is the zy-plane, hence the equation of plane $OBCD$ is $x = 0$ (or $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$). Hence we can focus only on the x-coordinate of the reflected point and equate it to 0.

By ratio theorem,

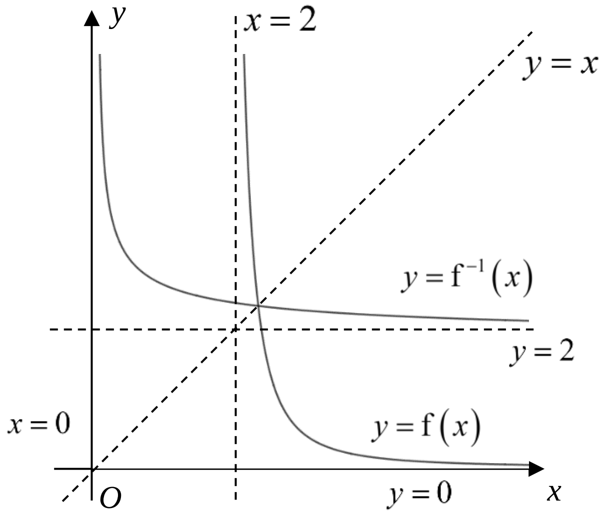
$$\overrightarrow{ON} = \frac{\overrightarrow{OE} + \overrightarrow{OE}'}{2}$$

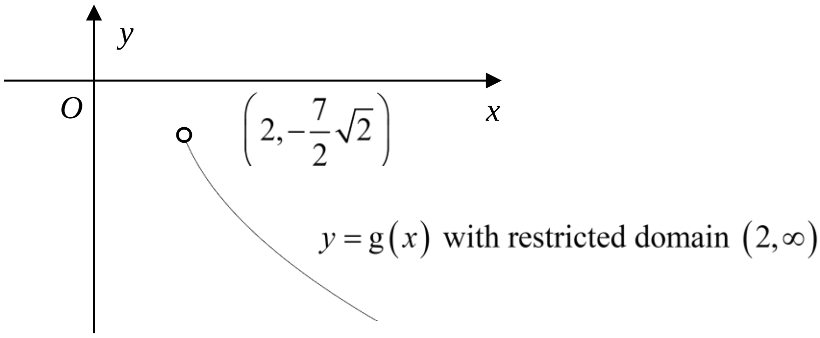
$$\frac{4}{\left(25k^2+1\right)} = \frac{1}{2}(4+x)$$

$$x = \frac{8}{\left(25k^2+1\right)} - 4$$

The foot of perpendicular found in (iii) is the midpoint of E and E'. Use midpoint theorem to find $\overrightarrow{ON}$.
We can focus on the x-coordinate.

	Alternatively, $\overrightarrow{OE}' = 2\overrightarrow{ON} - \overrightarrow{OE} = 2\begin{pmatrix} \frac{4}{(25k^2+1)} \\ 3 \\ 20-\frac{20k}{(25k^2+1)} \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 20 \end{pmatrix} = \begin{pmatrix} \frac{8}{(25k^2+1)}-4 \\ 3 \\ 20-\frac{40k}{(25k^2+1)} \end{pmatrix}$ x-coordinate of E' = $\frac{8}{(25k^2+1)}-4$ For the reflection of the plane *CDE* in the plane *CDF* to lie on the plane *OBCD*, *E'* has to be a point on the plane *OBCD*. Therefore, x-coordinate of E' = 0 $\frac{8}{(25k^2+1)}-4=0$ $\frac{8}{(25k^2+1)}=4$ $8=100k^2+4$ $100k^2=4$ $k^2=\frac{1}{25}$ $k=\pm\frac{1}{5}$ Since $0<k<1$, $k=\frac{1}{5}$

Qn	Solution
9	**Functions + Inequalities**
(i)	For f^{-1} to exist, f must be one-one function. Least value of $k = 2$.
(ii)	 The graphs of $y=\mathrm{f}(x)$, $y=\mathrm{f}^{-1}(x)$ and $y=x$ must intersect at the same point. The asymptotes $y=2$ and $x=2$ must intersect at the line $y=x$. The asymptotes $y=0$ and $x=0$ must intersect at the line $y=x$. The graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ must be a reflection of each other in the line $y=x$. $y=\mathrm{f}(x)=\dfrac{1}{(x-2)^2}$ has asymptotes $y=0$ and $x=2$, so $y=\mathrm{f}^{-1}(x)$ will have asymptotes $x=0$ and $y=2$.
(iii)	Let $\mathrm{f}^{-1}(5)=x$. $\mathrm{ff}^{-1}(5)=\mathrm{f}(x)$ $\mathrm{f}(x)=5$ $\dfrac{1}{(x-2)^2}=5$ $(x-2)^2=\dfrac{1}{5}$ $x-2=\dfrac{1}{\sqrt{5}} \quad (\because x>2)$ $x=2+\dfrac{1}{\sqrt{5}}$ $=2+\dfrac{\sqrt{5}}{5}$ First, we can compose the function f on both sides of the equation $\mathrm{f}^{-1}(5)=x$. And since $\mathrm{ff}^{-1}(5)=5$, we do not need to find the expression of f^{-1}.

	Alternatively, you may find the expression of f^{-1}. Let $y=\dfrac{1}{(x-2)^2}$ $(x-2)^2=\dfrac{1}{y}$ $x-2=\pm\dfrac{1}{\sqrt{y}}$ since $x>2$, $x=2+\dfrac{1}{\sqrt{y}}$ $\therefore \mathrm{f}^{-1}(x)=2+\dfrac{1}{\sqrt{x}}$ $\mathrm{f}^{-1}(5)=2+\dfrac{1}{\sqrt{5}}$ $\quad =2+\dfrac{\sqrt{5}}{5}$
(iv)	$\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=(2,\infty)$ and $\mathrm{D}_{\mathrm{g}}=(0,\infty)$ Since $\mathrm{R}_{\mathrm{f}^{-1}}\subseteq\mathrm{D}_{\mathrm{g}}$, gf^{-1} exists. Using mapping method,  $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=(0,\infty)\xrightarrow{\mathrm{f}^{-1}}\mathrm{R}_{\mathrm{f}^{-1}}=(2,\infty)\xrightarrow{\mathrm{g}}\mathrm{R}_{\mathrm{gf}^{-1}}=\left(-\infty,-\dfrac{7}{2}\sqrt{2}\right)$ Using $\mathrm{R}_{\mathrm{f}^{-1}}=(2,\infty)$ as the restricted domain on the graph of g, we subst when $x=2$ into $\mathrm{g}(x)=\dfrac{1}{\sqrt{x}}-4\sqrt{x}$, we will get $\mathrm{g}(2)=\dfrac{1}{\sqrt{2}}-4\sqrt{2}=\dfrac{\sqrt{2}}{2}-4\sqrt{2}=\sqrt{2}\left(\dfrac{1}{2}-4\right)=-\dfrac{7\sqrt{2}}{2}$ $\mathrm{R}_{\mathrm{gf}^{-1}}=\left(-\infty,-\dfrac{7}{2}\sqrt{2}\right)$

(v)

$$\mathrm{gf}(x) \le 0$$

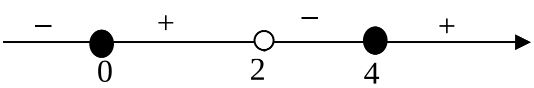
$$\mathrm{g}\left(\frac{1}{(x-2)^2}\right) \le 0$$

$$(x-2) - \frac{4}{(x-2)} \le 0 \quad (\because x > 2)$$

$$\frac{(x-2)^2 - 4}{(x-2)} \le 0$$

$$\frac{x(x-4)}{(x-2)} \le 0$$

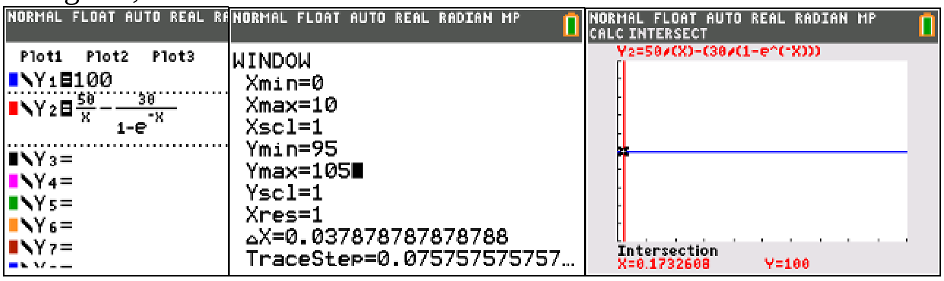
$$\boxed{\begin{aligned} \mathrm{g}\left(\frac{1}{(x-2)^2}\right) &= \frac{1}{\sqrt{\frac{1}{(x-2)^2}}} - 4\sqrt{\frac{1}{(x-2)^2}} \\ &= \sqrt{(x-2)^2} - \frac{4}{\sqrt{(x-2)^2}} \\ &= x - 2 - \frac{4}{x-2} \qquad (\text{since } x > 2) \end{aligned}}$$

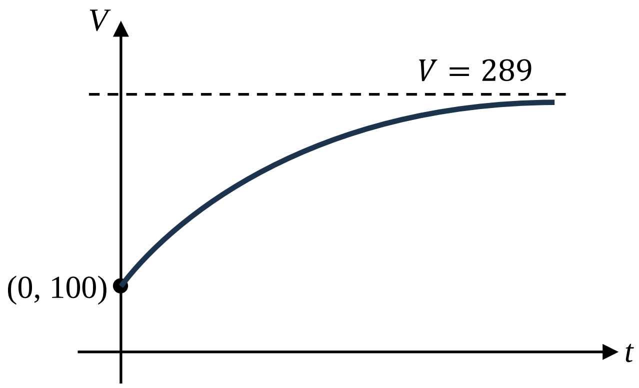


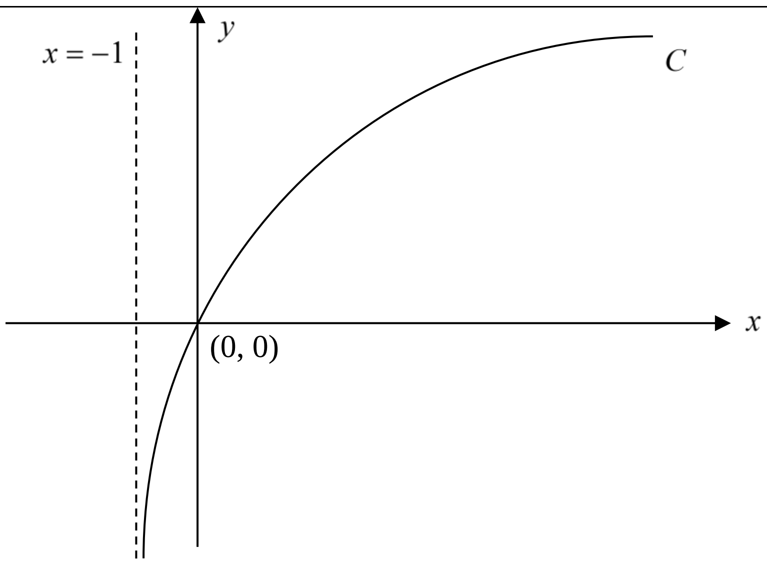
$x \le 0$ or $2 < x \le 4$

Since $x > 2$, we have $2 < x \le 4$.

Since $\mathrm{D}_{\mathrm{gf}} = \mathrm{D}_{\mathrm{f}}$, the final range of *x* must satisfy the domain of f (i.e. $x > 2$)

Qn	Solution
10	**Differential Equations**
(i)	$\frac{\mathrm{d}V}{\mathrm{d}t}=50-kV \quad ,k>0$
(ii)	$\int \frac{1}{50-kV}\,\mathrm{d}V=\int 1\,\mathrm{d}t$ $-\frac{1}{k}\ln\lvert 50-kV\rvert=t+c$ $\ln\lvert 50-kV\rvert=-kt-kc$ $50-kV=A\mathrm{e}^{-kt} \quad$, where $A=\pm\mathrm{e}^{-kc}$ $kV=50-A\mathrm{e}^{-kt}$ $V=\frac{50}{k}-\frac{A}{k}\mathrm{e}^{-kt}=\frac{50}{k}-B\mathrm{e}^{-kt} \quad ,B=\frac{A}{k}$ Use the variables separable method to ensure LHS is in terms of V, and the RHS is in terms of t Remember the modulus sign, and the golden rule when integrating
(iii)	When $t=0$, $V=100 \Rightarrow 100=\frac{50}{k}-B \quad ---(1)$ When $t=1$, $V=130 \Rightarrow 130=\frac{50}{k}-B\mathrm{e}^{-k} \quad ---(2)$ $(2)-(1)$: $30=-B\mathrm{e}^{-k}+B$ $B=\frac{30}{1-\mathrm{e}^{-k}}$ $\therefore 100=\frac{50}{k}-\frac{30}{1-\mathrm{e}^{-k}}$ Using GC, $k=0.17326$  $B=\left(\frac{30}{1-\mathrm{e}^{-0.17326}}\right)=188.58$ $V=\frac{50}{0.17326}-188.58\mathrm{e}^{-0.17326t}$ $=288.58-188.58\mathrm{e}^{-0.17326t}$ $=289-189\mathrm{e}^{-0.173t} \quad (3\text{ s.f.}) \quad$ (shown) Initially, the volume of water in the tank is 100 cm^3 ➔ When $t=0$, $V=100$ After a minute, the volume increased to 130 cm^3 ➔ When $t=1$, $V=130$ This is a SHOW question, please show all working clearly. Intermediate answers should be given in at least 5 s.f.
(iv)	At 95% capacity, $V=0.95(300)=285$ $285=289-189\mathrm{e}^{-0.173t}$ Using GC, $t=22.3 \quad (3\text{ s.f.})$ It will take about 22.3 minutes (or 22 minutes and 18 seconds) to reach 95% of the capacity. Alternatively, $285=288.58-188.58\mathrm{e}^{-0.17326t}$ Using GC, $t=22.9 \quad (3\text{ s.f.})$ It will take about 22.9 minutes (or 22 minutes and 54 seconds) to reach 95% of the capacity.

(v)	
(vi)	As $t \to \infty$, $V \to 289$ In the long run, the volume of the water in the tank **<u>will increase</u>** and **<u>approach</u>** 289 cm^3.

Qn	Solution
11	
(i)	 To get vertical asymptote → observe the parametric equation of C. Since $y = \ln(t+1)$ $\Rightarrow$ vertical asymptote happens when $t+1=0 \Rightarrow t=-1$. Hence, when $t=-1$, $x=(-1)^2+2(-1) \Rightarrow x=-1$ is a vertical asymptote of C.
(ii)	$\frac{dx}{dt} = 2t+2$ $\frac{dy}{dt} = \frac{1}{t+1}$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t+1}}{2t+2}$ $= \frac{1}{(2t+2)(t+1)}$ $= \frac{1}{2(t+1)^2}$ When $t=2$, **<u>Do not</u>** convert equation of C from parametric form to cartesian form as it is not required in the question. $\frac{\frac{1}{t+1}}{2t+2} = \frac{1}{t+1} \div (2t+2) = \frac{1}{t+1} \times \frac{1}{2t+2}$

$$\frac{dy}{dx}=\frac{1}{2(3)^2}=\frac{1}{18},\ y=\ln 3,\ x=8$$

Gradient of normal $=-18$

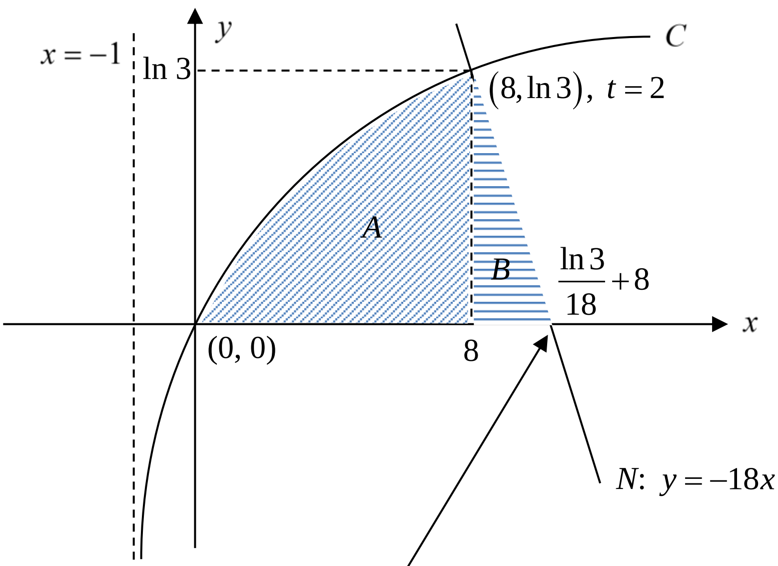
Equation of normal at $t=2$ is:

$$y-\ln 3=-18(x-8)$$

$$y=-18x+\ln 3+144$$

(iii) $\tan^{-1}18=86.8°>80°$, therefore N is not suitable in the design of the roller coaster drop as it is unsafe.

(iv)



Do not convert equation of C from parametric form to cartesian form as it is not required in the question.

When $y=0$,

$$0=-18x+\ln 3+144$$

$$x=\frac{\ln 3+144}{18}=\frac{1}{18}\ln 3+8$$

(Refer to graph)

Base $=\frac{\ln 3}{18}+8-8$

Height $=\ln 3$

Method 1: Using x-axis

Area required

= Area A + Area B (Triangle)

$$=\int_0^8 y\ \mathrm{d}x+\frac{1}{2}\left(\frac{1}{18}\ln 3+8-8\right)(\ln 3)$$

(Area A – change to t)

1) Limits: $x=0\Rightarrow t=0$ | $x=8\Rightarrow t=2$

2) Expression: $y=\ln(t+1)$

3) $\mathrm{d}x$: $\frac{\mathrm{d}x}{\mathrm{d}t}=2t+2\Rightarrow$ Let $\mathrm{d}x=(2t+2)\mathrm{d}t$

$$=\int_0^2 \ln(t+1)(2t+2)\ \mathrm{d}t+\frac{(\ln 3)^2}{36}$$

$$=\left[\left(\frac{2t^2}{2}+2t\right)\ln(t+1)\right]_0^2-\int_0^2\left(\frac{2t^2}{2}+2t\right)\frac{1}{t+1}\mathrm{d}t+\frac{(\ln 3)^2}{36}$$

Integration by parts (LIATE)

Choose $\ln(t+1)$ [L] for u

$u=\ln(t+1)$, $v=2t+2$

$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{t+1}$, $\int v\ \mathrm{d}x=\frac{2t^2}{2}+2t$

$$=\left[(t^2+2t)\ln(t+1)\right]_0^2-\int_0^2\frac{t^2+2t}{t+1}\mathrm{d}t+\frac{(\ln 3)^2}{36}$$

$$=8\ln 3-0-\int_0^2 t+1-\frac{1}{t+1}\mathrm{d}t+\frac{(\ln 3)^2}{36}$$

Long division

$$=8\ln 3-\left[\frac{t^2}{2}+t-\ln|t+1|\right]_0^2+\frac{(\ln 3)^2}{36}$$

$$=8\ln 3-\left[(4-\ln 3)-0\right]+\frac{(\ln 3)^2}{36}$$

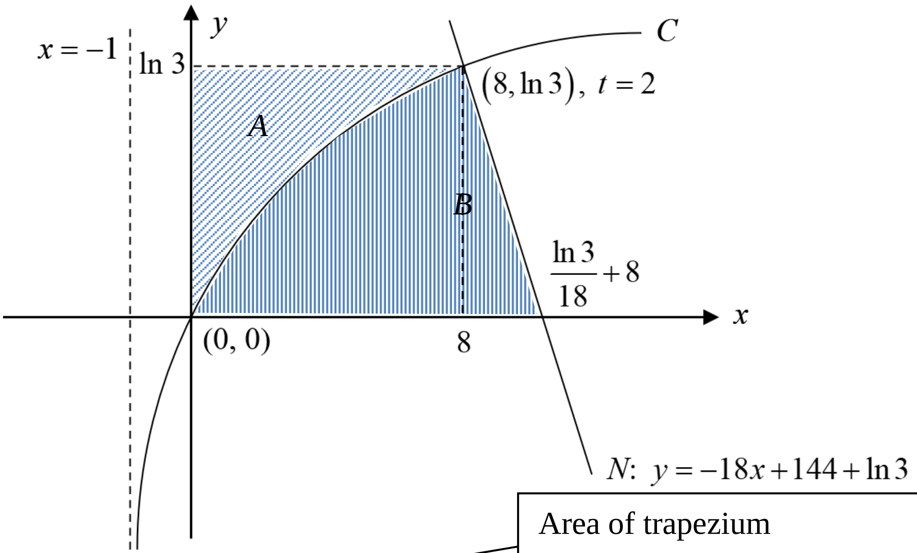
$$=\frac{1}{36}(\ln 3)^2+9\ln 3-4$$

$a=\dfrac{1}{36}$

$b=9$

$c=-4$

<u>Method 2:</u> Using y-axis



Area of trapezium

$=\dfrac{1}{2}\times$ Sum of 2 parallel lines $\times$ height

$=\dfrac{1}{2}\times\left[\left(\dfrac{\ln 3}{18}+8\right)+8\right]\times\ln 3$

Area required

= Area A+B (Trapezium) $-$ Area A

$$=\frac{1}{2}\left(\frac{1}{18}\ln 3+8+8\right)(\ln 3)-\int_0^{\ln 3}x\,\mathrm{d}y$$

$$=8\ln 3+\frac{(\ln 3)^2}{36}-\int_0^2\left(t^2+2t\right)\left(\frac{1}{t+1}\right)\mathrm{d}t$$

(Area A – change to t)

1) Limits: $y=0\Rightarrow t=0$ | $y=\ln 3\Rightarrow t=2$

2) Expression: $x=t^2+2t$

3) dx : $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{1}{t+1}\Rightarrow$ Let $\mathrm{d}y=\left(\dfrac{1}{t+1}\right)\mathrm{d}t$

$$=8\ln 3+\frac{(\ln 3)^2}{36}-\int_0^2\frac{t^2+2t}{t+1}\mathrm{d}t$$

$$=8\ln 3+\frac{(\ln 3)^2}{36}-\int_0^2 t+1-\frac{1}{t+1}\mathrm{d}t$$

Long division

$$=8\ln 3+\frac{(\ln 3)^2}{36}-\left[\frac{t^2}{2}+t-\ln|t+1|\right]_0^2$$

$$=8\ln 3+\frac{(\ln 3)^2}{36}-\left[(4-\ln 3)-0\right]$$

$$=\frac{1}{36}(\ln 3)^2+9\ln 3-4 \quad ,\ a=\frac{1}{36},\ b=9,\ c=-4$$