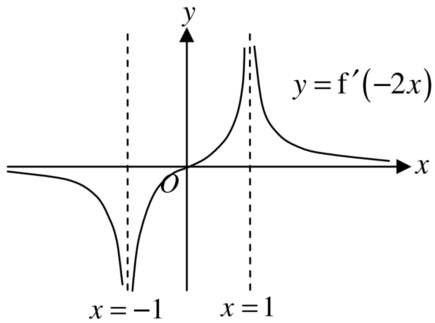
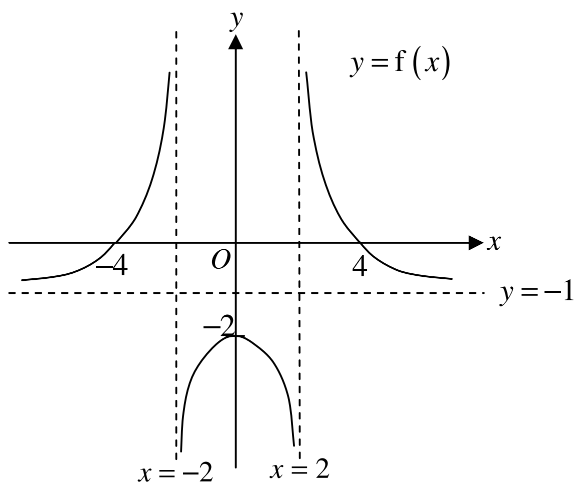
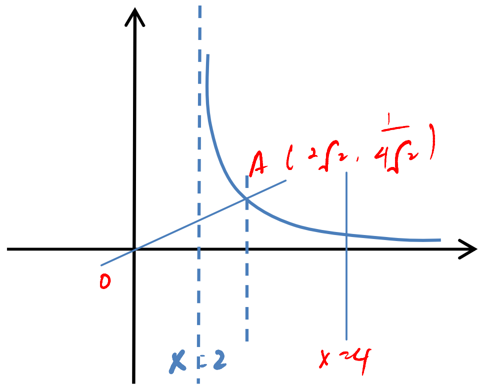
Solutions for 2012 H2 Maths Prelim 2 Paper 2

1(i)	
1(ii)	
2(i)	$\int_{\frac{1}{2\sqrt{2}}}^{\frac{1}{4}} \frac{u}{\sqrt{\left(\frac{1}{u}\right)^2-4}}\cdot -\frac{1}{u^2}\,\mathrm{d}u$ $=\int_{\frac{1}{2\sqrt{2}}}^{\frac{1}{4}} \frac{u}{\sqrt{\frac{1-4u^2}{u^2}}}\cdot -\frac{1}{u^2}\,\mathrm{d}u$ $=-\int_{\frac{1}{2\sqrt{2}}}^{\frac{1}{4}} \frac{1}{\sqrt{1-4u^2}}\,\mathrm{d}u$ $=-\frac{1}{2}\left[\sin^{-1}(2u)\right]_{\frac{1}{2\sqrt{2}}}^{\frac{1}{4}}$ $=-\frac{1}{2}\left[\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$ $=-\frac{1}{2}\left[\frac{\pi}{6}-\frac{\pi}{4}\right]=\frac{\pi}{24}$

2(ii)	 Area $= \frac{1}{2}\left(2\sqrt{2}\right)\left(\frac{1}{4\sqrt{2}}\right)+\int_{2\sqrt{2}}^{4}\frac{1}{x\sqrt{x^2-4}}\,\mathrm{d}x$ $= \frac{1}{4}+\frac{\pi}{24}$ unit^2
3(i)	$n\to\infty,\quad x_n\to l,\ x_{n+1}\to l.$ $l=\sqrt{10-3l}$ $l^2=10-3l$ $l^2+3l-10=0$ $(l-2)(l+5)=0$ Solving, $l=2$ or $l=-5$ (rejected)
3(ii)	$(x_{n+1})^2-l^2 = 10-3x_n-2^2$ $= 6-3x_n$ $= 3(l-x_n)$
3(iii)	Since $x_n<l$ $l-x_n>0$ $(x_{n+1})^2-l^2 = 3(l-x_n)$ $(x_{n+1})^2-l^2 > 0$ $(x_{n+1}-l)(x_{n+1}+l)>0$ $(x_{n+1}-l)>0$ (since $x_{n+1}+l>0$) $x_{n+1}>l$

<table>
<tr><td>4</td><td>Volume of cone $V=\frac{1}{3}\pi r^2 h$, where
$r=2\theta$ and $h=\sqrt{16\pi^2-4\theta^2}$.

$V=\frac{1}{3}\pi(2\theta)^2\sqrt{16\pi^2-4\theta^2}$
$V=\frac{4}{3}\pi\sqrt{\theta^4}\sqrt{4}\sqrt{4\pi^2-\theta^2}$
$V=\frac{8}{3}\pi\sqrt{4\pi^2\theta^4-\theta^6}$</td></tr>
<tr><td></td><td>Differentiating with respect to θ,
$\frac{\mathrm{d}V}{\mathrm{d}\theta}=\left(\frac{8}{3}\pi\right)\left(\frac{1}{2}\right)\left(4\pi^2\theta^4-\theta^6\right)^{-\frac{1}{2}}\left(16\pi^2\theta^3-6\theta^5\right)$
When $\frac{\mathrm{d}V}{\mathrm{d}\theta}=0$,
$16\pi^2\theta^3-6\theta^5=0$
$\theta^3\left(16\pi^2-6\theta^2\right)=0$
$16\pi^2=6\theta^2$
$\theta^2=\frac{8\pi^2}{3}$
$V=\frac{8}{3}\pi\sqrt{4\pi^2\left(\frac{8\pi^2}{3}\right)^2-\left(\frac{8\pi^2}{3}\right)^3}$
$V=\frac{8}{3}\pi\sqrt{\frac{256\pi^6}{9}-\frac{512\pi^6}{27}}$
$V=\frac{8}{3}\pi^4\sqrt{\frac{256}{27}}$
$V=\frac{128}{9\sqrt{3}}\pi^4$
$V=\frac{128\sqrt{3}}{27}\pi^4$
$p=128$, $q=27$

<table>
<tr><td>θ</td><td>$\sqrt{\frac{8\pi^2}{3}}^-$</td><td>$\sqrt{\frac{8\pi^2}{3}}$</td><td>$\sqrt{\frac{8\pi^2}{3}}^+$</td></tr>
<tr><td>gradient</td><td>+ ve</td><td>0</td><td>- ve</td></tr>
<tr><td>Shape</td><td>/</td><td>_</td><td>\</td></tr>
</table>

Therefore V is a maximum when $\theta=\sqrt{\frac{8\pi^2}{3}}$ (verified).</td></tr>
</table>

5	$\overrightarrow{OB}=\begin{pmatrix}12\\8\\0\end{pmatrix}, \overrightarrow{OD}=\begin{pmatrix}0\\4\\7\end{pmatrix}, \overrightarrow{DB}=\begin{pmatrix}12\\4\\-7\end{pmatrix}$ Equation of line DB: $\boldsymbol{r}=\begin{pmatrix}12\\8\\0\end{pmatrix}+\lambda\begin{pmatrix}12\\4\\-7\end{pmatrix}$ $\overrightarrow{OE}=\begin{pmatrix}12\\4\\5\end{pmatrix}$ Let *N* be the foot of perpendicular from *E* to line *DB*. $\overrightarrow{ON}=\begin{pmatrix}12+12k\\8+4k\\-7k\end{pmatrix}, \overrightarrow{EN}=\begin{pmatrix}12k\\4+4k\\-5-7k\end{pmatrix}$ $\overrightarrow{EN}.\boldsymbol{d}=0$ $\begin{pmatrix}12k\\4+4k\\-5-7k\end{pmatrix}.\begin{pmatrix}12\\4\\-7\end{pmatrix}=0$ $144k+16+16k+35+49k=0$ $k=-\frac{51}{209}$ $\overrightarrow{ON}=\frac{1}{209}\begin{pmatrix}1896\\1468\\357\end{pmatrix}$ The foot of perpendicular from E to DB is $\left(\frac{1896}{209},\frac{1468}{209},\frac{357}{209}\right)$.
(ii)	$\overrightarrow{OE}=\begin{pmatrix}12\\4\\5\end{pmatrix}$ $\overrightarrow{OB}\times\overrightarrow{OE}=\begin{pmatrix}12\\8\\0\end{pmatrix}\times\begin{pmatrix}12\\4\\5\end{pmatrix}=-4\begin{pmatrix}-10\\15\\12\end{pmatrix}$ $\boldsymbol{n}_{OBE}=\begin{pmatrix}-10\\15\\12\end{pmatrix}$

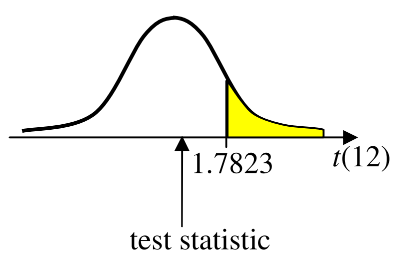
	Let θ be the angle between DB and OBE. $\theta = \sin^{-1}\left(\dfrac{\left\lvert\begin{pmatrix}12\\4\\-7\end{pmatrix}.\begin{pmatrix}-10\\15\\12\end{pmatrix}\right\rvert}{\sqrt{209}\sqrt{469}}\right) = \sin^{-1}\left(\dfrac{144}{\sqrt{209}\sqrt{469}}\right) = 27.4^o$
(iii)	$\overrightarrow{DE} = \begin{pmatrix}12\\0\\-2\end{pmatrix}, \overrightarrow{DB} = \begin{pmatrix}12\\4\\-7\end{pmatrix}$ Length of projection $= \dfrac{\begin{pmatrix}12\\0\\-2\end{pmatrix}.\begin{pmatrix}12\\4\\-7\end{pmatrix}}{\sqrt{209}} = \dfrac{158}{\sqrt{209}} = 10.929 = 10.9$ units
(iv)	As *DB* and *AC* are skew lines, they are not co-planar.

6(i)	Obtain a list of households according to the addresses. Select one household at random from the first 10 households on the list. Thereafter select every 10^{th} household on the list.
(ii)	<u>Stratified Sampling</u> Obtain a list of all households and divide the households according to the different types of housing. Select a random sample from each type of housing such that the sample size is proportional to the relative size of each type of housing. One advantage is that the sample obtained is a better representative of the population than systematic sampling. <u>Quota Sampling</u> Divide the households according to the different types of housing and set a quota for each type of housing. The interviewer interviews the households such that the quota is met. One advantage is that no sampling frame is needed.

7	Let X be the number of sixes obtained out of 25 throws of the biased die. Then $X \sim \mathrm{B}(25, p)$. Given that std. dev. of X is 1.5. So, $\mathrm{Var}(X) = 1.5^2$. $\therefore 25p(1-p) = 1.5^2$ $\Rightarrow p - p^2 = 0.09 \Rightarrow p^2 - p + 0.09 = 0$ From GC, $p = \frac{1}{10}$ or $\frac{9}{10}$ Since $p < \frac{1}{6}$, $\therefore p = \frac{1}{10}$ (shown) Hence $X \sim \mathrm{B}\left(25, \frac{1}{10}\right)$. Required probability = $\mathrm{P}(6 \le X < 10) = \mathrm{P}(X \le 9) - \mathrm{P}(X \le 5)$ $\approx 0.033321 = 0.0333$ (3 s.f.) Let Y be the number of sixes obtained out of 40 throws of the biased die. $Y \sim \mathrm{B}\left(40, \frac{1}{10}\right)$. From GC, $\mathrm{P}(X = 3) = 0.20032$ $\mathrm{P}(X = 4) = 0.20589$ $\mathrm{P}(X = 5) = 0.16471$ Hence, most likely number of sixes obtained is 4.

8(a)(i)	Let X be the number of genuine call-outs in a 2-week period. Then $X \sim \mathrm{Po}(4)$. $\mathrm{P}(X < 6) = \mathrm{P}(X \le 5) = 0.785$ (3 s.f.)
(ii)	Let T be the total number of call-outs in a 6-week period. Then $T \sim \mathrm{Po}(6(2+0.5))$, i.e., $T \sim \mathrm{Po}(15)$. Since $\lambda = 15$ (> 10), $T \sim \mathrm{N}(15, 15)$ approximately. $\mathrm{P}(T > 19) = \mathrm{P}(T > 19.5)$ (with continuity correction) $\approx 0.12264 = 0.123$ (3 s.f.)

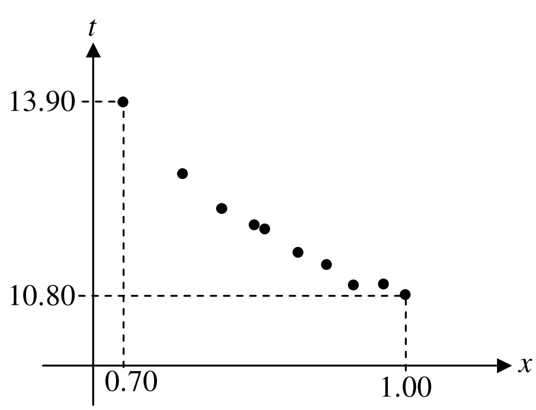
8(b)	Let W be the no. of genuine call-outs in a week at station B. $W \sim \text{Po}(m)$. $\text{P}(W \le 1) = 0.08$ $\Rightarrow \text{P}(W=0)+\text{P}(W=1)=0.08$ $\Rightarrow \text{e}^{-m}+m\text{e}^{-m}=0.08$ $\Rightarrow \text{e}^{-m}(1+m)=0.08$ From GC, $m = 4.17$ (3 s.f.)

9(i)	Unbiased estimate of population mean, $\bar{x}=\frac{\sum(x-12)}{13}+12$ $=\frac{6.09}{13}+12$ $= 12.4685$ $= 12.5$ (3 s.f.) Unbiased estimate of population variance, $s^2=\frac{1}{12}\left(20.853-\frac{6.09^2}{13}\right)$ $= 1.50$ (3 s.f.)
9(ii)	$H_0: \mu=\mu_0$ $H_1: \mu>\mu_0$ Significance level: 5% Under H_0, $T=\frac{\bar{X}-\mu_0}{\frac{S}{\sqrt{n}}} \sim t_{(12)}$ From GC, $\text{P}(T<1.7823)=0.95$ If H_0 is not rejected,  $\Rightarrow \frac{12.4685-\mu_0}{\sqrt{\frac{1.50}{13}}}<1.7823$ $\Rightarrow \mu_0>11.863$ $\Rightarrow \mu_0>11.9$ (3 s.f.) Thus, set of values of μ_0 for the null hypothesis not rejected is $\{\mu_0 \in \mathbb{R}: \mu_0>11.9\}$.

9(iii)	$H_0 : \mu = 12$ $H_1 : \mu > 12$ Significance level: $\alpha\%$ From GC, p-value = 0.09649 Significant evidence that the modified petrol does improve mileage in cars $\Rightarrow$ reject H_0 $\Rightarrow$ p-value $\le \dfrac{\alpha}{100}$ $\Rightarrow$ $\alpha \ge 9.649$ Thus least significance level = 9.65% (3 s.f.)
	Assumption: X is normally distributed. OR The mileage of the particular model of car is normally distributed.

10(i)	P(exactly two girls are chosen) $= \dfrac{15}{25}\times\dfrac{14}{24}\times\dfrac{10}{23}\times\dfrac{9}{22}\times\dfrac{4!}{2!2!}$ $= \dfrac{189}{506}$ (or $0.373518 \approx 0.374$) <u>Alt</u> P(exactly two girls are chosen) $= \dfrac{{}^{15}C_2\,{}^{10}C_2\times 4!}{{}^{25}C_4\times 4!} = \dfrac{189}{506}$ (or $0.373518 \approx 0.374$)
10(ii)	P(Chairperson and Vice-Chairperson are of opposite sex) $= \dfrac{15}{25}\times\dfrac{10}{24}\times 2$ $= 0.5$ <u>Alt</u> P(Chairperson and Vice-Chairperson are of opposite sex) $= \dfrac{{}^{15}C_1\,{}^{10}C_1\,{}^{23}C_2\times 2!\times 2}{{}^{25}C_4\times 4!}$ $= 0.5$
10(iii)	Reqd prob $= \dfrac{\text{P}(\text{boy, boy, boy, girl})+\text{P}(\text{boy, boy, girl, boy})}{\text{P(treasurer and sec are of opp sex)}}$

	$\text{P}(\text{boy, boy, boy, girl})+\text{P}(\text{boy, boy, girl, boy})$ $=\frac{10}{25}\times\frac{9}{24}\times\frac{8}{23}\times\frac{15}{22}\times2$ $=\frac{18}{253}$ or 0.071146 <u>Alt</u> $\text{P}(\text{boy, boy, boy, girl})+\text{P}(\text{boy, boy, girl, boy})$ $=\frac{{}^{10}C_3\,{}^{15}C_1\times3!\times2}{{}^{25}C_4\times4!}$ $=\frac{18}{253}$ or 0.071146 Reqd prob = $\frac{\left(\frac{18}{253}\right)}{0.5}$ or $\frac{0.071146}{0.5}$ $=\frac{36}{253}$ or $0.142292\approx0.142$
10	Let A be the event 'Chairperson and Vice-Chairperson are both boys', and B be the event 'Treasurer and Secretary are of opposite sex'. $\text{P}(A)=\frac{10}{25}\times\frac{9}{24}=0.15$ OR $\frac{{}^{10}C_2\,{}^{23}C_2\times2!\times2!}{{}^{25}C_4\times4!}=0.15$ From (iii), $\text{P}(A\mid B)=\frac{36}{253}$ (or 0.142) Since $\text{P}(A\mid B)\neq\text{P}(A)$, the events A and B are not independent.

11(i)	There will be no difference as the product moment correlation coefficient is independent of the units in which the data is measured.
11(ii)	The **regression line of *t* on *x*** should be used because the running time t is **dependent** on the leg length, x.
11(iii)	

11(iv)	**Yes**. Aaron has reason to disagree because the **scatter diagram suggests** that t and x has a **curvilinear relationship** rather than a linear one.
11(v)(a)	Product moment correlation coefficient between t and $\frac{1}{x^2}$ is **0.992** (3 s.f.) The new model is a **better model** because $\lvert 0.992 \rvert$ is **closer to 1** than $\lvert -0.963 \rvert = 0.963$.
11(v)(b)	Regression line is $t = 7.8603 + 2.8616\frac{1}{x^2}$ $t = 7.86 + 2.86\frac{1}{x^2}$ (3 s.f.) When $t = 10$, $10 = 7.8603 + 2.8616\frac{1}{x^2}$ $x^2 = \frac{2.8616}{2.1397}$ $x = 1.16$ (to 2 dec places) since $x > 0$ Thus minimum length of leg required is **1.16m**. This estimate **may not be reliable** as $t = 10$ is **outside the sample data range for *t***. OR Extrapolated values are unreliable.

12	Let X cm^3 and Y cm^3 be the volume of glass and wood in a paperweight respectively. Then $X \sim \mathrm{N}(56.5, 2.9^2)$, $Y \sim \mathrm{N}(38.4, \sigma^2)$. $\therefore X + Y \sim \mathrm{N}(94.9, 2.9^2 + \sigma^2)$ $\mathrm{P}(X + Y > 100) = 0.05$ $\Rightarrow \mathrm{P}(X + Y \le 100) = 0.95$ $\Rightarrow \mathrm{P}\left(Z \le \frac{100 - 94.9}{\sqrt{2.9^2 + \sigma^2}} \right) = 0.95$ $\Rightarrow \frac{5.1}{\sqrt{2.9^2 + \sigma^2}} = 1.6449$ $\Rightarrow 2.9^2 + \sigma^2 = \left(\frac{5.1}{1.6449} \right)^2$ $\Rightarrow \sigma = \sqrt{1.20305} = 1.09684$ $= 1.10$ (to 3 s.f.)

12(i)	Sample mean volume of glass for 20 paperweights, $\overline{X}=\dfrac{X_1+X_2+...+X_{20}}{20} \sim \text{N}\left(56.5, \dfrac{2.9^2}{20}\right)$ $\text{P}(\overline{X}<57.1)=0.823$ (3 s.f.)
(ii)	$Y_1-Y_2 \sim \text{N}(0, 2.42)$ $\text{P}\left(\lvert Y_1-Y_2\rvert \geq 0.07\right)$ $=\text{P}\left(Y_1-Y_2 \leq -0.07 \text{ or } Y_1-Y_2 \geq 0.07\right)$ $=\text{P}\left(Y_1-Y_2 \leq -0.07\right)+\text{P}(Y_1-Y_2 \geq 0.07)$ $=0.964$ (3 s.f.)
(iii)	Let W grams be the weight of a paperweight. Then $W = 3.1X + 0.8Y$ $\text{E}(W) = (3.1)(56.5)+0.8(38.4) = 205.87$ $\text{Var}(W) = 3.1^2(2.9^2) + 0.8^2(1.10^2) = 81.5945$ Then $W \sim \text{N}(205.87, 81.5945)$ $\text{P}(200 < W < 220) = 0.683$ (3 s.f.)