2023 Term 4 Timed Practice (Structured Remedial Session 3)

Question 1. (N14/2/4(b)(ii))

It is given that $w = \sqrt{3} - \mathrm{i}$.

Without using a calculator, find the three smallest positive whole number values of n for which $\dfrac{w^n}{w^*}$ is a real number. [4]

Solution	***Learning Points and self mark***
If a complex number is real, then: $\arg\left(\dfrac{w^n}{w^*}\right) = 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots$	1-mark for writing out the arguments for purely real complex number.
$n\arg(w) - \arg(w^*) = 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots$	1-mark for using result that $\arg\left(\dfrac{w^n}{w^*}\right) = n\arg(w) - \arg(w^*)$, or any appropriate method.
$n\arg(w) - \left[-\arg(w)\right] = 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots$ $n\arg(w) + \arg(w) = 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots$	
$\arg w = -\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$	1-mark for argument of *w*.
$n\left(-\dfrac{\pi}{6}\right) + \left(-\dfrac{\pi}{6}\right) = 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots$ $-n-1 = 0,\ \pm 6,\ \pm 12,\ \pm 18, \ldots$	
$n = 5,\ 11,\ 17$	1-mark for all answers of *n*.

Question 2. (N12/I/6, parts and modified)

The complex number z is given by $z = 1 - \mathrm{i}\sqrt{3}$.

Find the smallest positive integer n such that $\left|z^n\right| > 1000$. State the modulus and argument of z^n when n takes this value. [5]

Solution	***Learning Points and self mark***
use $z = 1 - \mathrm{i}\sqrt{3}$ $\left\|1 - \mathrm{i}\sqrt{3}\right\| = 2$ $\left\|z^n\right\| > 1000$ $\left\|z\right\|^n > 1000$ $\left\|1 - \mathrm{i}\sqrt{3}\right\|^n > 1000$ $2^n > 1000$ $n > 9.97$ Smallest positive value of n is 10.	1-mark for getting modulus of *z*. 1-mark for using result that $\left\|z^n\right\| = \left\|z\right\|^n$, or any appropriate method. 1-mark for correct answer.
$\left\|z^{10}\right\| = \left\|z\right\|^{10} = 2^{10} = 1024$ $\arg z^{10} = 10 \arg z^{10} = 10\left(-\frac{\pi}{3}\right) = -\frac{10\pi}{3}$ $= \frac{2\pi}{3}$	1-mark for getting 1024. 1-mark for getting $\frac{2\pi}{3}$.

Question 3. (N12/I/11, parts and modified)

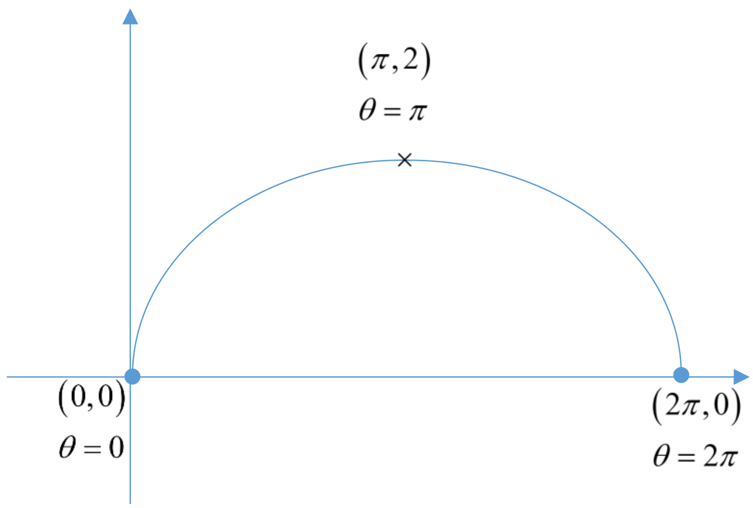
A curve C has parametric equations

$$x = \theta - \sin\theta, \qquad y = 1 - \cos\theta$$

where $0 \le \theta \le 2\pi$.

(i) Sketch C, showing clearly the features of the curve at the points where $\theta = 0$ and 2π. [2]

(ii) Without using the calculator, find the exact area of the region bounded by C and the x-axis. [5]

	Solution	***Learning Points and self mark***
(i)		1-mark for shape. 1-mark for **both** end-points written in coordinates form.
(ii)	$A = \int_0^{2\pi} y \, \mathrm{d}x$ $= \int_0^{2\pi} (1-\cos\theta)(1-\cos\theta) \, \mathrm{d}\theta$ $= \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta \, \mathrm{d}\theta$ $= \int_0^{2\pi} 1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta) \, \mathrm{d}\theta$	1-mark for expressing the area in terms of integral involving the parameter θ. 1-mark for expansion. 1-mark for using appropriate trigonometry identity that allows integration to take place.

	$=\left[\theta-2\sin\theta+\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}$ $=\left[2\pi+\frac{1}{2}(2\pi)\right]-[0]$	1-mark for correctly integrated expression.
	$=3\pi$	1-mark for answer.