Solutions (Techniques of Integrations)

1	$\int e^{2x}\tan^{-1}\left(e^{-2x}\right)\,\mathrm{d}x$ $=\frac{1}{2}e^{2x}\tan^{-1}\left(e^{-2x}\right)-\int\frac{1}{2}e^{2x}\frac{-2e^{-2x}}{1+\left(e^{-2x}\right)^2}\,\mathrm{d}x$ $=\frac{1}{2}e^{2x}\tan^{-1}\left(e^{-2x}\right)+\int\frac{1}{\left[1+e^{-4x}\right]}\,\mathrm{d}x$ $=\frac{1}{2}e^{2x}\tan^{-1}\left(e^{-2x}\right)+\int\frac{e^{4x}}{e^{4x}+1}\,\mathrm{d}x=\frac{1}{2}e^{2x}\tan^{-1}\left(e^{-2x}\right)+\frac{1}{4}\ln\left(e^{4x}+1\right)+C$
	$\int\frac{x}{\sqrt{1-4x-2x^2}}\,dx=-\frac{1}{4}\int\frac{-4-4x}{\sqrt{1-4x-2x^2}}\,dx-\int\frac{1}{\sqrt{1-4x-2x^2}}\,dx$ $=-\frac{1}{2}\sqrt{1-4x-2x^2}-\int\frac{1}{\sqrt{2}\sqrt{\frac{3}{2}-(x+1)^2}}\,dx$ $=-\frac{1}{2}\sqrt{1-4x-2x^2}-\frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}(x+1)}{\sqrt{3}}+C$

2 (i)	$\int x\sin x\,dx$ $=-x\cos x-\int-\cos x\,dx$ $=\sin x-x\cos x+c$ $dx=-2\sin u\cos u\,du$
(ii)	$\int\frac{1}{\cos^2 u\sqrt{1-\cos^2 u}}\bullet-2\sin u\cos u\,du$ $=-2\int\frac{1}{\cos u}\,du$ $=-2\int\sec u\,du$ $=-2\ln(\sec u+\tan u)+c$ $=-2\ln\left(\frac{1}{\sqrt{x}}+\sqrt{\frac{1-x}{x}}\right)+c$

3	(a) $\displaystyle\int_{\frac{\pi}{6}}^{0} \frac{5\sin x - 3\cos x}{\cos x - \sin x}\ \mathrm{d}x = \int_{\frac{\pi}{6}}^{0} \frac{(\cos x + \sin x) - 4(\cos x - \sin x)}{\cos x - \sin x}\ \mathrm{d}x$ $\displaystyle = \int_{\frac{\pi}{6}}^{0} \frac{\cos x + \sin x}{\cos x - \sin x} - 4\ \mathrm{d}x$ $\displaystyle = \Big[-\ln\lvert\cos x - \sin x\rvert - 4x\Big]_{\frac{\pi}{6}}^{0}$ $\displaystyle = \frac{2}{3}\pi + \ln\left(\frac{\sqrt{3}-1}{2}\right)$ (b) $\displaystyle \frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1-x^2\right)^{\frac{1}{2}}\right) = \frac{x}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x) + \left(1-x^2\right)^{\frac{1}{2}}$ $\displaystyle = \frac{x^2 + \left(1-x^2\right)}{\sqrt{1-x^2}}$ $\displaystyle = \frac{1-2x^2}{\sqrt{1-x^2}}$ $\displaystyle \int \frac{1-2x^2}{\sqrt{1-x^2}}\ \mathrm{d}x = \left(x\left(1-x^2\right)^{\frac{1}{2}}\right) + C$ $\displaystyle \int_{0}^{\frac{1}{2}} \frac{3-2x^2}{\sqrt{1-x^2}} - \frac{2}{\sqrt{1-x^2}}\ \mathrm{d}x = \Big[x\sqrt{1-x^2}\Big]_{0}^{\frac{1}{2}}$ $\displaystyle \int_{0}^{\frac{1}{2}} \frac{3-2x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = \Big[x\sqrt{1-x^2} + 2\sin^{-1} x\Big]_{0}^{\frac{1}{2}}$ $\displaystyle = \frac{\pi}{3} + \frac{\sqrt{3}}{4}$

4	$\displaystyle \int \frac{\ln x - \ln 2}{x\sqrt{\ln x - \ln 2 - 2}}\,\mathrm{d}x$ $\displaystyle = \int \frac{\ln 2e^t - \ln 2}{2e^t\sqrt{\ln 2e^t - \ln 2 - 2}}\ \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$ $\displaystyle = \int \frac{\ln 2 + t - \ln 2}{2e^t\sqrt{\ln 2 + t - \ln 2 - 2}}\ 2e^t\ \mathrm{d}t$ $\displaystyle = \int \frac{t}{\sqrt{t-2}}\ \mathrm{d}t$	$x = 2e^t$ $\displaystyle \frac{\mathrm{d}x}{\mathrm{d}t} = 2e^t$

	$=\int \dfrac{(t-2)+2}{\sqrt{t-2}}\ \mathrm{d}t$ $=\int (t-2)^{\frac{1}{2}}+\dfrac{2}{\sqrt{t-2}}\ \mathrm{d}t$ (shown) $\displaystyle\int_{2e^2}^{2e^4} \frac{\ln x-\ln 2}{x\sqrt{\ln x-\ln 2-2}}\ \mathrm{d}x$ $\displaystyle=\int_2^4 (t-2)^{\frac{1}{2}}+\frac{2}{\sqrt{t-2}}\ \mathrm{d}t$ $=\left[\dfrac{2}{3}(t-2)^{\frac{3}{2}}+4(t-2)^{\frac{1}{2}}\right]_2^4$ $=\left[\dfrac{2}{3}(4-2)^{\frac{3}{2}}+4(4-2)^{\frac{1}{2}}\right]$ $=\dfrac{2}{3}(2)^{\frac{3}{2}}+4(2)^{\frac{1}{2}}$ $=(2)^{\frac{1}{2}}\left[\dfrac{2}{3}(2)+4\right]$ $=\dfrac{16\sqrt{2}}{3}$ Therefore $a=16$, $b=3$

5	$\dfrac{\mathrm{d}}{\mathrm{d}x}e^{\cos x}=-e^{\cos x}\sin x$.
	$\int e^{\cos x}\sin 2x\ \mathrm{d}x$ $=\int e^{\cos x}(2\sin x\cos x)\ \mathrm{d}x$ $=\int(-e^{\cos x}\sin x)(-2\cos x)\ \mathrm{d}x$ $=-2e^{\cos x}\cos x-2\int e^{\cos x}\sin x\ \mathrm{d}x$ $=-2e^{\cos x}\cos x+2e^{\cos x}+C$

6 (a)	Let $\dfrac{2x^2-5x+13}{x^2-2x+5}=A+\dfrac{B(2x-2)+C}{x^2-2x+5}$ $\Rightarrow 2x^2-5x+13=Ax^2+(-2A+2B)x+(5A-2B+C)$ Comparing coeffs: $\begin{cases} A=2 \\ -2A+2B=-5 \Rightarrow B=-\dfrac{1}{2} \\ 5A-2B+C=13 \Rightarrow C=2 \end{cases}$

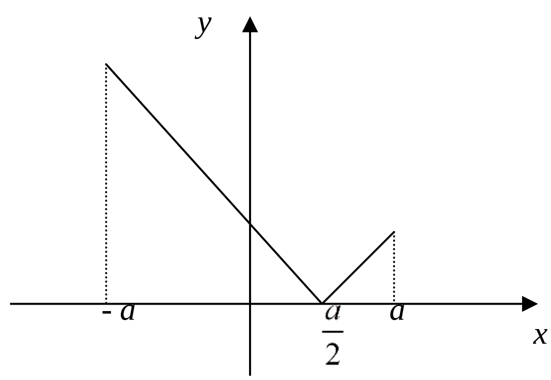
	$\therefore \int \frac{2x^2-5x+13}{x^2-2x+5}\,dx = \int 2\,dx - \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}\,dx + \int \frac{2}{(x-1)^2+2^2}\,dx$ $= 2x - \frac{1}{2}\ln(x^2-2x+5) + \tan^{-1}\frac{x-1}{2} + C$
(b)	$\int_1^{2e} x^{n-1}\ln x\,dx = \left[\ln x \cdot \frac{x^n}{n} - \int \frac{x^n}{n}\cdot\frac{1}{x}\,dx\right]_1^{2e}$ $= \left[\ln x \cdot \frac{x^n}{n} - \frac{1}{n^2}\cdot x^n\right]_1^{2e}$ $= \left[\ln(2e)\cdot\frac{(2e)^n}{n} - \frac{1}{n^2}(2e)^n\right] - \left[0 - \frac{1}{n^2}\right]$ $= \frac{1}{n^2}\left[n(2e)^n(\ln 2+1) - (2e)^n + 1\right]$

7	$\int_{-1}^{1} \left\| e^{2x} - \frac{1}{e^{2(x-1)}} \right\| \, dx$ $= -\int_{-1}^{1/2} e^{2x} - \frac{1}{e^{2(x-1)}}\,dx + \int_{1/2}^{1} e^{2x} - \frac{1}{e^{2(x-1)}}\,dx$ $= -\left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2(x-1)}\right]_{-1}^{1/2} + \left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2(x-1)}\right]_{1/2}^{1}$ $= \frac{1}{2}\left(e^4 + e^2 - 4e + e^{-2} + 1\right)$

8	$\int x\cos^{-1}x^2\,dx$ $= \frac{x^2}{2}\cos^{-1}x^2 - \int \frac{x^2}{2}\left(\frac{-2x}{\sqrt{1-x^4}}\right)dx$ $= \frac{x^2}{2}\cos^{-1}x^2 - \frac{1}{4}\int \frac{-4x^3}{\sqrt{1-x^4}}\,dx$ $= \frac{x^2}{2}\cos^{-1}x^2 - \frac{1}{2}\sqrt{1-x^4} + C$

9	$\int \frac{[\ln(2x)]^2}{x\{25-2[\ln(2x)]^2\}}\,dx$ $= \int \frac{2u^2}{e^u(25-2u^2)}\cdot\frac{1}{2}e^u\,du$	$x = \frac{1}{2}e^u \Rightarrow 2x = e^u$ $2\,dx = e^u\,du$

	$= \int \frac{u^2}{25-2u^2}\ \mathrm{d}u$ $= -\frac{1}{2}\int \frac{-2u^2+25-25}{25-2u^2}\ \mathrm{d}u$ $= -\frac{1}{2}\int 1-\frac{25}{25-2u^2}\ \mathrm{d}u$ $= -\frac{1}{2}\left(u-25\cdot\frac{1}{\sqrt{2}(2)(5)}\ln\left\|\frac{5+u\sqrt{2}}{5-u\sqrt{2}}\right\|\right)+c$ $= -\frac{1}{2}\left(u-\frac{5}{2\sqrt{2}}\ln\left\|\frac{5+u\sqrt{2}}{5-u\sqrt{2}}\right\|\right)+c$ $= \frac{1}{2}\left(\frac{5}{2\sqrt{2}}\ln\left\|\frac{5+\sqrt{2}\ln(2x)}{5-\sqrt{2}\ln(2x)}\right\|-\ln(2x)\right)+c$

10 (a)	 $\int_{-a}^{0}\left\|x-\frac{a}{2}\right\|\ \mathrm{d}x = k\int_{0}^{a}\left\|x-\frac{a}{2}\right\|\ \mathrm{d}x$ $\frac{1}{2}(\frac{3}{2}a+\frac{1}{2}a)(a) = k2\cdot\frac{1}{2}\frac{a}{2}(a-\frac{a}{2})$ $a^2 = \frac{1}{4}ka^2$ $k = 4$
(b)	**(i)** $\frac{\mathrm{d}}{\mathrm{d}x}(x^2\sin 2x) = 2x\sin 2x + 2x^2\cos 2x$ **(ii)** $\int(2x^2\cos 2x + 2x\sin 2x)\,\mathrm{d}x = x^2\sin 2x$

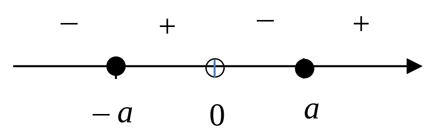
	$\int (2x^2 \cos 2x)\ \mathrm{d}x = -\int 2x \sin 2x\, \mathrm{d}x + x^2 \sin 2x$ $= -[-x\cos 2x - \int -\cos 2x\ \mathrm{d}x] + x^2 \sin 2x$ $= -[-x\cos 2x + \frac{1}{2}\sin 2x] + x^2 \sin 2x + C$ $= x\cos 2x - \frac{1}{2}\sin 2x + x^2 \sin 2x + C$ $\int (2x^2 \cos 2x)\ \mathrm{d}x = \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + \frac{1}{2}x^2 \sin 2x + C$

11(a) (i)	$\int x\tan\left(x^2\right)\mathrm{d}x = -\frac{1}{2}\int \frac{-2x\sin\left(x^2\right)}{\cos\left(x^2\right)}\mathrm{d}x$ $= -\frac{1}{2}\ln\left\|\cos\left(x^2\right)\right\| + c$
(ii)	$\int \frac{x}{x^2+x+3}\mathrm{d}x = \frac{1}{2}\int \frac{2x+1-1}{x^2+x+3}\mathrm{d}x$ $= \frac{1}{2}\int \frac{2x+1}{x^2+x+3}\mathrm{d}x - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{11}{4}}\mathrm{d}x$ $= \frac{1}{2}\ln\left\|x^2+x+3\right\| - \frac{1}{\sqrt{11}}\tan^{-1}\frac{(2x+1)}{\sqrt{11}} + c$
(b) (i)	$\int_0^{\frac{1}{\sqrt{2}}} x\sin^{-1}\left(x^2\right)\mathrm{d}x = \left[\frac{x^2}{2}\sin^{-1}\left(x^2\right)\right]_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^4}}\mathrm{d}x$ $= \left[\frac{x^2}{2}\sin^{-1}\left(x^2\right)\right]_0^{\frac{1}{\sqrt{2}}} + \frac{1}{4}\int_0^{\frac{1}{\sqrt{2}}} \frac{-4x^3}{\sqrt{1-x^4}}\mathrm{d}x$ $= \left[\frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{2}\sqrt{1-x^4}\right]_0^{\frac{1}{\sqrt{2}}}$ $= \frac{\pi}{24} + \frac{\sqrt{3}}{4} - \frac{1}{2}$
(ii)	Since $0 < b < 1$, $\int_0^1 x\left\|x-b\right\|dx = \int_0^b -x(x-b)dx + \int_b^1 x(x-b)dx$ $= -\left[\frac{x^3}{3} - \frac{bx^2}{2}\right]_0^b + \left[\frac{x^3}{3} - \frac{bx^2}{2}\right]_b^1$ $= \frac{b^3}{3} + \frac{1}{3} - \frac{b}{2}$

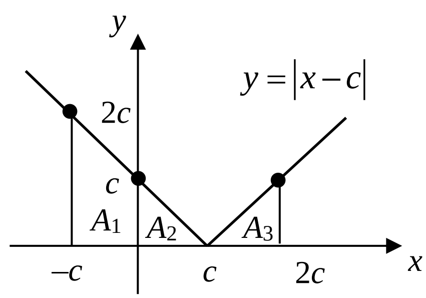
12	$\int_{\frac{1}{2}}^{n} \frac{(\tan^{-1}2x)^2}{1+4x^2}\,dx$ $=\frac{1}{2}\int_{\frac{1}{2}}^{n} 2\frac{(\tan^{-1}2x)^2}{1+4x^2}\,dx=\frac{1}{6}\left[(\tan^{-1}2x)^3\right]_{\frac{1}{2}}^{n}$ $=\frac{1}{6}\left[\left(\tan^{-1}2n\right)^3-\left(\frac{\pi}{4}\right)^3\right]$ As $n\to\infty,\ \tan^{-1}2n\to\frac{\pi}{2}$. $\therefore \int_{\frac{1}{2}}^{\infty} \frac{(\tan^{-1}2x)}{1+4x^2}\,dx=\frac{1}{6}\left[\left(\frac{\pi}{2}\right)^3-\left(\frac{\pi}{4}\right)^3\right]=\frac{7}{384}\pi^3$

13 (i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(2^{2x}\right)=2^{2x+1}\ln 2$
(ii)	$\int 2^{2x}\ln 2^{x}\mathrm{d}x$ $=\frac{1}{2}\int (x)\left(2^{2x+1}\ln 2\right)\mathrm{d}x$ $=\frac{1}{2}\left[2^{2x}x-\int 2^{2x}\mathrm{d}x\right]$ $=\frac{1}{2}\left[2^{2x}x-2^{2x}\frac{1}{2\ln 2}\right]+C$ $=2^{2x-1}\left(x-\frac{1}{2\ln 2}\right)+C$

14 (a)	$\int (\ln\frac{x}{2})^2\,\mathrm{d}x=[x(\ln\frac{x}{2})^2]-\int x[2(\ln\frac{x}{2})(\frac{2}{x})(\frac{1}{2})]\mathrm{d}x$ $=[x(\ln\frac{x}{2})^2]-2\int(\ln\frac{x}{2})\,\mathrm{d}x$ $=[x(\ln\frac{x}{2})^2]-2[x(\ln\frac{x}{2})-\int x(\frac{2}{x})(\frac{1}{2})\ \mathrm{d}x]$ $=x(\ln\frac{x}{2})^2-2x(\ln\frac{x}{2})+2x+c$
(b)	$x^3\geq\frac{a^4}{x}$ $\frac{x^4-a^4}{x}\geq 0$ $\frac{(x^2-a^2)(x^2+a^2)}{x}\geq 0$

	$\dfrac{(x-a)(x+a)(x^2+a^2)}{x} \geq 0$
	Since $x^2+a^2>0$, $\therefore$ $\dfrac{(x+a)(x-a)}{x} \geq 0$
	$-a \leq x < 0$ or $x \geq a$
	
	For $1<x<a$, $x^3-\dfrac{a^4}{x}<0$
	For $a<x<3$, $x^3-\dfrac{a^4}{x}>0$
	$\displaystyle\int_1^3 \left\lvert x^3-\frac{a^4}{x}\right\rvert \mathrm{d}x$
	$\displaystyle= \int_1^a -(x^3-\frac{a^4}{x})\,\mathrm{d}x + \int_a^3 (x^3-\frac{a^4}{x})\,\mathrm{d}x$
	$\displaystyle= -[\frac{x^4}{4}-a^4\ln\lvert x\rvert]_1^a + [\frac{x^4}{4}-a^4\ln\lvert x\rvert]_a^3$
	$\displaystyle= -[\frac{a^4}{4}-a^4\ln a-(\frac{1}{4})]+[\frac{81}{4}-a^4\ln 3-(\frac{a^4}{4}-a^4\ln a)]$
	$\displaystyle= 2a^4\ln a - a^4\ln 3-\frac{a^4}{2}+\frac{41}{2}$
	$\displaystyle= a^4\ln\left(\frac{a^2}{3}\right)-\frac{a^4}{2}+\frac{41}{2}$

15	$\displaystyle\int_{-c}^{0}\lvert x-c\rvert\mathrm{d}x = \int_{-c}^{0} c-x\mathrm{d}x$
	$\displaystyle= \left[cx-\frac{x^2}{2}\right]_{-c}^{0}$
	$\displaystyle= -\left[c(-c)-\frac{(-c)^2}{2}\right]$
	$\displaystyle= c^2+\frac{1}{2}c^2$
	$\displaystyle= \frac{3}{2}c^2$
	$\displaystyle\int_{0}^{2c}\lvert x-c\rvert\mathrm{d}x = \int_{0}^{c} c-x\mathrm{d}x + \int_{c}^{2c} x-c\mathrm{d}x$
	$\displaystyle= \left[cx-\frac{x^2}{2}\right]_0^c + \left[\frac{x^2}{2}-cx\right]_c^{2c}$

	$= c^2 - \frac{c^2}{2} + \left(\frac{4c^2}{2} - 2c^2\right) - \left(\frac{c^2}{2} - c^2\right)$ $= c^2$ $\int_{-c}^{0} \lvert x-c \rvert \, dx = k\int_{0}^{2c} \lvert x-c \rvert \, dx \Leftrightarrow \frac{3}{2}c^2 = kc^2$ $\therefore k = \frac{3}{2}$ <u>Alternative:</u>  $\int_{-c}^{0} \lvert x-c \rvert \, dx = k\int_{0}^{2c} \lvert x-c \rvert \, dx$ Area A_1 = k (Area A_2 + Area A_3) $\frac{1}{2}c(2c+c) = k\left(\frac{1}{2}c(c) + \frac{1}{2}c(c)\right)$ $\frac{1}{2}\left(3c^2\right) = kc^2$ $k = \frac{3}{2}$

16 (a)	$\int \frac{x^2}{(x-1)(x-2)} \, dx$ $= \int 1 - \frac{1}{x-1} + \frac{4}{x-2} \, dx$ $= x - \ln\lvert x-1 \rvert + 4\ln\lvert x-2 \rvert + c$
(b)(i)	$\frac{d}{dx}\sin^{-1}\left(x^2\right) = \frac{2x}{\sqrt{1-x^4}}$

(b)(ii)	$\int_0^n x\sin^{-1}\left(x^2\right)\,\mathrm{d}x$ $=\left[\frac{x^2}{2}\sin^{-1}\left(x^2\right)\right]_0^n-\int_0^n \frac{x^2}{2}\frac{2x}{\sqrt{1-x^4}}\,\mathrm{d}x$ $=\left[\frac{x^2}{2}\sin^{-1}\left(x^2\right)+2\left(\frac{1}{4}\right)\sqrt{1-x^4}\right]_0^n$ $=\frac{n^2}{2}\sin^{-1}\left(n^2\right)+\frac{1}{2}\sqrt{1-n^4}-\frac{1}{2}=\frac{\pi}{4}-\frac{1}{2}$ From GC or observation, $n = 1$ (reject $n=-1$ since $n\in\mathbb{Z}^+$)

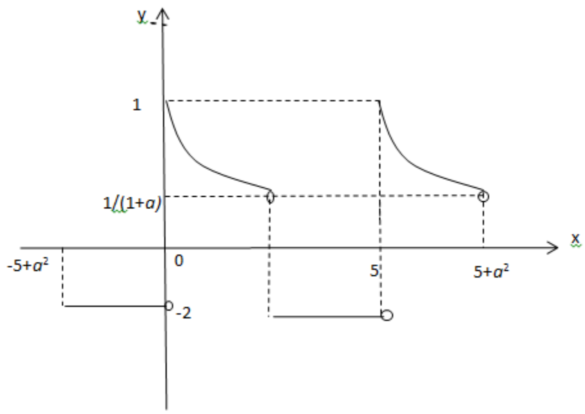
17(a)	$\int x\sec^2(x+a)\,\mathrm{d}x$ $=x\tan(x+a)-\int \tan(x+a)\,\mathrm{d}x$ $=x\tan(x+a)-\ln\left\|\sec(x+a)\right\|+C$ **OR**: $x\tan(x+a)+\ln\left\|\cos(x+a)\right\|+C$
(b)	$\int \frac{x-1}{x^2-2x+2}\,\mathrm{d}x=\frac{1}{2}\int \frac{2x-2}{x^2-2x+2}\,\mathrm{d}x$ $=\frac{1}{2}\ln(x^2-2x+2)+C$
(b)(i)	$\int_1^2 \frac{x-4}{x^2-2x+2}\,\mathrm{d}x$ $=\int_1^2 \frac{x-1}{x^2-2x+2}\,\mathrm{d}x-\int_1^2 \frac{3}{x^2-2x+2}\,\mathrm{d}x$ $=\int_1^2 \frac{x-1}{x^2-2x+2}\,\mathrm{d}x-\int_1^2 \frac{3}{(x-1)^2+1}\,\mathrm{d}x$ $=\frac{1}{2}\left[\ln\left(x^2-2x+2\right)\right]_1^2-3\left[\tan^{-1}(x-1)\right]_1^2$ $=\frac{1}{2}\left[\ln 2-\ln 1\right]-3\left[\tan^{-1}1-\tan^{-1}0\right]$ $=\frac{1}{2}\ln 2-\frac{3\pi}{4}$

(b)(ii)	Note that $\frac{x-1}{x^2-2x+2}=\frac{x-1}{(x-1)^2+1}$: $-$ $+$ (sign change at 1) $\int_{2-p}^{p}\left\|\frac{x-1}{x^2-2x+2}\right\|\,dx$ $=-\int_{2-p}^{1}\frac{x-1}{(x-1)^2+1}\,dx+\int_{1}^{p}\frac{x-1}{(x-1)^2+1}\,dx$ $=2\int_{1}^{p}\frac{x-1}{(x-1)^2+1}\,dx$ (by symmetry) $=2\left[\frac{1}{2}\ln(x^2-2x+2)\right]_1^p=\ln\left(p^2-2p+2\right)$

18	From $u=1-x$, $\frac{du}{dx}=-1$. Limits: when $x=0, u=1$, and when $x=1, u=0$. Therefore $\int_0^1 x^n(1-x)^m\,dx=\int_1^0(1-u)^n u^m\,(-du)$ $=\int_0^1(1-u)^n u^m\,du$ $=\int_0^1(1-x)^n x^m\,dx$ (by a change of dummy variables) By substituting $n=2$ and $m=\frac{1}{2}$ into the previous result: $\int_0^1 x^2(1-x)^{\frac{1}{2}}\,dx=\int_0^1(1-x)^2 x^{\frac{1}{2}}\,dx$ $=\int_0^1\left(1-2x+x^2\right)x^{\frac{1}{2}}\,dx$ $=\int_0^1 x^{\frac{1}{2}}-2x^{\frac{3}{2}}+x^{\frac{5}{2}}\,dx$ $=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{4}{5}x^{\frac{5}{2}}+\frac{2}{7}x^{\frac{7}{2}}\right]_0^1=\frac{16}{105}$

19(a) (i)	$u=\ln x$ $\qquad$ $\frac{dv}{dx}=\frac{1}{x^2}$ $\frac{du}{dx}=\frac{1}{x}$ $\qquad$ $v=-\frac{1}{x}$
	$\int_1^n\frac{1}{x^2}\ln x\,dx$ $=\left[-\frac{1}{x}\ln x\right]_1^n-\int_1^n-\frac{1}{x}\left(\frac{1}{x}\right)dx$

	$= -\frac{\ln n}{n} - \left[\frac{1}{x}\right]_1^n$ $= -\frac{\ln n}{n} - \left[\frac{1}{n} - 1\right]$ $= -\frac{\ln n}{n} - \frac{1}{n} + 1$
(a)(ii)	$\int_1^{\infty} \frac{1}{x^2} \ln x \, dx = \lim_{n\to\infty}\left[-\frac{\ln n}{n} - \frac{1}{n} + 1\right] = 1$
(b)	$x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$ When x = a, $\sec\theta = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$. When x = 2a, $\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$. $\int_a^{2a} \frac{\sqrt{x^2 - a^2}}{x} \, dx$ $= \int_0^{\frac{\pi}{3}} \frac{\sqrt{a^2\sec^2\theta - a^2}}{a\sec\theta} \, a\sec\theta\tan\theta \, d\theta$ $= a\int_0^{\frac{\pi}{3}} \tan^2\theta \, d\theta$ $= a\int_0^{\frac{\pi}{3}} \left(\sec^2\theta - 1\right) d\theta$ $= a\left[\tan\theta - \theta\right]_0^{\frac{\pi}{3}}$ $= a\left(\sqrt{3} - \frac{\pi}{3}\right)$

20(i)	$\mathrm{f}(100)=\mathrm{f}(0)=1$
(ii)	
(iii)	$u=1+\sqrt{x}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}=\frac{1}{2(u-1)}$ $\int_0^{a^2}\frac{1}{1+\sqrt{x}}\,\mathrm{d}x$ $=\int_1^{1+a}\frac{1}{u}\,2(u-1)\mathrm{d}u$ $=2\int_1^{1+a}1-\frac{1}{u}\,\mathrm{d}u=2\left[u-\ln\lvert u\rvert\right]_1^{1+a}=2\left(1+a-\ln\lvert 1+a\rvert-1+0\right)=2\left(a-\ln(1+a)\right)$

21(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{1-4x^2}}\right)=-\frac{1}{2}\left(1-4x^2\right)^{-\frac{3}{2}}\cdot(-4)(2x)$ $=\frac{4x}{\sqrt{\left(1-4x^2\right)^3}}$

(ii)	$\displaystyle\int \frac{x\sin^{-1}(2x)}{\sqrt{\left(1-4x^2\right)^3}}\ \mathrm{d}x$ $\displaystyle=\int \frac{4x}{\sqrt{\left(1-4x^2\right)^3}}\cdot\frac{1}{4}\sin^{-1}(2x)\ \mathrm{d}x$ $\displaystyle=\frac{1}{\sqrt{1-4x^2}}\cdot\frac{1}{4}\sin^{-1}(2x)\ -\int \frac{1}{\sqrt{1-4x^2}}\cdot\frac{1}{4}\frac{2}{\sqrt{1^2-(2x)^2}}\ \mathrm{d}x$ $\displaystyle=\frac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}}-\frac{1}{4}\int \frac{2}{1^2-(2x)^2}\ \mathrm{d}x$ $\displaystyle=\frac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}}-\frac{1}{8}\ln\left\|\frac{1+2x}{1-2x}\right\|+C$ or $\displaystyle\frac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}}-\frac{1}{8}\ln\frac{1+2x}{1-2x}+C$

22. Solutions
Given $u = 2x - 1$. Then $x=\frac{1}{2}(u+1)$ and $\frac{\mathrm{d}x}{\mathrm{d}u}=\frac{1}{2}$. $\displaystyle\int \frac{x}{\sqrt{1-(2x-1)^2}}\ \mathrm{d}x$ $\displaystyle=\int \frac{\frac{1}{2}(u+1)}{\sqrt{1-u^2}}\ \frac{1}{2}\mathrm{d}u$ $\displaystyle=\frac{1}{4}\left[\int \frac{u}{\sqrt{1-u^2}}\ \mathrm{d}u+\int \frac{1}{\sqrt{1-u^2}}\ \mathrm{d}u\right]$ $\displaystyle=\frac{1}{4}\int \left(-\frac{1}{2}\right)(-2u)\left(1-u^2\right)^{-\frac{1}{2}}\ \mathrm{d}u+\sin^{-1}u+C$ $\displaystyle=\frac{1}{4}\left[-\frac{1}{2}\frac{\left(1-u^2\right)^{\frac{1}{2}}}{\frac{1}{2}}+\sin^{-1}u\right]+C$ $\displaystyle=\frac{1}{4}\left[\sin^{-1}u-\sqrt{1-u^2}\right]++C$ $\displaystyle=\frac{1}{4}\sin^{-1}(2x-1)-\frac{1}{4}\sqrt{1-(2x-1)^2}+C$ where C is an arbitrary constant.

$$\int \sin^{-1}(2x-1)\,\mathrm{d}x$$

$$u = \sin^{-1}(2x-1) \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{\sqrt{1-(2x-1)^2}} \qquad v = x$$

$$\int \sin^{-1}(2x-1)\,\mathrm{d}x$$

$$= x\sin^{-1}(2x-1) - \int \frac{2x}{\sqrt{1-(2x-1)^2}}\,\mathrm{d}x$$

$$= x\sin^{-1}(2x-1) - \frac{1}{2}\sin^{-1}(2x-1) + \frac{1}{2}\sqrt{1-(2x-1)^2} + C$$

$$= \left(x - \frac{1}{2}\right)\sin^{-1}(2x-1) + \frac{1}{2}\sqrt{1-(2x-1)^2} + C$$

where C is an arbitrary constant.

23(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+x^2} = \frac{1}{2}\cdot\frac{2x}{\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}} \text{ (shown)}$$

$$\int \frac{3x^3}{\sqrt{1+x^2}}\,\mathrm{d}x = \int \left(3x^2\right)\cdot\frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$= (3x^2)\sqrt{1+x^2} - \int (6x)\sqrt{1+x^2}\,\mathrm{d}x$$

$$= (3x^2)\sqrt{1+x^2} - 3\int (2x)\sqrt{1+x^2}\,\mathrm{d}x$$

$$= (3x^2)\sqrt{1+x^2} - 3\cdot\frac{\left(1+x^2\right)^{3/2}}{3/2} + c$$

$$= (3x^2)\sqrt{1+x^2} - 2\left(1+x^2\right)^{3/2} + c$$

By parts:

$$u = 3x^2 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x}{\sqrt{1+x^2}}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x \qquad v = \sqrt{1+x^2}$$

(from previous part)

Power formula:

$$\int \mathrm{f}'(x)\left(\mathrm{f}(x)\right)^n\,\mathrm{d}x = \frac{\left(\mathrm{f}(x)\right)^{n+1}}{n+1}$$

$$\mathrm{f}(x) = 1+x^2$$

$$\mathrm{f}'(x) = 2x$$

23(bi)	$\int \cos 2mx \cos 2nx \, dx$ $= \frac{1}{2}\int \cos(2mx+2nx)+\cos(2mx-2nx) \, dx$ $= \frac{1}{2}\int \cos(2m+2n)x+\cos(2m-2n)x \, dx$ $= \frac{1}{2}\frac{\sin(2m+2n)x}{2m+2n}+\frac{1}{2}\frac{\sin(2m-2n)x}{2m-2n}+C$ $= \frac{\sin(2m+2n)x}{4m+4n}+\frac{\sin(2m-2n)x}{4m-4n}+C$ From MF26: $\cos P+\cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$ $\cos\underbrace{\frac{1}{2}(P+Q)}_{2mx}\cos\underbrace{\frac{1}{2}(P-Q)}_{2nx} = \frac{1}{2}(\cos P+\cos Q)$ $\frac{1}{2}(P+Q)=2mx-(1) \quad \frac{1}{2}(P-Q)=2nx-(2)$ $(1)+(2): \quad P=2mx+2nx$ $(1)-(2): \quad Q=2mx-2nx$ $\therefore \cos 2mx \cos 2nx$ $= \frac{1}{2}(\cos(2mx+2nx)+\cos(2mx-2nx))$
23(bii)	$\int_0^{\pi}(f(x))^2 \, dx$ $= \int_0^{\pi}(\cos 2mx+\cos 2nx)^2 \, dx$ $= \int_0^{\pi}\cos^2(2mx)+2\cos(2mx)\cos(2nx)+\cos^2(2nx) dx$ $= \int_0^{\pi}\left[\frac{1}{2}(\cos(2)(2mx)+1)\right]+2\left[\cos(2mx)\cos(2nx)\right]+\left[\frac{1}{2}(\cos(2)(2nx)+1)\right]dx$ $= \int_0^{\pi}\left[\frac{1}{2}\cos(4mx)+\frac{1}{2}\right]+2\left[\cos(2mx)\cos(2nx)\right]+\left[\frac{1}{2}\cos(4nx)+\frac{1}{2}\right]dx$ $= \left[\frac{\sin(4mx)}{8m}+\frac{x}{2}\right]_0^{\pi}+2\left[\frac{\sin(2m+2n)x}{4m+4n}+\frac{\sin(2m-2n)x}{4m-4n}\right]_0^{\pi}+\left[\frac{\sin(4nx)}{8n}+\frac{x}{2}\right]_0^{\pi}$ $= \frac{\pi}{2}+0+\frac{\pi}{2}$ $= \pi$

24. DHS/2022/I/Q3

(a) Differentiate $e^{\sin^2 2x}$ with respect to x. [2]

(b) Find $\int \frac{e^{\sin^2 2x}\sin 4x}{\sqrt{1+e^{\sin^2 2x}}} \, dx$. [2]

(c) Find the exact value of $\int_0^{\frac{\pi}{4}} e^{\sin^2 2x} \sin 4x \cos^2 2x \, dx$. [3]

DHS Prelim 9758/2022/01/Q3

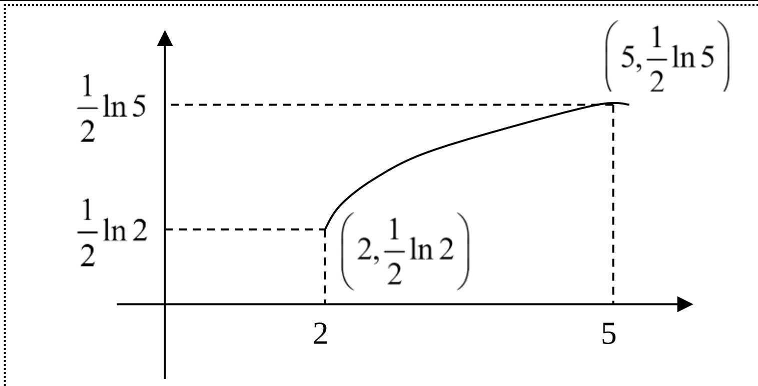
Qn	Suggested Solution
3(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\sin^2 2x}=4\mathrm{e}^{\sin^2 2x}\sin 2x\cos 2x=2\mathrm{e}^{\sin^2 2x}\sin 4x$
(b)	$\int \frac{\mathrm{e}^{\sin^2 2x}\sin 4x}{\sqrt{1+\mathrm{e}^{\sin^2 2x}}}\,\mathrm{d}x$ $=\frac{1}{2}\int\left(2\mathrm{e}^{\sin^2 2x}\sin 4x\right)\left(1+\mathrm{e}^{\sin^2 2x}\right)^{-\frac{1}{2}}\,\mathrm{d}x$ $=\left(1+\mathrm{e}^{\sin^2 2x}\right)^{\frac{1}{2}}+c$ $=\sqrt{1+\mathrm{e}^{\sin^2 2x}}+c$
(c)	$\int_0^{\frac{\pi}{4}}\left(\mathrm{e}^{\sin^2 2x}\sin 4x\right)\cos^2 2x\,\mathrm{d}x$ $=\left[\frac{1}{2}\mathrm{e}^{\sin^2 2x}\cos^2 2x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{1}{2}\mathrm{e}^{\sin^2 2x}\left(-4\cos 2x\sin 2x\right)\,\mathrm{d}x$ $=-\frac{1}{2}+\int_0^{\frac{\pi}{4}}\mathrm{e}^{\sin^2 2x}\sin 4x\,\mathrm{d}x$ $=-\frac{1}{2}+\left[\frac{1}{2}\mathrm{e}^{\sin^2 2x}\right]_0^{\frac{\pi}{4}}$ $=-\frac{1}{2}+\frac{1}{2}\mathrm{e}-\frac{1}{2}$ $=\frac{1}{2}\mathrm{e}-1$

Solutions (Areas & Volumes)

1	(i) By G.C Intersection point (1.05395, -0.947453), (4.3919, 0.47976) $\text{Area} = \int_{1.05395}^{4.3919} \ln(x) - e^{x-4} dx$ *[M1 - correct limits ; M1 - correct form]* $\text{Area} = 1.68$ (ii) $V_x = \pi \int_0^b (x^2)^2 dx$ $V_x = \pi \left[\frac{x^5}{5} \right]_0^b = \pi \frac{b^5}{5}$ $V_y = \pi (b^2)(b^2) - \pi \int_0^{b^2} y dy$ *[M1 - Vol of cylinder ; M1 - Vol of revolution abt y-axis]* $V_y = \pi b^4 - \pi \left[\frac{y^2}{2} \right]_0^{b^2} = \pi \frac{b^4}{2}$ $\pi \frac{b^5}{5} = \pi \frac{b^4}{2}$ $b^4 \left(\frac{b}{5} - \frac{1}{2} \right) = 0$ $b = 0 \ (rejected)$ $b = \frac{5}{2}$ <u>Alternative Solution</u> $V_x = \pi \int_0^b (x^2)^2 dx$ $V_x = \pi \left[\frac{x^5}{5} \right]_0^b = \pi \frac{b^5}{5}$ $V_y = \pi (b^2)(b^2) - \pi \int_1^{b^2+1} y - 1 dy$ $V_y = \pi b^4 - \pi \left[\frac{y^2}{2} - y \right]_1^{b^2+1} = \pi b^4 - \pi \left[\frac{(b^2+1)^2}{2} - (b^2+1) - \frac{1}{2} + 1 \right]$

	$V_y = \pi b^4 - \pi\left[\frac{y^2}{2} - y\right]_1^{b^2+1} = \pi b^4 - \pi\frac{b^4}{2} = \pi\frac{b^4}{2}$ Therefore, $b = 0$ (*rejected*) or $b = \frac{5}{2}$

2 (i)



(ii) Required area $= \int_2^5 y\ dx$

$= \int_{\sqrt{2}}^{\sqrt{5}} (\ln t) 2t\ dt$

$= 2\left\{\left[\frac{t^2}{2}\ln t\right]_{\sqrt{2}}^{\sqrt{5}} - \int_{\sqrt{2}}^{\sqrt{5}} \frac{t^2}{2}.\frac{1}{t}\ dt\right\}$

$= \frac{5}{2}\ln 5 - \ln 2 - \frac{3}{2}$

Therefore, $\alpha = \frac{5}{2}$, $\beta = -1$, $\gamma = -\frac{3}{2}$

(iii)

Required volume $= \pi\int_0^5 \left(\frac{1}{2}\ln 5\right)^2 \mathrm{d}x - \pi\int_0^2 \left(\frac{1}{2}\ln 2\right)^2 \mathrm{d}x - \pi\int_2^5 y^2 \mathrm{d}x$

$= 10.17205 - 0.75469 - \pi\int_{\sqrt{2}}^{\sqrt{5}} (\ln t)^2\, 2t\, \mathrm{d}t$

$= 5.75$ units^3

3

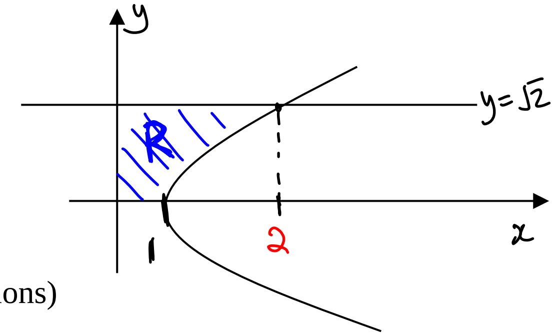
(i) Let $u = \sqrt{x-1} \Rightarrow x = u^2 + 1 \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2u$

When $x = 1$, $u = 0$, When $x = 2$, $u = 1$

$\int_1^2 x\sqrt{x-1}\ \mathrm{d}x = \int_0^1 (u^2+1)\left(\sqrt{u^2+1-1}\right)(2u)\ \mathrm{d}u$

$= \int_0^1 (u^2+1)(u)(2u)\ \mathrm{d}u$

$= \int_0^1 (2u^4 + 2u^2)\ \mathrm{d}u$

	$= \left[\frac{2}{5}u^5 + \frac{2}{3}u^3\right]_0^1 = \frac{16}{15}$ **(ii)** When $y=\sqrt{2}$, $\left(\sqrt{2}\right)^2 = x\sqrt{x-1}$ $\Rightarrow 2 = x\sqrt{x-1}$ $\Rightarrow x^2(x-1) = 2^2$ $\Rightarrow x^3 - x^2 - 4 = 0$ $\Rightarrow x = 2$ (no other real solutions)  Required volume $=$ volume of cylinder $-\pi\int_1^2 y^2 \, dx$ $= \pi\left(\sqrt{2}\right)^2(2) - \pi\int_1^2 x\sqrt{x-1}\, dx = 4\pi - \frac{16}{15}\pi = \frac{44}{15}\pi$

4(i)	$u = e^x \quad \Rightarrow \frac{du}{dx} = e^x \quad \Rightarrow \frac{du}{dx} = u$ $\int_0^{\ln 2} \frac{e^x}{e^x + 3e^{-x}} dx = \int_1^2 \frac{u}{u^2+3} du = \frac{1}{2}\left[\ln(u^2+3)\right]_1^2 = \frac{1}{2}\ln(\frac{7}{4})$
(ii)	Area $= \int_0^{\ln 2} \frac{7e^x}{e^x + 3e^{-x}} dx - \frac{1}{2}(\ln 2)(4) = \frac{7}{2}\ln(\frac{7}{4}) - 2\ln 2$
(iii)	$\text{Vol}_{(x)} = \pi\int_0^{\ln 2}\left(\frac{7e^x}{e^x+3e^{-x}}\right)^2 dx - \frac{1}{3}\pi(4)^2(\ln 2) = 6.72 \text{ unit}^3$ (3s.f)
(iv)	$y = \frac{7e^{\frac{1}{3}x}}{e^{\frac{1}{3}x} + 3e^{-\frac{1}{3}x}} - 5$

5(a)	$\int x\tan^{-1}(2x^2)\,dx$ $= \frac{1}{2}x^2\tan^{-1}(2x^2) - \int \frac{2x^3}{1+4x^4}dx$ $= \frac{1}{2}x^2\tan^{-1}(2x^2) - \frac{1}{8}\int \frac{16x^3}{1+4x^4}dx$ $= \frac{1}{2}x^2\tan^{-1}(2x^2) - \frac{1}{8}\ln(1+4x^4) + C$
(b)	$3\int_0^m \frac{1}{\pi\sqrt{1-9x^2}}dx = \frac{3}{3\pi}\int_0^m \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}dx$

	$=\frac{1}{\pi}\left[\sin^{-1}(3x)\right]_0^m=\frac{1}{\pi}\sin^{-1}(3m)$ $\frac{1}{\pi}\sin^{-1}(3m)=\frac{1}{4}$ $\Rightarrow \sin^{-1}(3m)=\frac{\pi}{4}\Rightarrow 3m=\frac{\sqrt{2}}{2}$ $\therefore m=\frac{\sqrt{2}}{6}$
(c)	$5k^2-3x^2=2x^2\Rightarrow x=\pm k$ Volume $=2\pi\int_0^k\left(5k^2-3x^2\right)^2-\left(2x^2\right)^2\,dx$ $=2\pi\int_0^k\left(25k^4-30k^2x^2+9x^4\right)-4x^4\,dx$ $=2\pi\int_0^k\left(25k^4-30k^2x^2+5x^4\right)dx$ $=2\pi\left[25k^4x-10k^2x^3+x^5\right]_0^k$ $=2\pi\left(25k^5-10k^5+k^5\right)$ $=32\pi k^5$

6(a)	$x^2+(y-a)^2=a^2$ $\Rightarrow x^2=a^2-(y-a)^2$ Volume formed $=\pi\int_0^{2a}\left[a^2-(y-a)^2\right]dy$ $=\pi\int_0^{2a}\left[a^2-\left(y^2-2ay+a^2\right)\right]dy$ OR $=\pi\left[a^2y-\frac{1}{3}(y-a)^3\right]_0^{2a}$ $=\pi\int_0^{2a}\left[a^2-\left(y^2-2ay+a^2\right)\right]dy$ $\quad=\pi\left[\left(2a^3-\frac{1}{3}a^3\right)-\left(0-\frac{1}{3}\left(-a^3\right)\right)\right]$ $=\pi\int_0^{2a}\left[-y^2+2ay\right]dy$ $\quad=\frac{4}{3}\pi a^3$ $=\pi\left[-\frac{y^3}{3}+ay^2\right]_0^{2a}$ $=\pi\left[-\frac{8a^3}{3}+4a^3\right]$ $=\frac{4}{3}\pi a^3$

	Volume of sphere with radius a is $=\frac{4}{3}\pi a^3$ Therefore volume of the semi-circle obtained when rotated 2π radian about the y-axis is equal to the volume of a sphere with radius a
(b)	When $x=\ln\left(\frac{\sqrt{3}}{2}\right) \Rightarrow t=2$ $x=\ln\left(\frac{\sqrt{24}}{5}\right) \Rightarrow t=5$ $x=\ln\frac{\left(t^2-1\right)^{\frac{1}{2}}}{t}=\frac{1}{2}\ln\left(t^2-1\right)-\ln t$ $\frac{dx}{dt}=\frac{1}{2}\left(\frac{2t}{t^2-1}\right)-\frac{1}{t}$ $\frac{dx}{dt}=\frac{1}{t\left(t^2-1\right)}$ Area of required region $=\int_{\ln\frac{\sqrt{3}}{2}}^{\ln\frac{\sqrt{24}}{5}} y\,dx$ $=\int_2^5 y\frac{dx}{dt}\,dt$ $=\int_2^5 t\left(5t^2-8\right)\times\frac{1}{t\left(t^2-1\right)}\,dt$ $=\int_2^5 \frac{5t^2-8}{\left(t^2-1\right)}\,dt$ $=\int_2^5 5-\frac{3}{\left(t^2-1\right)}\,dt$ $=\left[5t-\frac{3}{2}\ln\left(\frac{t-1}{t+1}\right)\right]_2^5$ $=\left[25-\frac{3}{2}\ln\frac{4}{6}\right]-\left[10-\frac{3}{2}\ln\frac{1}{3}\right]$ $=15+\frac{3}{2}\ln\frac{1}{2}$ OR $=15-\frac{3}{2}\ln 2$

7	Total area of four rectangles $=\frac{1}{4}\left[\frac{2}{1+5/4}+\frac{2}{1+6/4}+\frac{2}{1+7/4}+\frac{2}{1+2}\right]$ $=\frac{1}{4}\left[\frac{2}{1+5/4}+\frac{2}{1+6/4}+\frac{2}{1+7/4}+\frac{2}{1+8/4}\right]$

$$=\frac{1}{4}\left[\frac{2(4)}{9}+\frac{2(4)}{10}+\frac{2(4)}{11}+\frac{2(4)}{12}\right]$$

$$=\frac{2}{9}+\frac{2}{10}+\frac{2}{11}+\frac{2}{12}=\sum_{r=1}^{4}\frac{2}{8+r}$$

Total area of n rectangles $=\frac{1}{n}\left[\frac{2}{1+\left(1+\frac{1}{n}\right)}+\frac{2}{1+\left(1+\frac{2}{n}\right)}+\ldots+\frac{2}{1+\left(1+\frac{n-1}{n}\right)}+\frac{2}{1+2}\right]$

$$=\frac{1}{n}\left[\frac{2}{1+\left(\frac{n+1}{n}\right)}+\frac{2}{1+\left(\frac{n+2}{n}\right)}+\ldots+\frac{2}{1+\left(\frac{2n-1}{n}\right)}+\frac{2}{1+\left(\frac{2n}{n}\right)}\right]$$

$$=\frac{1}{n}\left[\frac{2n}{2n+1}+\frac{2n}{2n+2}+\ldots+\frac{2n}{2n+n-1}+\frac{2n}{2n+n}\right]$$

$$=\frac{2}{2n+1}+\frac{2}{2n+2}+\ldots+\frac{2}{2n+n-1}+\frac{2}{2n+n}$$

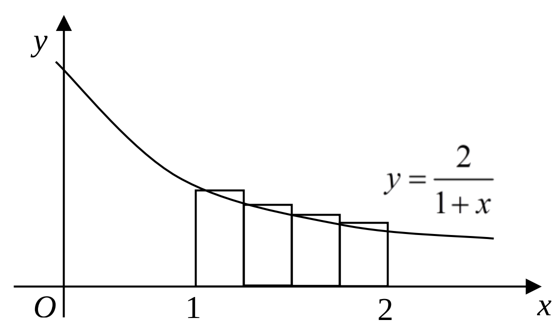
$$=\sum_{r=1}^{n}\frac{2}{2n+r}$$

Area under graph $=\int_1^2\frac{2}{1+x}\,dx=\left[2\ln|1+x|\right]_1^2=2\ln 3-2\ln 2=2\ln\frac{3}{2}$

Since Sum of Area of Rectangles < Area under graph

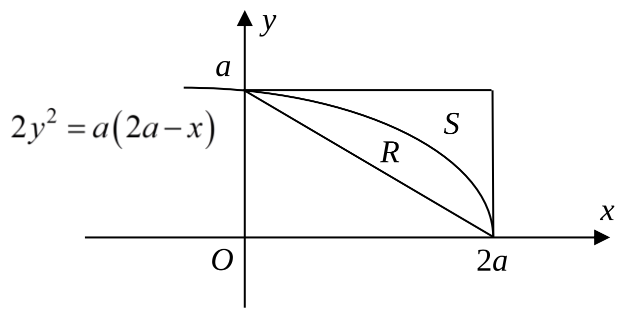
$$\Rightarrow\sum_{r=1}^{n}\frac{2}{2n+r}<2\ln\left(\frac{3}{2}\right) \text{ ---------- (1)}$$

Consider rectangles as seen in the diagram,
Total area of n rectangles



$$=\frac{1}{n}\left[\frac{2}{1+1}+\frac{2}{1+\left(1+\frac{1}{n}\right)}+\ldots+\frac{2}{1+\left(1+\frac{n-1}{n}\right)}\right]=\frac{1}{n}\left[\frac{2n}{2n}+\frac{2n}{2n+1}+\ldots+\frac{2n}{2n+n-1}\right]$$

$$=\sum_{r=1}^{n}\frac{2}{2n+r-1}$$

	Since Sum of Area of Rectangles > Area under graph $$\Rightarrow \sum_{r=1}^{n} \frac{2}{2n+r-1} > 2\ln\left(\frac{3}{2}\right) \text{ -------- (2)}$$ Considering (1) and (2), $$\sum_{r=1}^{n} \frac{2}{2n+r} < 2\ln\left(\frac{3}{2}\right) < \sum_{r=1}^{n} \frac{2}{2n+r-1} \text{. (deduced)}$$
8	(i)  (ii) When S is rotated completely about the x-axis, $$\text{Required volume} = \pi a^2(2a) - \pi\int_0^{2a} \frac{a}{2}(2a-x)\,dx$$ $$= 2\pi a^3 - \frac{\pi a}{2}\left[\frac{(2a-x)^2}{-2}\right]_0^{2a}$$ $$= 2\pi a^3 - \frac{\pi a}{2}(2a^2)$$ $$= \pi a^3 \text{ cu. units}$$ (iii) After a translation of $2a$ units in the negative x-direction, New equation is $2y^2 = a(2a-(x+2a)) \Rightarrow x = -\frac{2y^2}{a}$ When R is rotated completely about the line $x = 2a$, $$\text{Required volume} = \frac{1}{3}\pi(2a)^2(a) - \pi\int_0^{a}\left(-\frac{2y^2}{a}\right)^2 dy$$ $$= \frac{4}{3}\pi a^3 - \pi\left[\frac{4y^5}{5a^2}\right]_0^{a}$$ $$= \frac{4}{3}\pi a^3 - \frac{4}{5}\pi a^3 = \frac{8}{15}\pi a^3 \text{ cu. units}$$

9

(i) $t=\tan x \Rightarrow \dfrac{dt}{dx}=\sec^2 x=1+t^2$

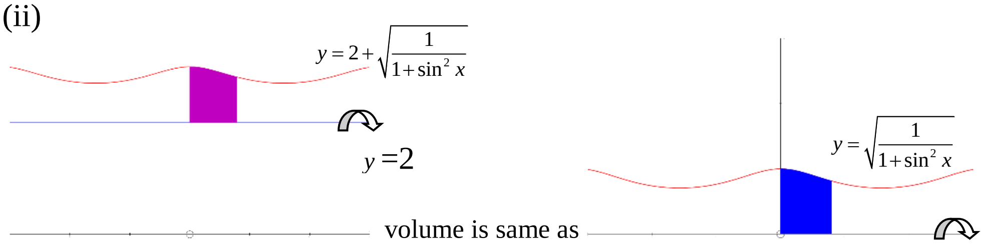
$$=\int \frac{1}{1+\frac{t^2}{1+t^2}}\left(\frac{1}{1+t^2}\right)dt$$

$$=\int \frac{1}{1+2t^2}dt$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}t+c$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}\tan x+c$$

(ii)



Exact volume = $\pi\int_0^{\frac{\pi}{4}}\left(\sqrt{\dfrac{1}{1+\sin^2 x}}\right)^2 \mathrm{d}x$

$$=\pi\left[\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}\tan x\right]_0^{\frac{\pi}{4}}$$

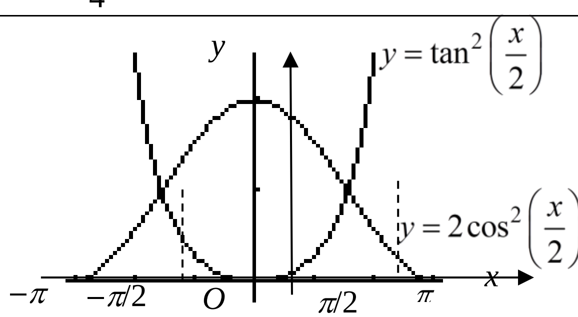
$$=\frac{\pi}{\sqrt{2}}\tan^{-1}\sqrt{2}$$

10(i)

$$\tan^2\frac{\pi}{4}-2\cos^2\frac{\pi}{4}=1-2\left(\frac{1}{\sqrt{2}}\right)^2=0$$

$\therefore\ \theta=\dfrac{\pi}{4}$ is a root of the equation.

(ii)



	$\tan^2\left(\frac{x}{2}\right) > 2\cos^2\left(\frac{x}{2}\right) \Rightarrow -\pi < x < -\frac{\pi}{2}$ or $\frac{\pi}{2} < x < \pi$
(iii)	$\int_0^{\frac{2\pi}{3}} \left\lvert \tan^2\left(\frac{x}{2}\right) - 2\cos^2\left(\frac{x}{2}\right) \right\rvert \, dx$ $= -\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{x}{2}\right) - 2\cos^2\left(\frac{x}{2}\right) \, dx + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \tan^2\left(\frac{x}{2}\right) - 2\cos^2\left(\frac{x}{2}\right) \, dx$ $= -\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{x}{2}\right) - 1 - [1+\cos x] \, dx + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sec^2\left(\frac{x}{2}\right) - 1 - [1+\cos x] \, dx$ $= -\left[2\tan\frac{x}{2} - 2x - \sin x\right]_0^{\frac{\pi}{2}} + \left[2\tan\frac{x}{2} - 2x - \sin x\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$ $= -(2-\pi-1) + \left[\left(2\sqrt{3} - \frac{4\pi}{3} - \frac{\sqrt{3}}{2}\right) - (2-\pi-1)\right]$ $= \frac{3\sqrt{3}}{2} + \frac{2\pi}{3} - 2$

11	(i) $\int \frac{x^4}{1+x^2} \, dx = \int \frac{x^4}{1+x^2} \, dx$ $= \int (x^2 - 1 + \frac{1}{1+x^2}) \, dx$ $= \frac{1}{3}x^3 - x + \tan^{-1} x + C$ (ii) Let $u = \tan x$ $\frac{du}{dx} = \sec^2 x = \tan^2 x + 1 = u^2 + 1$ $dx = \frac{1}{u^2+1} du$ When $x = \frac{\pi}{4}, u = \tan\frac{\pi}{4} = 1$ $x = 0, u = \tan 0 = 0$

$$\int_0^{\frac{\pi}{4}} \tan^4 x \, dx = \int_0^1 \frac{u^4}{1+u^2} \, du$$

$$= [\frac{1}{3}u^3 - u + \tan^{-1} u]_0^1$$

$$= \frac{1}{3} - 1 + \tan^{-1} 1 = \frac{\pi}{4} - \frac{2}{3}$$

(iii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^4 x \, dx = 2\int_0^{\frac{\pi}{4}} \tan^4 x \, dx = 2(\frac{\pi}{4} - \frac{2}{3}) = \frac{\pi}{2} - \frac{4}{3}$

A parametric $y = \tan\theta, x = \sec^2\theta$, where $0 \le \theta \le 2\pi$.

(iv) When $y = 1, \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, x = 2$

Area of region $R = 2\int_1^2 y \, dx$

$$= 2\int_0^{\frac{\pi}{4}} \tan\theta \cdot 2\sec^2\theta \tan\theta \, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \tan^2\theta \sec^2\theta \, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \tan^2\theta(\tan^2\theta + 1) \, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} (\tan^4\theta + \tan^2\theta) \, d\theta \quad \text{(shown)}$$

$$= 4\int_0^{\frac{\pi}{4}} (\tan^4\theta) \, d\theta + 4\int_0^{\frac{\pi}{4}} (\tan^2\theta) \, d\theta$$

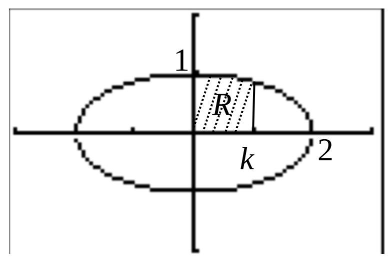
$$= 4(\frac{\pi}{4} - \frac{2}{3}) + 4\int_0^{\frac{\pi}{4}} (\sec^2\theta - 1) \, d\theta$$

$$= \pi - \frac{8}{3} + 4[\tan\theta - \theta]_0^{\frac{\pi}{4}}$$

$$= \pi - \frac{8}{3} + 4[1 - \frac{\pi}{4}] = \frac{4}{3}$$

(v) $V_y = \pi(2)^2 2 - 2\pi\int_0^1 x^2 \, dy$

$$= 8\pi - 2\pi\int_0^{\frac{\pi}{4}} (\sec^2\theta)^2 \cdot \sec^2\theta \, d\theta = 13.4$$

12	**(i)**
	$$x\left[\frac{1}{2}\frac{-2x}{\sqrt{4-x^2}}\right]+\sqrt{4-x^2}+4\left(\frac{1}{2}\right)\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$
	$$=\frac{-x^2+4-x^2}{\sqrt{4-x^2}}+\frac{4}{\sqrt{4-x^2}}$$
	$$=2\sqrt{4-x^2}$$
	(ii)
	$$\frac{1}{2}\int_0^k \sqrt{4-x^2}\,\mathrm{d}x=\frac{1}{2}\left[\frac{1}{2}\left(x\sqrt{4-x^2}+4\sin^{-1}\left(\frac{x}{2}\right)\right)\right]_0^k$$
	$$=\frac{1}{4}\left[k\sqrt{4-k^2}+4\sin^{-1}\left(\frac{k}{2}\right)\right] \quad \Rightarrow \quad a=\frac{k}{4}$$
	(iii)
	$$4y^2+x^2=4$$
	$$y^2+\frac{x^2}{2^2}=1$$
	$$R=\frac{1}{2}\int_0^k \sqrt{4-x^2}\,\mathrm{d}x$$
	
	(iv)
	Required area $=4R$ with $k=1$
	$$=\sqrt{3}+4\sin^{-1}\left(\frac{1}{2}\right)$$
	$$=\sqrt{3}+4\left(\frac{\pi}{6}\right)$$
	$$=\sqrt{3}+\frac{2\pi}{3}$$

13	(a)
	$$x=2\sec\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta}=2\sec\theta\tan\theta$$
	$$\int \frac{1}{x^2\sqrt{x^2-4}}\,\mathrm{d}x=\int \frac{1}{4\sec^2\theta\sqrt{4(\sec^2\theta-1)}}\,2\sec\theta\tan\theta\,\mathrm{d}\theta$$

$$= \int \frac{1}{2\sec\theta\sqrt{4\tan^2\theta}} \tan\theta \, \mathrm{d}\theta$$

$$= \int \frac{1}{2\sec\theta\left(2\tan\theta\right)} \tan\theta \, \mathrm{d}\theta$$

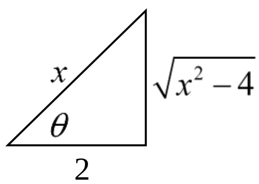
$$= \int \frac{1}{4\sec\theta} \, \mathrm{d}\theta$$

$$= \frac{1}{4}\int \cos\theta \, \mathrm{d}\theta$$

$$= \frac{1}{4}\sin\theta + C$$

$$= \frac{\sqrt{x^2-4}}{4x} + C$$

Note: $x = 2\sec\theta \Rightarrow \cos\theta = \dfrac{2}{x}$



(b) $V = \pi \int_{-\sqrt{\frac{3}{2}}}^{-\frac{1}{\sqrt{2}}} \left(\frac{1}{\sqrt{1+2x^2}}\right)^2 \mathrm{d}x$

$$= \pi \int_{-\sqrt{\frac{3}{2}}}^{-\frac{1}{\sqrt{2}}} \frac{1}{1+2x^2} \, \mathrm{d}x$$

$$= \frac{\pi}{2} \int_{-\sqrt{\frac{3}{2}}}^{-\frac{1}{\sqrt{2}}} \frac{1}{\frac{1}{2}+x^2} \, \mathrm{d}x$$

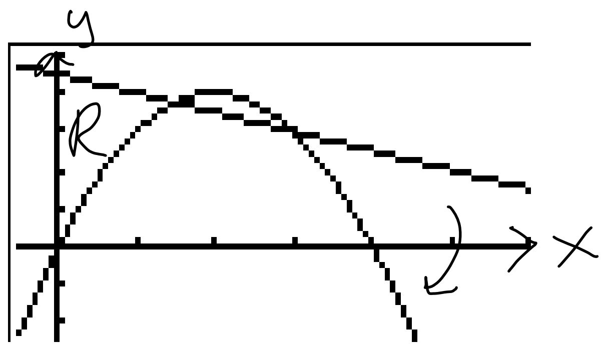
$$= \frac{\pi}{2}\left[\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)\right]_{-\sqrt{\frac{3}{2}}}^{-\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{2}\left[\sqrt{2}\tan^{-1}\left(\sqrt{2}x\right)\right]_{-\sqrt{\frac{3}{2}}}^{-\frac{1}{\sqrt{2}}}$$

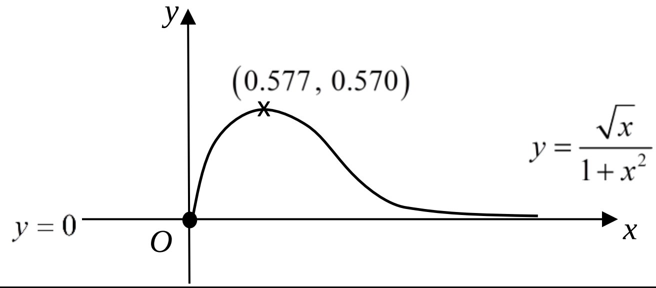
$$= \frac{\pi}{2}\left[\sqrt{2}\tan^{-1}(-1) - \sqrt{2}\tan^{-1}\left(-\sqrt{3}\right)\right]$$

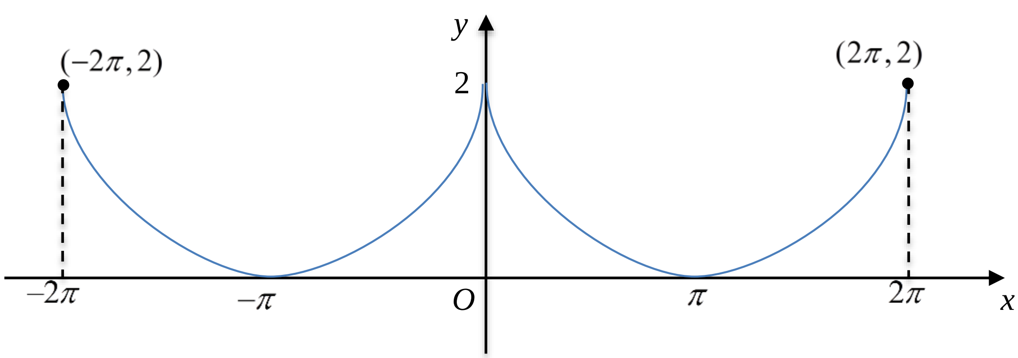
$$= \frac{\pi}{2}\left[\sqrt{2}\left(-\frac{\pi}{4}\right) - \sqrt{2}\left(-\frac{\pi}{3}\right)\right]$$

	$=\frac{\sqrt{2}}{2}\pi^2\left[-\frac{1}{4}+\frac{1}{3}\right]$ $=\frac{\sqrt{2}}{24}\pi^2$

14	 To find point of intersection: $y=4x-x^2 ---(1)$ $2y=9-x----(2)$ Solving (1) & (2) by G.C. $x=\frac{3}{2}$ or $x=3$ (NA) Volume of R about x-axis $=\pi\int_0^{\frac{3}{2}}\left(\frac{9}{2}-\frac{x}{2}\right)^2 \mathrm{d}x-\pi\int_0^{\frac{3}{2}}\left(4x-x^2\right)^2 \mathrm{d}x=50.89 \text{ units}^3$

15 (a)	$\int_0^{\frac{1}{p}}\frac{1}{1+p^2x^2}\,\mathrm{d}x=\int_1^{e^2}\ln x\,\mathrm{d}x$ $\left[\frac{1}{p}\tan^{-1}(px)\right]_0^{\frac{1}{p}}=\left[x\ln x\right]_1^{e^2}-\int_1^{e^2}\left(\frac{1}{x}\right)(x)\,\mathrm{d}x$ $\frac{1}{p}\left[\tan^{-1}1-\tan^{-1}0\right]=e^2\ln e^2-\left[x\right]_1^{e^2}$ $\frac{1}{p}\left(\frac{\pi}{4}\right)=2e^2-e^2+1$ $\frac{\pi}{4p}=e^2+1$ $p=\frac{\pi}{4\left(e^2+1\right)}$

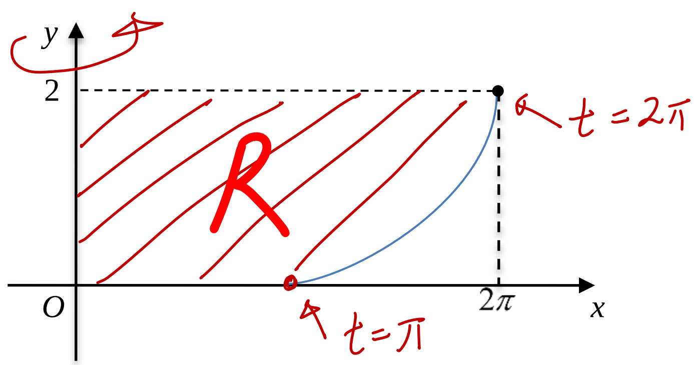
(b)(i)	
(b)(ii)	$x=0 \Rightarrow u=0$ $x=n \Rightarrow u=n^2$ $u=x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=2x$ $\int_0^n \frac{x}{\left(1+x^2\right)^2}\,\mathrm{d}x$ $=\int_0^{n^2} \frac{1}{(1+u)^2}\left(\frac{1}{2}\right)\mathrm{d}u$ $=\frac{1}{2}\int_0^{n^2} (1+u)^{-2}\,\mathrm{d}u$ $=\frac{1}{2}\left[-(1+u)^{-1}\right]_0^{n^2}$ $=\frac{1}{2}\left(1-\frac{1}{1+n^2}\right)$
(b)(iii)	Volume of the revolution $=\lim_{n\to\infty}\left[\pi\int_0^n \frac{x}{\left(1+x^2\right)^2}\,\mathrm{d}x\right]$ $=\frac{1}{2}\pi(1-0)=\frac{1}{2}\pi$ units^3

16	(i)  (ii) By symmetry

$$\text{Area} = 4\int_0^{\pi} y\, dx$$
$$= 4\int_0^{\pi} (1+\cos\theta)(1-\cos\theta)\, d\theta$$
$$= 4\int_0^{\pi} (1-\cos^2\theta)\, d\theta$$
$$= 4\int_0^{\pi} \sin^2\theta\, d\theta$$
$$= 2\int_0^{\pi} (1-\cos 2\theta)\, d\theta$$
$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$
$$= 2\pi$$

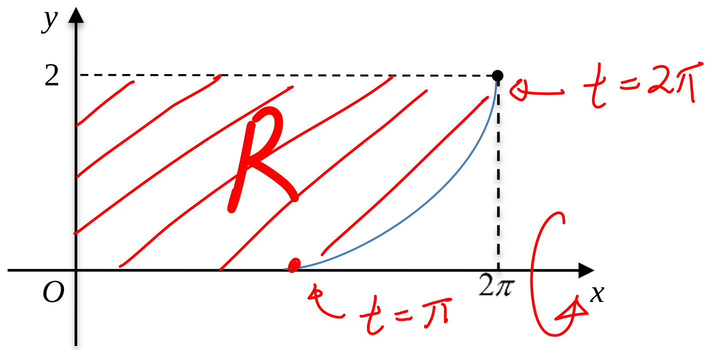
(iii) The area is $4\pi - \frac{\pi}{2}$ units^2.

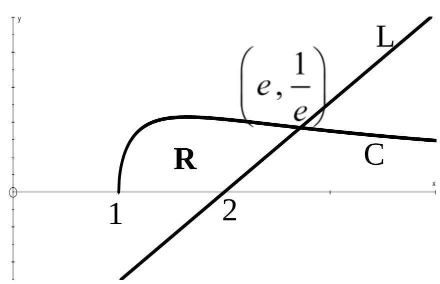
(iv) We have



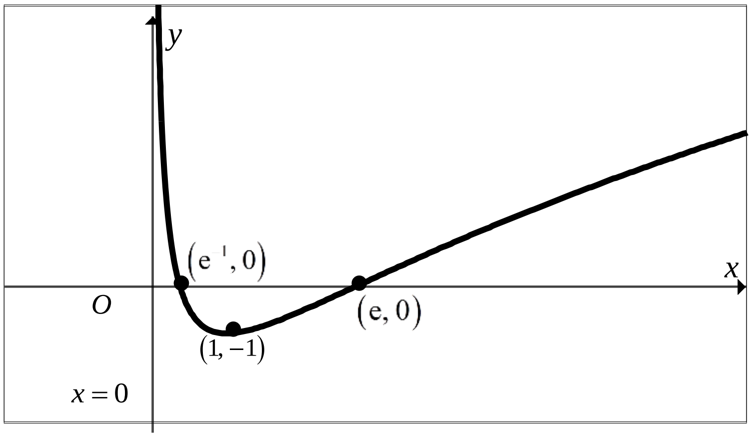
$$\text{Volume} = \pi\int_0^{2} x^2\, dy$$
$$= \pi\int_{\pi}^{2\pi} (\theta - \sin\theta)^2(-\sin\theta)\, d\theta$$
$$= 193.2$$

(v) We have

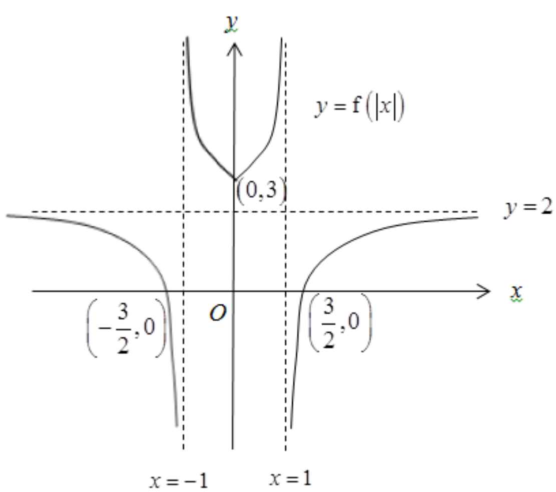
	 $\text{Volume} = \pi(2^2)(2\pi) - \pi\int_{\pi}^{2\pi} y^2\, dx$ $= 8\pi^2 - \pi\int_{\pi}^{2\pi} (1+\cos\theta)^2(1-\cos\theta)\, d\theta$ $= 8\pi^2 - \pi\int_{\pi}^{2\pi} \sin^2\theta(1+\cos\theta)\, d\theta$ $= 74.02$

17a	$\int \frac{6+2x}{\sqrt{1-4x-x^2}}\, \mathrm{d}x = \int \frac{2-(-4-2x)}{\sqrt{1-4x-x^2}}\, \mathrm{d}x$ $= \int \frac{2}{\sqrt{1-4x-x^2}}\, \mathrm{d}x - \int \frac{(-4-2x)}{\sqrt{1-4x-x^2}}\, \mathrm{d}x$ $= \int \frac{2}{\sqrt{5-(x+2)^2}}\, \mathrm{d}x - \int \frac{-4-2x}{\sqrt{1-4x-x^2}}\, \mathrm{d}x$ $= 2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) - 2\sqrt{1-4x-x^2} + c$
b	Point of intersection: $\left(e, \frac{1}{e}\right)$  Volume $= \pi\int_1^e \left(\frac{\sqrt{\ln x}}{x}\right)^2 \mathrm{d}x - \frac{\pi}{3}\left(\frac{1}{e}\right)^2(e-2)$ $= \pi\int_1^e \frac{\ln x}{x^2}\, \mathrm{d}x - \frac{\pi(e-2)}{3e^2}$ $= \pi\left[(\ln x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\frac{1}{x}\, \mathrm{d}x\right]_1^e - \frac{\pi(e-2)}{3e^2}$ $= \pi\left[\left(-\frac{\ln x}{x}\right) - \frac{1}{x}\right]_1^e - \frac{\pi(e-2)}{3e^2}$ $= \pi\left[1 - \frac{2}{e}\right] - \frac{\pi(e-2)}{3e^2}$

	$= \pi - \frac{2\pi}{e} - \frac{\pi}{3e} + \frac{2\pi}{3e^2}$ $= \pi\left(1 - \frac{7}{3e} + \frac{2}{3e^2}\right)$

18(i)	 To obtain x-intercepts, let $y = 0$ $\Rightarrow (\ln x)^2 = 1$ $\Rightarrow \ln x = \pm 1$ $\Rightarrow x = \mathrm{e}^1$ or e^{-1} To obtain the turning point, find $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln x$. Let $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 2\ln x = 0 \Rightarrow x = 1$ Thus coordinates of turning point is $(1, -1)$.
(ii)	Area of region R $= \int_{\mathrm{e}^{-1}}^{\mathrm{e}} -\left((\ln x)^2 - 1\right) \mathrm{d}x$ $= -\left[x(\ln x)^2\right]_{\mathrm{e}^{-1}}^{\mathrm{e}} + \int_{\mathrm{e}^{-1}}^{\mathrm{e}} x \frac{2\ln x}{x} \mathrm{d}x + [x]_{\mathrm{e}^{-1}}^{\mathrm{e}}$ $= -\left(\mathrm{e} - \mathrm{e}^{-1}\right) + 2\left([x\ln x]_{\mathrm{e}^{-1}}^{\mathrm{e}} - \int_{\mathrm{e}^{-1}}^{\mathrm{e}} 1 \, \mathrm{d}x\right) + \left(\mathrm{e} - \mathrm{e}^{-1}\right)$ $= -\left(\mathrm{e} - \mathrm{e}^{-1}\right) + 2\left(\left(\mathrm{e} + \mathrm{e}^{-1}\right) - \left(\mathrm{e} - \mathrm{e}^{-1}\right)\right) + \left(\mathrm{e} - \mathrm{e}^{-1}\right)$ $= 4\mathrm{e}^{-1}$
(iii)	Make x the subject: $y = (\ln x)^2 - 1$ $\Rightarrow \ln x = \pm\sqrt{y+1}$ $\Rightarrow x = \mathrm{e}^{\pm\sqrt{y+1}}$ Thus the volume obtained $= \pi \int_{-1}^{0} \left(\mathrm{e}^{\sqrt{y+1}}\right)^2 - \left(\mathrm{e}^{-\sqrt{y+1}}\right)^2 \mathrm{d}y = 12.2$ (to 3 s.f.)

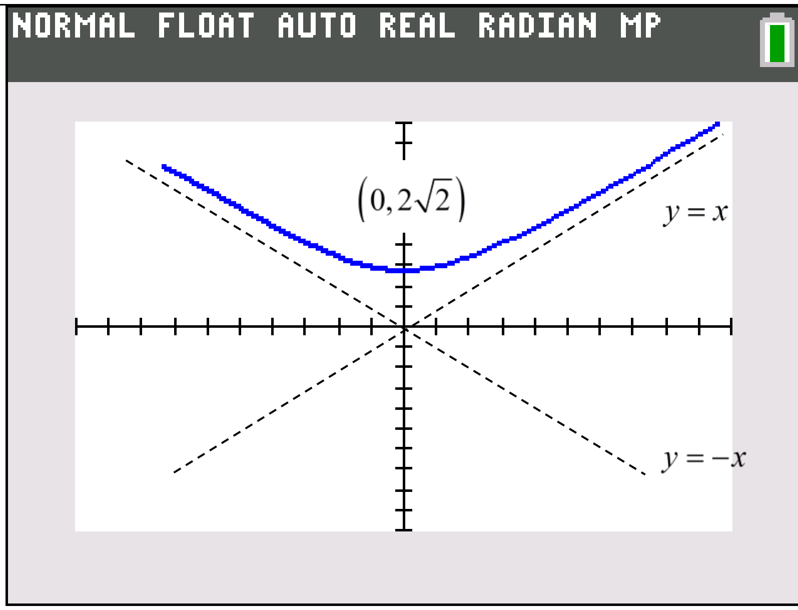
19 (i)	Area of region R $= 5\times\sqrt{3} - \int_0^{\sqrt{3}} \left(\frac{2}{\sqrt{4-x^2}}+3\right) \mathrm{d}x$ $= 1.37$ (to 3 s.f.)
(ii)	Equation of new curve $y = \frac{2}{\sqrt{4-x^2}}+3-5$ $y = \frac{2}{\sqrt{4-x^2}}-2$
(iii)	Volume of revolution $= \pi\int_0^{\sqrt{3}} \left(\frac{2}{\sqrt{4-x^2}}-2\right)^2 \mathrm{d}x$ $= \pi\int_0^{\sqrt{3}} \left(\frac{4}{4-x^2}-\frac{8}{\sqrt{4-x^2}}+4\right) \mathrm{d}x$ $= 4\pi\int_0^{\sqrt{3}} \left(\frac{1}{4-x^2}-\frac{2}{\sqrt{4-x^2}}+1\right) \mathrm{d}x$ (Shown) $= 4\pi\left[\frac{1}{2(2)}\ln\left\|\frac{2+x}{2-x}\right\| - 2\sin^{-1}\frac{x}{2} + x \right]_0^{\sqrt{3}}$ $= 4\pi\left[\frac{1}{4}\ln\left\|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right\| - \frac{2\pi}{3} + \sqrt{3} \right]$

20(i)	

(ii)	Required area $= \int_{\frac{5}{4}}^{2} \mathrm{f}\left(\lvert x\rvert\right)\,\mathrm{d}x$ $= -\int_{\frac{5}{4}}^{\frac{3}{2}} \left(2-\frac{1}{x-1}\right)\,\mathrm{d}x + \int_{\frac{3}{2}}^{2} \left(2-\frac{1}{x-1}\right)\,\mathrm{d}x$ $= -\left[2x-\ln\lvert x-1\rvert\right]_{\frac{5}{4}}^{\frac{3}{2}} + \left[2x-\ln\lvert x-1\rvert\right]_{\frac{3}{2}}^{2}$ $= -\left[\left(3-\ln\frac{1}{2}\right)-\left(\frac{5}{2}-\ln\frac{1}{4}\right)\right]+\left[\left(4-\ln 1\right)-\left(3-\ln\frac{1}{2}\right)\right]$ $= -\left(\frac{1}{2}-\ln 2\right)+\left[1-\ln 2\right]$ $= -\frac{1}{2}+\ln 2+\left[1-\ln 2\right] = \frac{1}{2}$
(iii)	Need to find the equation of the reflected portion of the graph: $y = -\left(2-\frac{1}{x-1}\right)$ $\frac{1}{x-1} = 2+y$ $x = \frac{1}{2+y}+1$ Required vol $= \pi\left(\frac{3}{2}\right)^2(2) - \pi\int_0^2 \left(\frac{1}{2+y}+1\right)^2\,\mathrm{d}y$ $= 2.71$

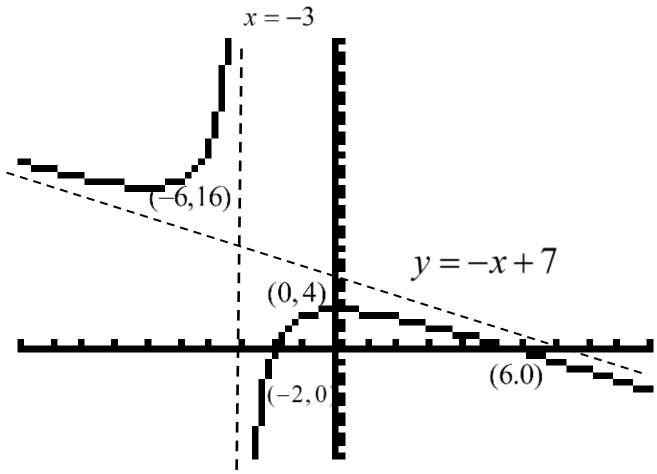
21(a)	C_1 is a circle centred at (0,0) with radius 5. $C_2 : \frac{x^2}{100/a}+\frac{y^2}{100/b}=1$ is an ellipse centred at (0,0) with length of the horizontal axis $2(\frac{10}{\sqrt{a}})$ and vertical axis $2(\frac{10}{\sqrt{b}})$. **Note :** $a<b \Rightarrow$ length of the horizontal axis > length of vertical axis To get 4 points of intersection, we need : $\frac{10}{\sqrt{a}} > 5 \Rightarrow 0<a<4$ and $\frac{10}{\sqrt{b}} < 5 \Rightarrow b>4$ **OR** Compare $C_1 : x^2+y^2=25 \Rightarrow \frac{x^2}{25}+\frac{y^2}{25}=1$ with $C_2 : \frac{x^2}{100/a}+\frac{y^2}{100/b}=1$. For them to intersects at 4 points, $\frac{100}{b}<25$ and $\frac{100}{a}>25$ $b>4$ and $0<a<4$ since $a>0$ is given.

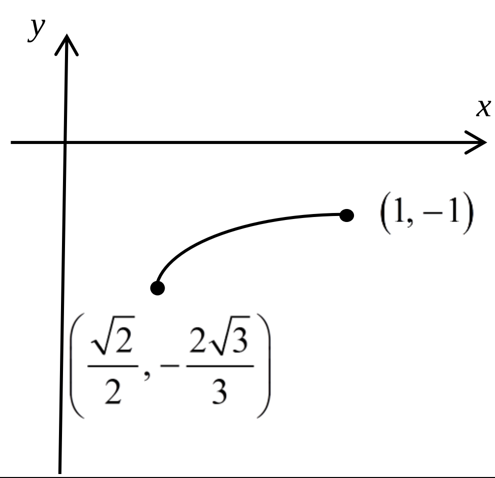
(b)	$C_1: \quad x^2+y^2=25 \quad \Rightarrow \quad y=\pm\sqrt{25-x^2}$ $C_2: \dfrac{x^2}{10^2}+\dfrac{y^2}{\left(10/3\right)^2}=1 \quad \Rightarrow \quad y=\pm\dfrac{\sqrt{100-x^2}}{3}$ C_1 and C_2 intersect at $x=\pm 3.9528$ (5 s.f.) (from GC) Thus area of the required region $=2\left[\int_{-10}^{-3.9528}\dfrac{\sqrt{100-x^2}}{3}\ \mathrm{d}x-\int_{-5}^{-3.9528}\sqrt{25-x^2}\ \mathrm{d}x\right]$ $=22.3$ (3 s.f.) **OR** $C_1: \quad x^2+y^2=25 \quad \Rightarrow \quad x=\pm\sqrt{25-y^2}$ $C_2: x^2+9y^2=100 \quad \Rightarrow \quad x=\pm\sqrt{100-9y^2}$ C_1 and C_2 intersect at $y=\pm 3.0619$ (5 s.f.) (from GC) Thus area of the required region $=2\left[\int_{0}^{3.0619}\sqrt{100-9y^2}\ -\sqrt{25-y^2}\ \mathrm{d}y\right]$ $=22.3$ (3 s.f.)
(c)	$C_1: \quad x^2+y^2=25 \quad \Rightarrow \quad x^2=25-y^2$ $C_2: \dfrac{x^2}{10^2}+\dfrac{y^2}{\left(10/2\right)^2}=1 \quad \Rightarrow \quad x^2=100-4y^2$ $=\pi\int_{-5}^{5}x^2\ \mathrm{d}y-\dfrac{4}{3}\pi(5)^3$ **Note :** $\dfrac{4}{3}\pi(5)^3$ is the volume of sphere $=\pi\int_{-5}^{5}\left(100-4y^2\right)\ \mathrm{d}y-\dfrac{500}{3}\pi$ formed when rotating the circle Required Volume $=\pi\left[100y-\dfrac{4}{3}y^3\right]_{-5}^{5}-\dfrac{500}{3}\pi$ about the $y-$axis. $=\pi\left[\left(500-\dfrac{500}{3}\right)-\left(-500+\dfrac{500}{3}\right)\right]-\dfrac{500}{3}\pi$ $=500\pi$

22(i)	$x^2=\left(2t-\frac{1}{t}\right)^2=4t^2+\frac{1}{t^2}-4$ $y^2=\left(2t+\frac{1}{t}\right)^2=4t^2+\frac{1}{t^2}+4$ $y^2-x^2=8$
(ii)	 Since $y^2=x^2+8$ As $x\to\pm\infty,\ \ y^2\to x^2$ $\therefore y\to\pm x$ $x=0$ $2t-\frac{1}{t}=0$ $t=\frac{1}{\sqrt{2}}$ $y=2t+\frac{1}{t}=\frac{2}{\sqrt{2}}+\sqrt{2}=2\sqrt{2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $2t^2-1=0$ $t=\frac{1}{\sqrt{2}}$ Min point = y intercept = $\left(0,2\sqrt{2}\right)$

(iii)	$\frac{dx}{dt} = 2 + \frac{1}{t^2}$; $\frac{dy}{dt} = 2 - \frac{1}{t^2}$ $\frac{dy}{dx} = \frac{2 - \frac{1}{t^2}}{2 + \frac{1}{t^2}} = \frac{2t^2 - 1}{2t^2 + 1}$
(iv)	Equation of tangent at P: $y - \left(2p + \frac{1}{p}\right) = \frac{2p^2 - 1}{2p^2 + 1}\left(x - \left(2p - \frac{1}{p}\right)\right)$ substitute $x = 0, y = 1$ $1 - \left(2p + \frac{1}{p}\right) = \frac{2p^2 - 1}{2p^2 + 1}\left(0 - \left(2p - \frac{1}{p}\right)\right)$ $-1 + 2p + \frac{1}{p} = \frac{2p^2 - 1}{2p^2 + 1}\left(\frac{2p^2 - 1}{p}\right)$ $\left(-p + 2p^2 + 1\right)\left(2p^2 + 1\right) = \left(2p^2 - 1\right)^2$ $-2p^3 - p + 4p^4 + 2p^2 + 2p^2 + 1 = 4p^4 - 4p^2 + 1$ $2p^3 - 8p^2 + p = 0$ $p(2p^2 - 8p + 1) = 0$ $p = 0$ (reject as p>0), $2p^2 - 8p + 1 = 0$ $p = \frac{8 \pm \sqrt{56}}{4} = 2 + \frac{\sqrt{14}}{2}$ or $2 - \frac{\sqrt{14}}{2}$ (reject since the point P is in the first quadrant) x-coordinate of the point P = $2\left(2 + \frac{\sqrt{14}}{2}\right) - \frac{1}{2 + \frac{\sqrt{14}}{2}} = 4 + \sqrt{14} - \frac{2}{4 + \sqrt{14}} = 2\sqrt{14}$ Required area $= \int_0^{2\sqrt{14}} y \, dx = \int_{\frac{1}{\sqrt{2}}}^{2 + \frac{\sqrt{14}}{2}} \left(2t + \frac{1}{t}\right) \frac{dx}{dt} dt = \int_{\frac{1}{\sqrt{2}}}^{2 + \frac{\sqrt{14}}{2}} \left(2t + \frac{1}{t}\right)\left(2 + \frac{1}{t^2}\right) dt$ $= 36.7$ units^2 (correct to 3 s.f.)

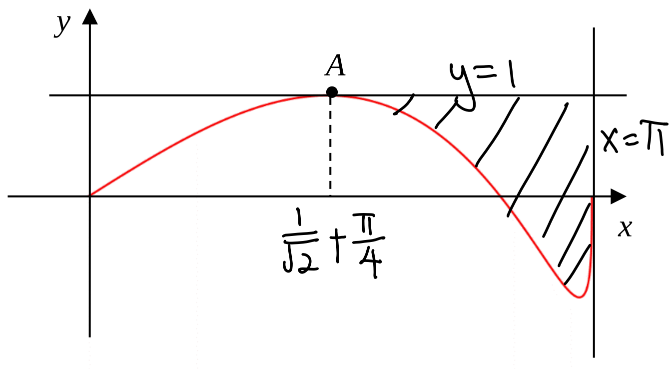
23(i)	$y = \frac{-x^2 + 4x + 12}{x + 3}$ $x^2 + \left(y - 4\right)x + 3y - 12 = 0$ When C does not exist, there is no real x. Discriminant < 0

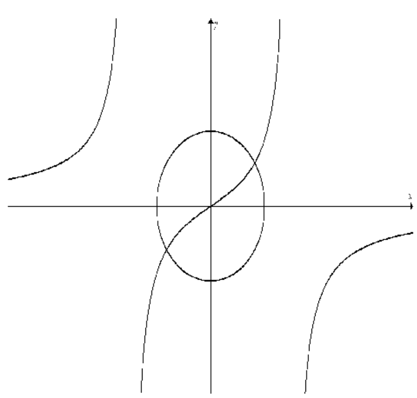
	$(y-4)^2-4(1)(3y-12)<0$ $y^2-8y+16-12y+48<0$ $y^2-20y+64<0$ $(y-16)(y-4)<0$ $4<y<16$
(ii)	$y=\frac{-x^2+4x+12}{x+3}=-x+7-\frac{9}{x+3}$  Asymptotes: $y=-x+7$, $x=-3$
(iii)	Volume generated $=\pi(4)^2(2)-\pi\int_{-2}^{0}\left(\frac{-x^2+4x+12}{x+3}\right)^2\,dx$ = 34.8 units^3 (to 3 s.f)

24	**(i)** 
	(ii) Required area $=-\int_{\frac{\sqrt{2}}{2}}^{1} y\,dx=-\int_{\frac{5\pi}{12}}^{\frac{\pi}{2}}\sec 2\theta(\cos\theta+\sin\theta)\,d\theta$

	$= -\int_{\frac{5\pi}{12}}^{\frac{\pi}{2}} \frac{\cos\theta + \sin\theta}{\cos^2\theta - \sin^2\theta}\, \mathrm{d}\theta$ $= -\int_{\frac{5\pi}{12}}^{\frac{\pi}{2}} \frac{\cos\theta + \sin\theta}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}\, \mathrm{d}\theta$ $= \int_{\frac{5\pi}{12}}^{\frac{\pi}{2}} \frac{1}{\sin\theta - \cos\theta}\, \mathrm{d}\theta$ [Shown]
	(iii) $\text{RHS} = \sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$ $= \sqrt{2}\left(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)$ $= \sin\theta - \cos\theta$ $= \text{LHS}$
	(iv) Hence required area $= \int_{\frac{5\pi}{12}}^{\frac{\pi}{2}} \frac{1}{\sin\theta - \cos\theta}\, \mathrm{d}\theta$ $= \int_{\frac{5\pi}{12}}^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)}\, \mathrm{d}\theta$ $= \frac{\sqrt{2}}{2}\int_{\frac{5\pi}{12}}^{\frac{\pi}{2}} \operatorname{cosec}\left(\theta - \frac{\pi}{4}\right)\, \mathrm{d}\theta$ $= \frac{\sqrt{2}}{2}\left[-\ln\left(\operatorname{cosec}\left(\theta - \frac{\pi}{4}\right) + \cot\left(\theta - \frac{\pi}{4}\right)\right)\right]_{\frac{5\pi}{12}}^{\frac{\pi}{2}}$ $= \frac{\sqrt{2}}{2}\left[-\ln\left(\operatorname{cosec}\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right)\right) + \ln\left(\operatorname{cosec}\left(\frac{\pi}{6}\right) + \cot\left(\frac{\pi}{6}\right)\right)\right]$ $= \frac{\sqrt{2}}{2}\left[-\ln\left(\sqrt{2}+1\right) + \ln\left(2+\sqrt{3}\right)\right] = \frac{\sqrt{2}}{2}\ln\left[\frac{\left(2+\sqrt{3}\right)\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}\right]$ $= \frac{\sqrt{2}}{2}\ln\left(\sqrt{2}-1\right)\left(2+\sqrt{3}\right)$

25 (a)	**Method 1:** $\int \sin 2x \cos x \, \mathrm{d}x$ $= \frac{1}{2}\int \sin 3x + \sin x \, \mathrm{d}x = \frac{1}{2}\left(-\frac{\cos 3x}{3} - \cos x\right) + C = -\frac{1}{2}\left(\frac{\cos 3x}{3} + \cos x\right) + C$ **Method 2:** $\int \sin 2x \cos x \, \mathrm{d}x$ $= \int 2\sin x \cos^2 x \, \mathrm{d}x$ [use $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n \mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c$] $= -\frac{2}{3}\cos^3 x + C$
(b)	**(i)** $\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t + 1, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2\cos 2t$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{2\cos 2t}{\cos t + 1}$ When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$ $\cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow x = \frac{1}{\sqrt{2}} + \frac{\pi}{4}, \quad \frac{1}{\sqrt{2}} + \frac{3\pi}{4}$ At point A, $x = \frac{1}{\sqrt{2}} + \frac{\pi}{4}, \quad y = 1$ $\therefore y = 1$ is the equation of the tangent to the curve at point A. Or Since $0 \le t \le \pi$, the maximum and minimum values of y (i.e. $y = \sin 2t$) is 1 and –1. The y-coordinate of point A is 1 and since the tangent to this max pt is a horizontal line ($\frac{\mathrm{d}y}{\mathrm{d}x} = 0$,), therefore the equation of the tangent to the curve at point A is y = 1.

	(ii)  Area $= \int_{\frac{1}{\sqrt{2}}+\frac{\pi}{4}}^{\pi} 1-y \, dx$ $= \frac{3\pi}{4} - \frac{1}{\sqrt{2}} - \int_{\frac{\pi}{4}}^{\pi} \sin 2t\left(\cos t+1\right) dt$ $= \frac{3\pi}{4} - \frac{1}{\sqrt{2}} - \int_{\frac{\pi}{4}}^{\pi} \sin 2t \cos t + \sin 2t \, dt$ $= \frac{3\pi}{4} - \frac{1}{\sqrt{2}} - \left[-\frac{1}{2}\left(\frac{\cos 3x}{3}+\cos x\right)\right]_{\frac{\pi}{4}}^{\pi} - \left[-\frac{\cos 2t}{2}\right]_{\frac{\pi}{4}}^{\pi}$ $= \frac{3\pi}{4} - \frac{1}{\sqrt{2}} - \frac{2}{3} - \frac{1}{3\sqrt{2}} + \frac{1}{2}$ $= \frac{3\pi}{4} - \frac{1}{6} - \frac{2\sqrt{2}}{3}$

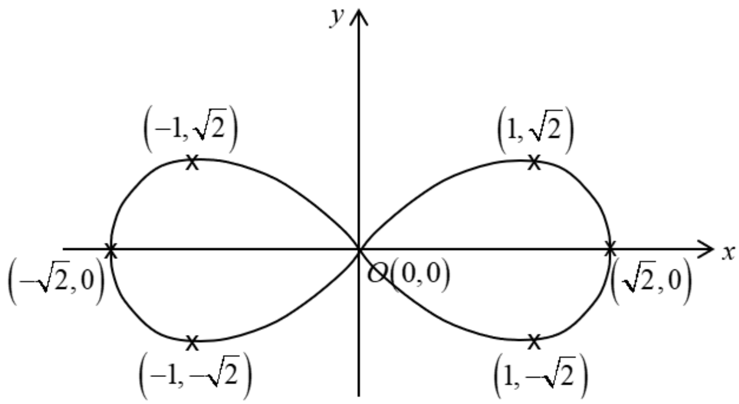
26(i)	$\frac{x^2}{\left(\frac{4}{3}\right)^2} + \frac{y^2}{2^2} = 1$ $9x^2 + 4y^2 = 16$
(ii)	

(iii)	Required volume $= \pi\int_0^{1.08729} \frac{16-9x^2}{4}\,\mathrm{d}x - \pi\int_0^{1.08729}\left(\frac{3x}{4-x^2}\right)^2 \mathrm{d}x = 9.487$ (3 d.p.)

27 (i)	$x=\frac{1}{2}\tan t \Rightarrow \frac{dx}{dt}=\frac{1}{2}\sec^2 t$ Area $R = \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\left(1+4x^2\right)^2}\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{3}} \frac{1}{\left(1+\tan^2 t\right)^2}\left(\frac{1}{2}\sec^2 t\right)\mathrm{d}t$ $=\frac{1}{2}\int_0^{\frac{\pi}{3}} \cos^2 t \ \ \mathrm{d}t$ $=\frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1+\cos 2t}{2}\ \ \mathrm{d}t$ $=\frac{1}{4}\left[t+\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{3}} = \frac{\pi}{12}+\frac{\sqrt{3}}{16}$ units^2
20(ii)	Volume = $\pi\int_{\frac{1}{16}}^{1} \frac{1}{4}\left(\frac{1}{\sqrt{y}}-1\right)\mathrm{d}y + \pi\left(\frac{\sqrt{3}}{2}\right)^2 \frac{1}{16}$ $= \frac{3\pi}{16} = 0.589$ units^3

28 (i)	$x=\sqrt{2}\cos\frac{t}{2} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\sqrt{2}}{2}\sin\frac{t}{2}$ $y=\sqrt{2}\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=\sqrt{2}\cos t$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\cos t}{\sin\frac{t}{2}}$ At $t=\frac{\pi}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\cos\frac{\pi}{2}}{\sin\frac{\pi}{4}} = 0$ (verified) When $t=\frac{\pi}{2}$, $x=\sqrt{2}\cos\left(\frac{\pi}{4}\right)=1$ Equation of normal: $x=1$

(ii)



(iii)

$$\begin{aligned}\text{Area} &= 4\int_0^{\sqrt{2}} y \, \mathrm{d}x \\ &= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t \\ &= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2} \, \mathrm{d}t \\ &= 8\int_0^\pi \sin^2\frac{t}{2}\cos\frac{t}{2} \, \mathrm{d}t \\ &= 8\left[\frac{2}{3}\sin^3\frac{t}{2}\right]_0^\pi \\ &= \frac{16}{3} \text{ units}^2\end{aligned}$$

Alternative Method

$$\begin{aligned}\text{Area} &= 4\int_0^{\sqrt{2}} y \, \mathrm{d}x \\ &= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t \\ &= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2} \, \mathrm{d}t \\ &= -2\int_0^\pi \cos\frac{3t}{2} - \cos\frac{t}{2} \, \mathrm{d}t \\ &= -2\left[\frac{2}{3}\sin\frac{3t}{2} - 2\sin\frac{t}{2}\right]_0^\pi \\ &= \frac{16}{3} \text{ units}^2\end{aligned}$$

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29 (i)

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

(ii)

Total area of rectangles $=\frac{1}{2(2+1)}+\frac{1}{3(3+1)}+\ldots+\frac{1}{n(n+1)}$

$=\sum_{x=2}^{n}\frac{1}{x(x+1)}$ so $a = 2$, $b = n$

$=\sum_{x=2}^{n}\left(\frac{1}{x}-\frac{1}{x+1}\right)$

$=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$=\frac{1}{2}-\frac{1}{n+1}$

Actual area $=\int_1^n \frac{1}{x(x+1)}\,\mathrm{d}x=\int_1^n \frac{1}{x}-\frac{1}{(x+1)}\,\mathrm{d}x$

$=\left[\ln x-\ln(x+1)\right]_1^n$
$=\ln n-\ln(n+1)-\ln 1+\ln 2$
$=\ln n-\ln(n+1)+\ln 2$

Area of rectangles < actual area

$\therefore \quad \frac{1}{2}-\frac{1}{n+1}<\ln n-\ln(n+1)+\ln 2$

$\frac{1}{2}-\ln 2<\frac{1}{n+1}+\ln\left(\frac{n}{n+1}\right)$

$\frac{1}{2}-\ln 2<\frac{1}{n+1}+\ln\left(1-\frac{1}{n+1}\right) \quad (shown)$

Using MF26,

$\ln\left(1-\frac{1}{n+1}\right)=-\frac{1}{n+1}-\frac{1}{2}\left(\frac{1}{n+1}\right)^2-\ldots-\frac{1}{r}\left(\frac{1}{n+1}\right)^r-\ldots$

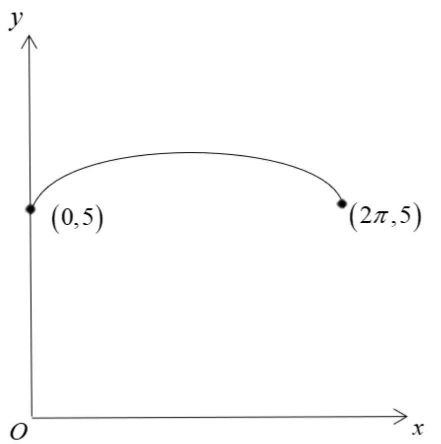
$\therefore \frac{1}{2}-\ln 2<\frac{1}{n+1}-\frac{1}{n+1}-\frac{1}{2}\left(\frac{1}{n+1}\right)^2-\ldots-\frac{1}{r}\left(\frac{1}{n+1}\right)^r-\ldots$

$\therefore \frac{1}{2}-\ln 2<-\frac{1}{2}\frac{1}{(n+1)^2}-\ldots-\frac{1}{r}\frac{1}{(n+1)^r}-\ldots$

$\Rightarrow \frac{1}{2}-\ln 2<\sum_{r=2}^{\infty}\frac{-1}{r(n+1)^r} \quad (shown)$

30. Suggested Solutions
$\int_0^{x_0} \sqrt{9-x^2}\ \mathrm{d}x$ $= \int_0^{\sin^{-1}\frac{x_0}{3}} \sqrt{9-9\sin^2\theta}\ (3\cos\theta\ \mathrm{d}\theta)$ $= \int_0^{\sin^{-1}\frac{x_0}{3}} 3\cos\theta \cdot 3\cos\theta\ \mathrm{d}\theta$ $= 9\int_0^{\sin^{-1}\frac{x_0}{3}} \cos^2\theta\ \mathrm{d}\theta$ $= \frac{9}{2}\int_0^{\sin^{-1}\frac{x_0}{3}} \cos 2\theta + 1\ \mathrm{d}\theta$ $= \frac{9}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\sin^{-1}\frac{x_0}{3}}$ $= \frac{9}{2}\left[\sin\theta\cos\theta + \theta\right]_0^{\sin^{-1}\frac{x_0}{3}}$ $= \frac{9}{2}\left[\frac{x_0}{3}\sqrt{1-\frac{x_0{}^2}{9}} + \sin^{-1}\frac{x_0}{3}\right]$ $= \frac{x_0}{2}\sqrt{9-(x_0)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x_0}{3}\right)$ (shown) $x = 3\sin\theta$ $\mathrm{d}x = 3\cos\theta\ \mathrm{d}\theta$ $x = x_0, 3\sin\theta = x_0$ $\Rightarrow \theta = \sin^{-1}\frac{x_0}{3}$ $x = 0, 3\sin\theta = 0$ $\Rightarrow \theta = 0$
Since $\tan\alpha = \frac{2}{3}$, Equation of line above *x*-axis: $y = \frac{2}{3}x$ --- (1) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ --- (2) Substitute (1) into (2): $\frac{x^2}{9} + \frac{x^2}{9} = 1 \Rightarrow 2x^2 = 9$ $x = \frac{3}{\sqrt{2}}$ (since $x > 0$) , $y = \frac{2}{3} \times \frac{3}{\sqrt{2}} = \sqrt{2}$ $\therefore A\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$ (shown)
$\int_0^{\frac{3}{\sqrt{2}}} \sqrt{9-x^2}\ \mathrm{d}x = \frac{3}{2\sqrt{2}}\sqrt{9-\frac{9}{2}} + \frac{9}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{9}{4} + \frac{9\pi}{8}$ Area of shaded region $= 2\left\{\frac{2}{3}\int_0^{\frac{3}{\sqrt{2}}} \sqrt{9-x^2}\ \mathrm{d}x - \frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)\left(\frac{3}{\sqrt{2}} \cdot \frac{2}{3}\right)\right\}$ $= 2\left\{\frac{2}{3}\left[\frac{9}{4} + \frac{9\pi}{8}\right] - \frac{3}{2}\right\}$ $= 3 + \frac{3\pi}{2} - 3$ $= \frac{3}{2}\pi$ where $k = \frac{3}{2}$.

$\text{Volume} = 2\left[\pi\int_{\sqrt{2}}^{2} 9\left(1-\frac{y^2}{4}\right)\,\mathrm{d}y + \frac{1}{3}\pi\left(\frac{3}{\sqrt{2}}\right)^2\sqrt{2}\right]$ $= 22.1 \text{ unit}^3$ Exact answer: $24\pi\left(1-\frac{1}{\sqrt{2}}\right)$
The smallest possible dimensions of the cylindrical container will be of radius $\frac{3}{\sqrt{2}}$ and height 4 units.

31(i)	
31(ii)	$\int \sin^2 t\left(1-\cos 2t\right)\,\mathrm{d}t$ $=\frac{1}{2}\int\left(1-\cos 2t\right)^2\,\mathrm{d}t$ $=\frac{1}{2}\int 1-2\cos 2t+\cos^2 2t\,\mathrm{d}t$ $=\frac{1}{2}\int 1-2\cos 2t+\frac{1}{2}\left(1+\cos 4t\right)\mathrm{d}t$ $=\frac{1}{2}\left(\frac{3}{2}t-\sin 2t+\frac{1}{8}\sin 4t\right)+C,\ C\in\mathbb{R}$
31(iii)	$x = 2t-\sin 2t$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 2-2\cos 2t$ $\text{Area} = \int_0^{2\pi} y\,\mathrm{d}x$ $=\int_0^{\pi}\left(5+2\sin^2 t\right)\left(2-2\cos 2t\right)\mathrm{d}t$ $=2\int_0^{\pi} 5-5\cos 2t+2\sin^2 t\left(1-\cos 2t\right)\mathrm{d}t$ $=2\left[5t-\frac{5}{2}\sin 2t+\frac{3}{2}t-\sin 2t+\frac{1}{8}\sin 4t\right]_0^{\pi}$ (from part (ii)) $=13\pi \text{ m}^2$

	Alternative method $\text{Area} = 5\times 2\pi + \int_0^{2\pi} y - 5\ \mathrm{d}x$ $= 10\pi + \int_0^{\pi} \left(2\sin^2 t\right)\left(2 - 2\cos 2t\right) \mathrm{d}t$ $= 10\pi + 2\left[\frac{3}{2}t - \sin 2t + \frac{1}{8}\sin 4t\right]_0^{\pi}$ (from part (ii)) $= 13\pi\ \mathrm{m}^2$
31(iv)	$y = 5 + 2\sin^2 t$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 4\sin t\cos t = 2\sin 2t$ Surface area $= \frac{\pi}{4}\int_0^{\pi} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$ $= \frac{\pi}{4}\int_0^{\pi}\left(5 + 2\sin^2 t\right)\sqrt{\left(2 - 2\cos 2t\right)^2 + \left(2\sin 2t\right)^2}\ \mathrm{d}t$ Note that $\sqrt{\left(2 - 2\cos 2t\right)^2 + \left(2\sin 2t\right)^2} = 2\sqrt{1 - 2\cos 2t + \cos^2 2t + \sin^2 2t}$ $= 2\sqrt{2 - 2\cos 2t}$ $= 2\sqrt{2 - 2\left(1 - 2\sin^2 t\right)}$ $= 4\sqrt{\sin^2 t}$ $= 4\sin t$ (since $\sin t \ge 0$ for $0 \le t \le \pi$) Surface area $= \pi\int_0^{\pi}\left(5 + 2\sin^2 t\right)\sin t\ \mathrm{d}t$ $= \pi\int_0^{\pi}\left(7 - 2\cos^2 t\right)\sin t\ \mathrm{d}t$ $= \pi\int_0^{\pi} 7\sin t - 2\sin t\cos^2 t\ \mathrm{d}t$ $= \pi\left[-7\cos t + \frac{2}{3}\cos^3 t\right]_0^{\pi}$ $= \pi\left(7 - \frac{2}{3} - \left(-7 + \frac{2}{3}\right)\right)$ $= \frac{38}{3}\pi\ \ \mathrm{m}^2$

32. ACJC/2022/I/Q4

(a) The continuous function $\mathrm{f}(x)$, where $\mathrm{f}(x) > 0$, is strictly decreasing for $x \geq 1$. Sketch the curve $y = \mathrm{f}(x)$ for $k < x < k+1$, where k is an integer and $k \geq 1$.

By comparing the areas of appropriate rectangles and the area under the curve $y = \mathrm{f}(x)$, show that for any integer $k \geq 1$,

$$\mathrm{f}(k+1) < \int_k^{k+1} \mathrm{f}(x)\,\mathrm{d}x < \mathrm{f}(k).$$ [2]

(b) The region under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 10$, is split into 9 vertical strips of equal width. Use the result in part **(a)** to prove

(i) $\int_1^{10} \frac{1}{x}\,\mathrm{d}x < \sum_{k=1}^{9} \frac{1}{k}$, [1]

(ii) $\sum_{k=1}^{9} \frac{1}{k} < 1 + \int_1^{9} \frac{1}{x}\,\mathrm{d}x$. [2]

Hence show that $\ln 10 < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{9} < 1 + \ln 9$. [1]

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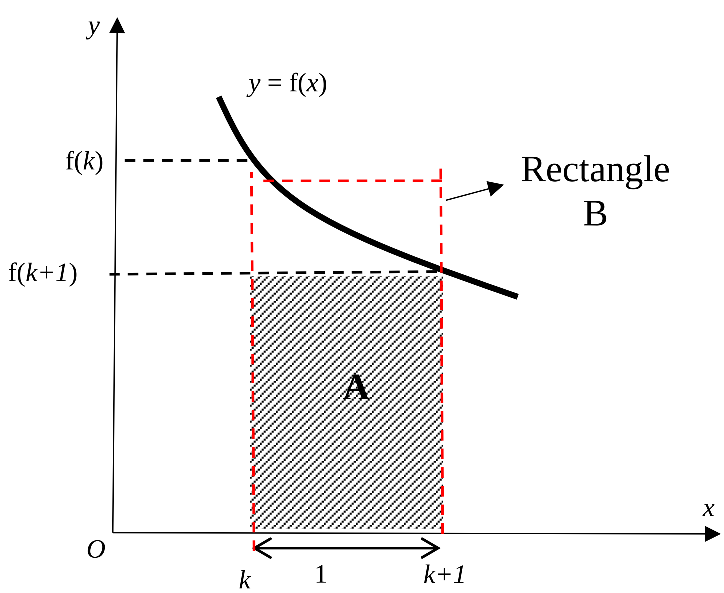
(a)	

	Area under curve = $\int_{k}^{k+1} \mathrm{f}(x)\,\mathrm{d}x$ Area of rectangle A= $\mathrm{f}(k+1)$ x 1 Area of rectangle B= $\mathrm{f}(k)$ x 1 As seen from diagram: $\mathrm{f}(k+1) < \int_{k}^{k+1} \mathrm{f}(x)\,\mathrm{d}x < \mathrm{f}(k)$
(b) (i)	$\int_1^{10} \frac{1}{x}\mathrm{d}x = \int_1^2 \frac{1}{x}\mathrm{d}x + \int_2^3 \frac{1}{x}\mathrm{d}x + \int_3^4 \frac{1}{x}\mathrm{d}x + ... + \int_9^{10} \frac{1}{x}\mathrm{d}x$ $< \mathrm{f(1)+f(2)+f(3)+.....+f(9)}$ $= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{9}$ $= \sum_{k=1}^{9} \frac{1}{k}$
(ii)	$\int_1^{9} \frac{1}{x}\mathrm{d}x = \int_1^2 \frac{1}{x}\mathrm{d}x + \int_2^3 \frac{1}{x}\mathrm{d}x + \int_3^4 \frac{1}{x}\mathrm{d}x + ... + \int_8^{9} \frac{1}{x}\mathrm{d}x$ $> \mathrm{f(2)+f(3)+f(4)+.....+f(9)}$ $= \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{9} = \sum_{k=1}^{9} \frac{1}{k} - 1$ $1 + \int_1^9 \frac{1}{x}\mathrm{d}x > \sum_{k=1}^{9} \frac{1}{k}$ $\therefore \sum_{k=1}^{9} \frac{1}{k} < 1 + \int_1^9 \frac{1}{x}\mathrm{d}x$
	$\int_1^{10} \frac{1}{x}\mathrm{d}x < \sum_{k=1}^{9} \frac{1}{k} < 1 + \int_1^9 \frac{1}{x}\mathrm{d}x$ $[\ln x]_1^{10} < 1 + \frac{1}{2} + \frac{1}{3} + ... \frac{1}{9} < 1 + [\ln x]_1^{9}$ $\ln 10 < 1 + \frac{1}{2} + \frac{1}{3} + ... \frac{1}{9} < 1 + \ln 9$