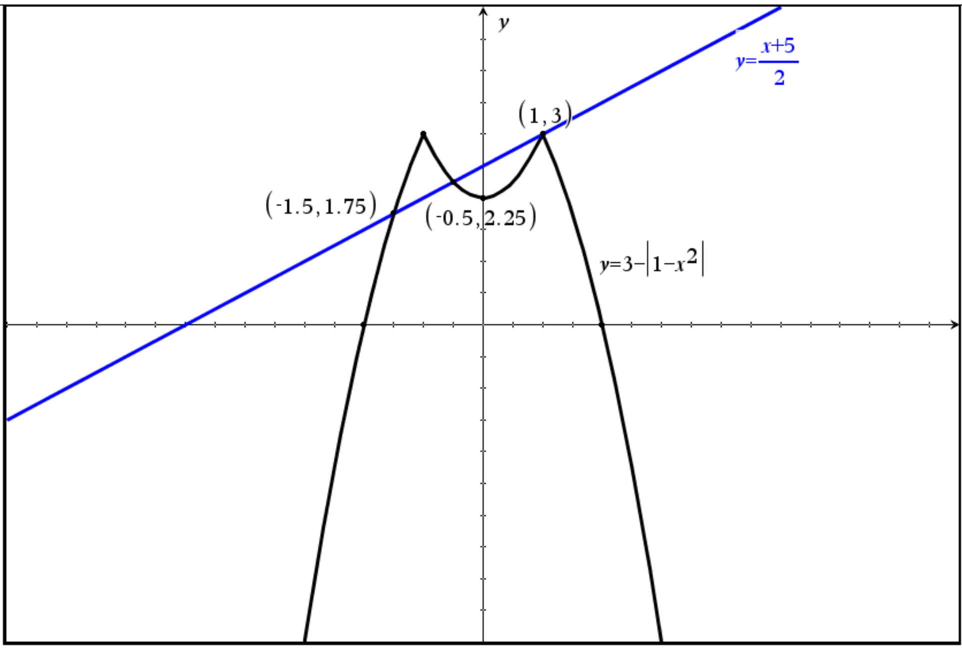
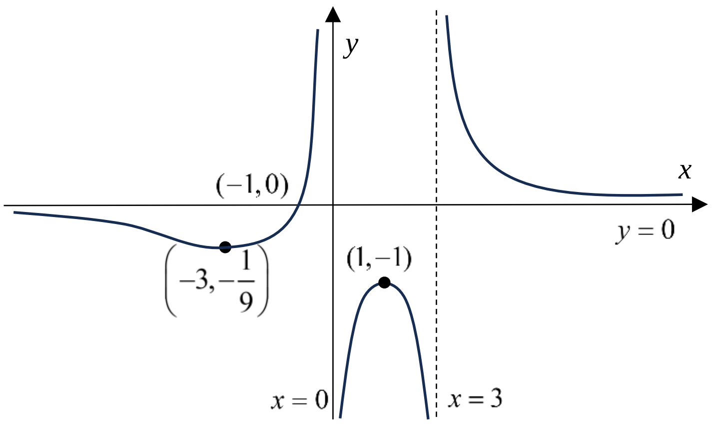
2024 EJC JC1 Promo Solutions

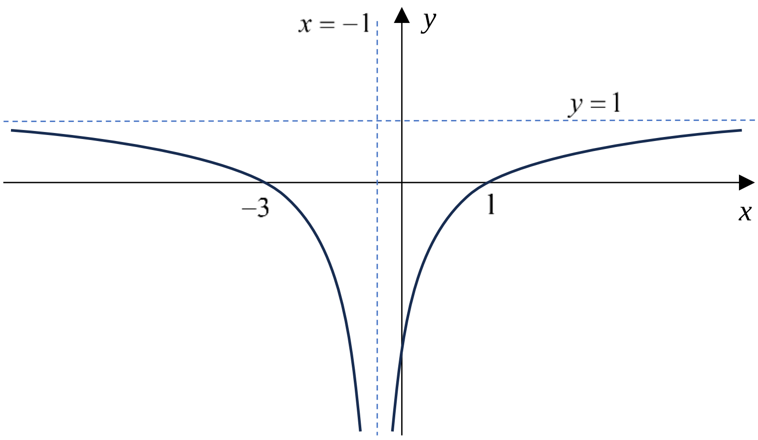
Q1	Let S, C and T be the number of stools, chairs and tables produced respectively. $\because$ 68 kg of metal is used in all, $S+4C+10T=68$. $\because$ 32 kg of plastic is used in all, $2S+2C+4T=32$. Solving this system of linear equations produces $\begin{cases} S=-\frac{4}{3}+\frac{2}{3}T \\ C=\frac{52}{3}-\frac{8}{3}T \end{cases}$ Since $S\geq 1$, $-\frac{4}{3}+\frac{2}{3}T\geq 1 \Rightarrow T\geq 3.5$ Since $C\geq 1$, $\frac{52}{3}-\frac{8}{3}T\geq 1 \Rightarrow T\leq 6.125$ So $T=4,5$ or 6. Testing all cases, • When $T=4$, S is not integer. • When $T=5$, $S=2$, $C=4$. • When $T=6$, S is not integer. So the only solution is $S=2, C=4, T=5$. $\therefore$ The number of stools, chairs and tables that could be produced is 2, 4, and 5 respectively.
Q2	 $\therefore -1.5\leq x\leq -0.5$ or $x=1$.

Q3

(a)



(b)



Q4

Let the length of *BC* be *x* cm and angle *BAC* be θ.

By cosine rule $x^2 = 5^2 + 4^2 - 2(5)(4)\cos\theta$

$x^2 = 41 - 40\cos\theta$ --- (1)

When $\theta = \frac{\pi}{3}$, $x^2 = 41 - 40\left(\frac{1}{2}\right) = 21$, $x = \sqrt{21}$ (since *x* is positive)

Differentiating (1) with respect to time *t*,

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} = -40(-\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Substituting in $\theta = \frac{\pi}{3}$, $x = \sqrt{21}$, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -0.2$,

$$2\sqrt{21}\frac{\mathrm{d}x}{\mathrm{d}t} = -40\left(-\frac{\sqrt{3}}{2}\right)(-0.2)$$

From GC, $\frac{\mathrm{d}x}{\mathrm{d}t} = -0.756$ (3 s.f.)

At that instant, *BC* is decreasing at a rate of 0.756 cm/s.

Q5 (a)	Required length of projection is $\left\|\|\mathbf{a}\|\cos\frac{5\pi}{6}\right\| = \left\|\sqrt{2}\times\frac{-\sqrt{3}}{2}\right\| = \frac{\sqrt{6}}{2}$
(b)	$\|3\mathbf{a}+2\mathbf{b}\|^2 = (3\mathbf{a}+2\mathbf{b}).(3\mathbf{a}+2\mathbf{b})$ $= (3\mathbf{a}).(3\mathbf{a})+(2\mathbf{b}).(3\mathbf{a})+(3\mathbf{a}).(2\mathbf{b})+(2\mathbf{b}).(2\mathbf{b})$ $= 9\|\mathbf{a}\|^2 + 12\mathbf{a}.\mathbf{b} + 4\|\mathbf{b}\|^2$ $= 9(2)+12(\sqrt{2})(\sqrt{6})\left(\frac{-\sqrt{3}}{2}\right)+4(6)$ $= 6$ So $\|3\mathbf{a}+2\mathbf{b}\| = \sqrt{6}$

Q6 (a)	$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^n\theta) = n\cos\theta\sin^{n-1}\theta$
(b)	$x = 2\cos^2\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -4\cos\theta\sin\theta$ $\int x\sqrt{1-\frac{x}{2}}\,\mathrm{d}x$ $= \int 2\cos^2\theta\sqrt{1-\cos^2\theta}\,(-4\cos\theta\sin\theta)\,\mathrm{d}\theta$ $= \int -8\cos^3\theta\sin\theta\sqrt{\sin^2\theta}\,\mathrm{d}\theta$ $= \int -8\cos^3\theta\sin^2\theta\,\mathrm{d}\theta$ $= \int -8\cos\theta\cos^2\theta\sin^2\theta\,\mathrm{d}\theta$ $= \int -8\cos\theta(1-\sin^2\theta)\sin^2\theta\,\mathrm{d}\theta$ $= 8\int \cos\theta\sin^4\theta - \cos\theta\sin^2\theta\,\mathrm{d}\theta$ (Shown)
(c)	Using part (a), $8\int \cos\theta\sin^4\theta - \cos\theta\sin^2\theta\,\mathrm{d}\theta = 8\left(\frac{\sin^5\theta}{5} - \frac{\sin^3\theta}{3}\right) + C$ Since $x = 2\cos^2\theta$,

	$\therefore \sin^2\theta = 1-\cos^2\theta = 1-\frac{x}{2}$ $\therefore \sin\theta = \sqrt{1-\frac{x}{2}}$ $\therefore \int x\sqrt{1-\frac{x}{2}}\,\mathrm{d}x = \frac{8}{5}\left(1-\frac{x}{2}\right)^{\frac{5}{2}} - \frac{8}{3}\left(1-\frac{x}{2}\right)^{\frac{3}{2}} + C$

Q7 (ai)	If $r=1$, all the terms of the sequence will be equal to the first term a. Then the sum of the first 10 terms will be $10a$, which is only 2 times the sum of the first 5 terms $5a$.
(aii)	Using sum of G.P. formula, $a\times\frac{1-r^{10}}{1-r} = 33\times a\times\frac{1-r^5}{1-r}$ $1-r^{10} = 33-33r^5$ $\left(1-r^5\right)\left(1+r^5\right) = 33\left(1-r^5\right)$ $\left(1-r^5\right)\left(1+r^5-33\right) = 0$ $\left(1-r^5\right)\left(r^5-32\right) = 0$ $r^5 = 1$ (reject since $r \neq 1$) or $r^5 = 32$ $r = \sqrt[5]{32} = 2$ (shown)
(aiii)	$u_6 \leq 11$ and $u_7 > 11$ $a(2)^5 \leq 11$ and $a(2)^6 > 11$ $32a \leq 11$ and $64a > 11$ $\frac{11}{64} < a \leq \frac{11}{32}$
(b)	$\sum_{r=4}^{n}\frac{1}{(2r+3)(2r+5)}$ $= \sum_{r-2=4}^{r-2=n}\frac{1}{(2r-1)(2r+1)}$ (replace r with $r-2$) $= \sum_{r=6}^{n+2}\frac{1}{(2r-1)(2r+1)}$ $= \sum_{r=1}^{n+2}\frac{1}{(2r-1)(2r+1)} - \sum_{r=1}^{5}\frac{1}{(2r-1)(2r+1)}$

	$=\left(\frac{1}{2}-\frac{1}{4n+10}\right)-\left(\frac{1}{2}-\frac{1}{22}\right)$ $=\frac{1}{22}-\frac{1}{4n+10}$

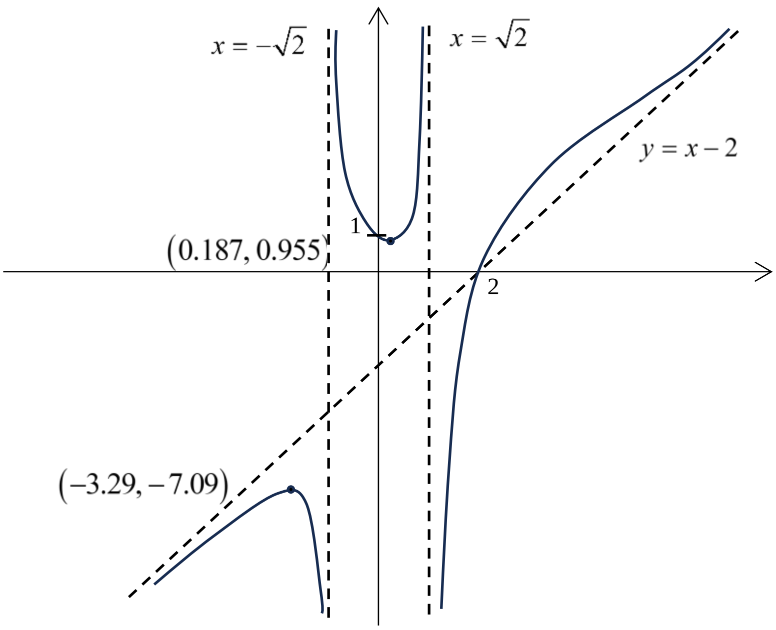
Q8 (a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\sec^2\left[\ln(1+2x)\right]\times\frac{1}{1+2x}\times 2=\frac{2}{1+2x}\left\{1+\tan^2\left[\ln(1+2x)\right]\right\}=\frac{2}{1+2x}\left(1+y^2\right)$ Rearranging, $(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x}=2\left(1+y^2\right)$ (shown)
(b)	Differentiating the given result, $(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}(2)=2(2y)\frac{\mathrm{d}y}{\mathrm{d}x}$ When $x=0$, $y=\tan\left[\ln(1+0)\right]=\tan 0=0$, $(1+0)\frac{\mathrm{d}y}{\mathrm{d}x}=2(1+0)\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=2$, $(1+0)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(2)=2(0)(2)\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4$, so $\tan\left[\ln(1+2x)\right]=(0)+(2)x+\frac{(-4)}{2!}x^2+...=2x-2x^2+...$
(c)	$\mathrm{e}^{\frac{1}{2}y}=\mathrm{e}^{\frac{2x-2x^2+...}{2}}=\mathrm{e}^{\left(x-x^2+...\right)}$ $=1+\left(x-x^2+...\right)+\frac{\left(x-x^2+...\right)^2}{2!}+...$ $=1+x-\frac{1}{2}x^2+...$
(d)	$\lim_{x\to 0}\left[\frac{\mathrm{e}^{\frac{1}{2}y}-1}{3x}\right]=\lim_{x\to 0}\left[\frac{\left(1+x-\frac{1}{2}x^2+...\right)-1}{3x}\right]$ $=\lim_{x\to 0}\left[\frac{x-\frac{1}{2}x^2+...}{3x}\right]$ $=\lim_{x\to 0}\left[\frac{1}{3}-\frac{1}{6}x+...\right]$ $=\frac{1}{3}$

Q9 (a)	$\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = 2t^2 - 2a$ $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{t^2 - a}{t}$ When $t = a$, $x = a^2 - 1$ $y = \frac{2}{3}a^3 - 2a^2$ $\frac{dy}{dx} = \frac{a^2 - a}{a} = a - 1$ Equation of tangent: $y - \left(\frac{2}{3}a^3 - 2a^2\right) = (a-1)[x - (a^2 - 1)]$ $y = (a-1)x + \left(-\frac{1}{3}a^3 - a^2 + a - 1\right)$
(b)	From part (a), the tangent at point Q has gradient $a - 1$. $\frac{t^2 - a}{t} = a - 1$ $t^2 - a = at - t$ $t^2 + t - at - a = 0$ $(t-a)(t+1) = 0$ $t = -1$ (since $t = a$ is point P) When $t = -1$, $x = (-1)^2 - 1 = 0$ $y = \frac{2}{3}(-1)^3 - 2a(-1) = 2a - \frac{2}{3}$ The coordinates of point Q are $\left(0, 2a - \frac{2}{3}\right)$.
(c)	When $y = 0$, $\frac{2}{3}t^3 - 2at = 0$ $\frac{2}{3}t\left(t^2 - 3a\right) = 0$ $t = 0$ or $t^2 = 3a$ $x = -1$ or $x = 3a - 1$ So we need $3a - 1 > 0$, i.e. $a > \frac{1}{3}$

Q10 (a)	[0, 8]
(b)(i)	$\because\ \mathrm{f}(4)=\mathrm{f}(8)=8$ f is a many-to-one function and f^{-1} does not exist
(b)(ii)	$0<c\le 2$
(c)(i)	[0, 9]
(c)(ii)	When $c=1$: From graph, For $0\le x\le 1,\ 0\le \mathrm{f}(x)\le 2.75$; For $1<x\le 8,\ 4.5<\mathrm{f}(x)\le 8$. Hence, the range is $[0,2.75]\cup(4.5,8]$.
(c)(iii)	For f^2 to exist, we need the range of f to be a subset of $[0,8]$ the domain of f, i.e. $R_{\mathrm{f}}\subseteq D_{\mathrm{f}}=[0,8]$ From graph, we need $0<c\le 4$.

Q11 (ai)	$u_2=\frac{1}{5}u_1-3=\frac{1}{5}(2)-3=-2.6$ or $-\frac{13}{5}$ $u_3=\frac{1}{5}u_2-3=\frac{1}{5}(-2.6)-3=-3.52$ or $-\frac{88}{25}$ $u_4=\frac{1}{5}u_3-3=\frac{1}{5}(-3.52)-3=-3.704$ or $-\frac{463}{125}$
(aii)	Since sequence converges to l, $u_n,u_{n+1}\approx l$ when n is large. Solving $l=\frac{l}{5}-3$, we get $l=-3.75$ or $-\frac{15}{4}$.
(bi)	$\sum_{r=1}^{n}u_{r+4}$ is the sum of n terms of an arithmetic progression with first term $u_5=2-3(4)=-10$ and common difference $d=-3$. $\therefore \sum_{r=1}^{n}u_{r+4}=\frac{n}{2}\left[2(-10)+(n-1)(-3)\right]=-\frac{n(3n+17)}{2}$.
(bii)	Since $u_n=2+(n-1)(-3)=5-3n$, $u_{3n+4}=5-3(3n+4)=-7-9n$ $u_{3(n+1)+4}=u_{3n+7}=5-3(3n+7)=-16-9n$ So $u_{3n+7}-u_{3n+4}=(-16-9n)-(-7-9n)=-9$ This is a constant independent of n, so the sequence is an A.P.
(biii)	$\sum_{r=1}^{100}u_{3r+4}=u_7+u_{10}+\ldots+u_{304}$ is the sum of 100 terms of an A.P. with first term $u_7=2-3(6)=-16$ and common difference -9. $\therefore \sum_{r=1}^{100}u_{3r+4}=\frac{100}{2}\left[2(-16)+99(-9)\right]=-46150$

Q12 (a)	When $x = 0$, $y=-2+\frac{-6k}{-2k}=-2+3=1$. When $y = 0$, we have $(x-2)(x^2-2k)+3k(x-2)=0$ $(x-2)(x^2-2k+3k)=0$ $\Rightarrow\ (x-2)(x^2+k)=0$ $\Rightarrow\ x=2$ The coordinates are (0, 1) and (2, 0).
(b)	As $x\to\infty$, $y\to x-2$ Vertical asymptotes: $x=\sqrt{2k}$ and $x=-\sqrt{2k}$ Oblique asymptote: $y=x-2$
(c)	$y=x-2+\frac{3(x-2)}{x^2-2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1+3\left[\frac{(x^2-2)-(x-2)(2x)}{(x^2-2)^2}\right]$ $=1+\frac{3(x^2-2-2x^2+4x)}{(x^2-2)^2}$ $=1+\frac{3(-x^2+4x-2)}{(x^2-2)^2}$ At turning points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$. Hence we have $(x^2-2)^2+3(-x^2+4x-2)=0$ $x^4-4x^2+4-3x^2+12x-6=0$ $\Rightarrow x^4-7x^2+12x-2=0$ (shown) Using GC, $x = -3.29$ or $x = 0.187$

(d)	

Q13 (a)	$\mathbf{v.n} = \begin{pmatrix}0\\0\\-1\end{pmatrix}.\frac{1}{5}\begin{pmatrix}3\\0\\4\end{pmatrix} = -\frac{4}{5}$ Thus $\mathbf{w} = \begin{pmatrix}0\\0\\-1\end{pmatrix} - 2\left(-\frac{4}{5}\right)\left(\frac{1}{5}\right)\begin{pmatrix}3\\0\\4\end{pmatrix} = \frac{1}{25}\begin{pmatrix}24\\0\\7\end{pmatrix}$
(b)	$\lvert\mathbf{w}\rvert = \frac{1}{25}\sqrt{24^2+7^2} = \frac{1}{25}\sqrt{576+49} = 1$ so **w** is a unit vector.
(c)	$\overrightarrow{MC} = \overrightarrow{OC} - \overrightarrow{OM} = \begin{pmatrix}10\\5\\10\end{pmatrix}$ So $\mathbf{w} = \frac{1}{\sqrt{10^2+5^2+10^2}}\begin{pmatrix}10\\5\\10\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}$
(d)	Let $\mathbf{n} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. Then $\mathbf{v.n} = \begin{pmatrix}0\\0\\-1\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = -z$, so substituting into (*) we get

	$\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - 2(-z)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 2xz \\ 2yz \\ 2z^2 - 1 \end{pmatrix}$ Solving, $2z^2 - 1 = \frac{2}{3}$ gives $z = \pm\sqrt{\frac{5}{6}}$. Substituting each value into the other components to solve for *x* and *y*, $\mathbf{n} = \begin{pmatrix} \sqrt{\frac{2}{15}} \\ \sqrt{\frac{1}{30}} \\ \sqrt{\frac{5}{6}} \end{pmatrix} = \frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ or $\mathbf{n} = -\frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ So $\frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ is a unit normal to plane *P*.
(e)	Since *P* passes through *M*, the equation of *P* is $\mathbf{r}.\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -10 \\ -5 \\ 0 \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = -25$ $2x + y + 5z = -25$
(f)	Required angle is $\cos^{-1}\left\lvert \frac{\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}.\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left\lvert\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}\right\rvert\left\lvert\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\rvert} \right\rvert = \cos^{-1}\frac{5}{\sqrt{30}} = 24.1^\circ$ (3 s.f.)
(g)	The incoming beam of sunlight cannot be parallel to the plane of the mirror.