Raffles Institution
H2 Mathematics
Solution for 2016 A-Level Paper 1

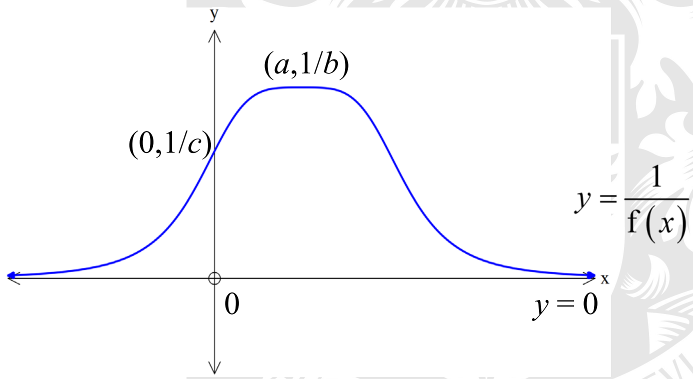
Question 1

No.	Suggested Solution	Remarks for Student
	$\frac{4x^2+4x-14}{x-4}-(x+3)=\frac{4x^2+4x-14-(x+3)(x-4)}{x-4}$ $=\frac{4x^2+4x-14-(x^2-x-12)}{x-4}$ $=\frac{3x^2+5x-2}{x-4}$ $=\frac{(3x-1)(x+2)}{x-4}$ $\frac{4x^2+4x-14}{x-4}<(x+3)$ $\frac{(3x-1)(x+2)}{x-4}<0$ $(3x-1)(x+2)(x-4)<0$ $x<-2$ or $\frac{1}{3}<x<4$	Detailed working needed. We can use GC to check our answers, though question states "without using a calculator".

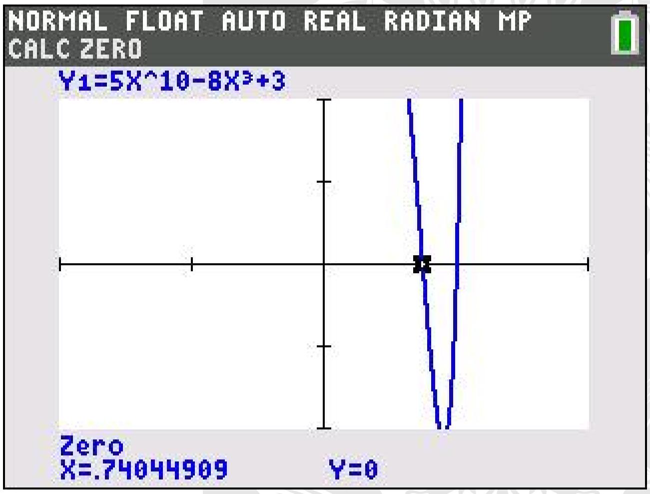
Question 2

No.	Suggested Solution	Remarks for Student
(i)	$y = 2^{\cos x}$ $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right\|_{x=0} = 0$ $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right\|_{x=\frac{\pi}{2}} = -0.6931471 = -0.693$ (3 s.f.)	Can obtain answer from GC directly. No working needed.
(ii)	Tangent at $x = 0$: $y = 2$ Tangent at $x = \dfrac{\pi}{2}$: $\dfrac{y-1}{x-\frac{\pi}{2}} = -0.69315$ $\Rightarrow y = -0.693x + 2.09$(3sf) When $y = 2$, $\dfrac{2-1}{x-\frac{\pi}{2}} = -0.69315 \Rightarrow x = 0.12811$ Required coordinates (0.128, 2).	

Question 3

No.	Suggested Solution	Remarks for Student
	$\mathrm{f}(x)=k(x-l)^4+m$ $\mathrm{f}'(x)=4k(x-l)^3$ $\mathrm{f}'(a)=0 \quad \Rightarrow 4k(a-l)^3=0$ $\Rightarrow l=a$ since $k\neq 0$ Thus $\mathrm{f}(x)=k(x-a)^4+m$ $\mathrm{f}(a)=b \quad \Rightarrow k(a-a)^4+m=b$ $\Rightarrow m=b$ Thus $\mathrm{f}(x)=k(x-a)^4+b$ $\mathrm{f}(0)=c \quad \Rightarrow k(0-a)^4+b=c$ $\Rightarrow k=\dfrac{c-b}{a^4}$	
		

Question 4

No.	Suggested Solution	Remarks for Student
	$a+3d=br^4 \quad ...(1)$ $a+8d=br^7 \quad ...(2)$ $a+11d=br^{14} \quad ...(3)$	
(i)	(3) – (1): $8d=br^{14}-br^4=br^4\left(r^{10}-1\right) \quad ...(4)$ (2) – (1): $5d=br^7-br^4=br^4\left(r^3-1\right) \quad ...(5)$ $\frac{(4)}{(5)}: \ \frac{8}{5}=\frac{r^{10}-1}{r^3-1} \Rightarrow 5r^{10}-5=8r^3-8$ $\Rightarrow 5r^{10}-8r^3+3=0$ Since $\|r\|<1$, using GC, $r=0.74045\approx 0.74$(2 d.p.) 	
(ii)	$\frac{br^n}{1-r}=3.85b\left(0.74\right)^n$	

Question 5

No.	Suggested Solution	Remarks for Student
	$\underset{\sim}{u}=\begin{pmatrix}2\\-1\\2\end{pmatrix},\qquad \underset{\sim}{v}=\begin{pmatrix}a\\0\\b\end{pmatrix}$	
(i)	$\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\times\left(\underset{\sim}{u}-\underset{\sim}{v}\right)$ $=\underset{\sim}{u}\times\underset{\sim}{u}+\underset{\sim}{v}\times\underset{\sim}{u}-\underset{\sim}{u}\times\underset{\sim}{v}-\underset{\sim}{v}\times\underset{\sim}{v}$ $=2\underset{\sim}{v}\times\underset{\sim}{u}$ $=2\begin{pmatrix}a\\0\\b\end{pmatrix}\times\begin{pmatrix}2\\-1\\2\end{pmatrix}$ $=2\begin{pmatrix}b\\2b-2a\\-a\end{pmatrix}=2b\underset{\sim}{i}+\left(4b-4a\right)\underset{\sim}{j}-2a\underset{\sim}{k}$	Note that $\underset{\sim}{u}\times\underset{\sim}{u}=\underset{\sim}{v}\times\underset{\sim}{v}=\underset{\sim}{0}$ $-\underset{\sim}{u}\times\underset{\sim}{v}=\underset{\sim}{v}\times\underset{\sim}{u}$
(ii)	Given that $b=-a$ $\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\times\left(\underset{\sim}{u}-\underset{\sim}{v}\right)=2a\begin{pmatrix}-1\\-4\\-1\end{pmatrix}$ $\left\|\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\times\left(\underset{\sim}{u}-\underset{\sim}{v}\right)\right\|=1\Rightarrow\left\|2a\right\|\sqrt{18}=1\Rightarrow a=\pm\frac{1}{2\sqrt{18}}$	
(iii)	$\left(\underset{\sim}{u}+\underset{\sim}{v}\right).\left(\underset{\sim}{u}-\underset{\sim}{v}\right)=0\Rightarrow\underset{\sim}{u}.\underset{\sim}{u}-\underset{\sim}{v}.\underset{\sim}{v}=0$ $\Rightarrow\left\|\underset{\sim}{u}\right\|^2-\left\|\underset{\sim}{v}\right\|^2=0$ $\Rightarrow\left\|\underset{\sim}{v}\right\|=\left\|\underset{\sim}{u}\right\|=\sqrt{2^2+\left(-1\right)^2+2^2}=3$	

Question 6

No.	Suggested Solution	Remarks for Student
(i)	Let P_n be the statement $\sum_{r=1}^{n} r\left(r^2+1\right)=\frac{1}{4}n(n+1)\left(n^2+n+2\right)$ for $n \in \mathbb{Z}^+$. For $n=1$, $LHS=\sum_{r=1}^{1} r\left(r^2+1\right)=(1)\left(1^2+1\right)=2$ $RHS=\frac{1}{4}(1)(1+1)\left(1^2+1+2\right)=\frac{8}{4}=2=LHS$ Therefore, P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} r\left(r^2+1\right)=\frac{1}{4}k(k+1)\left(k^2+k+2\right)$ We need to show P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} r\left(r^2+1\right)=\frac{1}{4}(k+1)(k+1+1)\left((k+1)^2+k+1+2\right)$ $=\frac{1}{4}(k+1)(k+2)\left(k^2+3k+4\right)$ For $n=k+1$, $\sum_{r=1}^{k+1} r\left(r^2+1\right)$ $=\sum_{r=1}^{k} r\left(r^2+1\right)+(k+1)\left((k+1)^2+1\right)$ $=\frac{1}{4}k(k+1)\left(k^2+k+2\right)+(k+1)\left(k^2+2k+2\right)$ $=\frac{1}{4}(k+1)\left(k^3+k^2+2k+4k^2+8k+8\right)$ $=\frac{1}{4}(k+1)\left(k^3+5k^2+10k+8\right)$ $=\frac{1}{4}(k+1)(k+2)\left(k^2+3k+4\right)$ Therefore $P_k \Rightarrow P_{k+1}$ is true. Since P_1 is also true, P_n is true for $n \in \mathbb{Z}^+$ by Mathematical Induction.	

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(ii)	$u_n = u_{n-1} + n^3 + n$ $u_0 = 2$ $u_1 = 4$ $u_2 = 14$ $u_3 = 44$	
(iii)	$\sum_{r=1}^{n}(u_r - u_{r-1}) = u_1 - u_0$ $+u_2 - u_1$ $+u_3 - u_2$ $\ldots$ $+u_{n-1} - u_{n-2}$ $+u_n - u_{n-1}$ $= u_n - u_0$ $\therefore u_n = u_0 + \sum_{r=1}^{n}(u_r - u_{r-1})$ $= 2 + \sum_{r=1}^{n}(r^3 + r)$ $= 2 + \sum_{r=1}^{n} r(r^2 + 1)$ $= 2 + \frac{1}{4}n(n+1)(n^2+n+2)$ from (i)	

Question 7

No.	Suggested Solution	Remarks for Student
(a)	$w^2+(-1-8\text{i})w+(-17+7\text{i})=0 \quad ...(1)$ $(-1+5\text{i})^2+(-1-8\text{i})(-1+5\text{i})+(-17+7\text{i})$ $=(-24-10\text{i})+(41+3\text{i})+(-17+7\text{i})$ $=(-24+41-17)+(-10+3+7)\text{i}$ $=0$ Thus, $-1+5\text{i}$ is a root of (1) Let $x+\text{i}y$ be the second root, then $w^2+(-1-8\text{i})w+(-17+7\text{i})=(w-(-1+5\text{i}))(w-(x+y\text{i}))$ $-17+7\text{i}=(-1+5\text{i})(x+y\text{i})$ $-17+7\text{i}=-x-5y+(5x-y)\text{i}$ Comparing Re and Im parts, $x+5y=17$ $5x-y=7$ Thus, $x=2, y=3$ Second root is $2+3\text{i}$ OR Let $x+\text{i}y$ be the second root, then $(x+\text{i}y)^2+(-1-8\text{i})(x+\text{i}y)+(-17+7\text{i})=0$ $(x^2-y^2+2xy\text{i})+(-x+8y-8x\text{i}-y\text{i})+(-17+7\text{i})=0$ Comparing Re and Im parts: $x^2-y^2-x+8y-17=0 \quad ...(2)$ $2xy-8x-y+7=0 \quad ...(3)$ Solving to get the same answers...	
(b)	Since coefficients are real, both 1 + *a*i and 1 – *a*i are roots. $z^3-5z^2+16z+k=(z-(1+a\text{i}))(z-(1-a\text{i}))(z-b)$ $z^3-5z^2+16z+k=(z^2-2z+(1+a^2))(z-b)$ Comparing coefficients and constant term, $-b-2=-5 \Rightarrow b=3$ $2b+(1+a^2)=16 \Rightarrow a=3$ since $a>0$ $k=-(1+a^2)b=-30$	

Question 8

No.	Suggested Solution	Remarks for Student
	$y=\mathrm{f}(x)=\tan(ax+b)$	
(i)	$\mathrm{f}'(x)=a\sec^2(ax+b)=a\left[1+\tan^2(ax+b)\right]=a+ay^2$ $\mathrm{f}''(x)=2ay\mathrm{f}'(x)=2ay\left(a+ay^2\right)=2a^2y+2a^2y^3$ $\begin{aligned}\mathrm{f}'''(x)&=2a^2\mathrm{f}'(x)+6a^2y^2\mathrm{f}'(x)\\&=2a^2\left(a+ay^2\right)+6a^2y^2\left(a+ay^2\right)\\&=2a^3+2a^3y^2+6a^3y^2+6a^3y^4\\&=2a^3+8a^3y^2+6a^3y^4\end{aligned}$	
(ii)	$y=\mathrm{f}(x)=\tan\left(ax+\frac{\pi}{4}\right)$ $\mathrm{f}(0)=\tan\left(\frac{\pi}{4}\right)=1$ $\mathrm{f}'(0)=a+a(1)^2=2a$ $\mathrm{f}''(0)=2a^2+2a^2=4a^2$ $\mathrm{f}'''(0)=16a^3$ $\mathrm{f}(x)=1+2ax+2a^2x^2+\frac{8a^3}{3}x^3+\ldots$	

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(iii)	Using part (i) with $a = 2$ and $b = 0$, $y = \mathrm{f}(x) = \tan(2x)$ $\mathrm{f}(0) = 0$ $\mathrm{f}'(0) = 2 + 2(0)^2 = 2$ $\mathrm{f}''(0) = 0$ $\mathrm{f}'''(0) = 16$ $\mathrm{f}(x) = 2x + \frac{8}{3}x^3 + \ldots$ OR Using part (ii) $\tan\left(2x + \frac{\pi}{4}\right) = 1 + 4x + 8x^2 + \frac{64}{3}x^3 + \ldots$ $\frac{\tan 2x + \tan\frac{\pi}{4}}{1 - \tan 2x \tan\frac{\pi}{4}} = 1 + 4x + 8x^2 + \frac{64}{3}x^3 + \ldots$ $\frac{\tan 2x + 1}{1 - \tan 2x} = 1 + 4x + 8x^2 + \frac{64}{3}x^3 + \ldots$ We can rewrite $\frac{\tan 2x + 1}{1 - \tan 2x} = \frac{\tan 2x - 1 + 2}{1 - \tan 2x} = -1 + \frac{2}{1 - \tan 2x}$ $\frac{2}{1 - \tan 2x} = 2 + 4x + 8x^2 + \frac{64}{3}x^3 + \ldots$ $\frac{1}{1 - \tan 2x} = 1 + 2x + 4x^2 + \frac{32}{3}x^3 + \ldots$ $1 - \tan 2x = \left(1 + 2x + 4x^2 + \frac{32}{3}x^3 + \ldots\right)^{-1}$ $= 1 - \left(2x + 4x^2 + \frac{32}{3}x^3\right) + \left(2x + 4x^2 + \frac{32}{3}x^3\right)^2 - \left(2x + 4x^2 + \frac{32}{3}x^3\right)^3 + \ldots$ $= 1 - 2x - \frac{8}{3}x^3 + \ldots$ $\tan 2x = 2x + \frac{8}{3}x^3 + \ldots$	

Question 9

No.	Suggested Solution	Remarks for Student
(i)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}=10$	
(a)	$y=\frac{\mathrm{d}x}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}=10\ \Rightarrow\frac{\mathrm{d}y}{\mathrm{d}t}+2y=10$ $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}t}=10-2y$	
(b)	$\frac{1}{10-2y}\frac{\mathrm{d}y}{\mathrm{d}t}=1$ $\int\frac{1}{10-2y}\,\mathrm{d}y=\int\mathrm{d}t$ $-\frac{1}{2}\ln\lvert 5-y\rvert=t+A$ $\ln\lvert 5-y\rvert=-2t-2A$ $\lvert 5-y\rvert=\mathrm{e}^{-2t-2A}$ $y=5-B\mathrm{e}^{-2t},\ B=\pm\mathrm{e}^{-2A}$ is an arbitrary constant $\frac{\mathrm{d}x}{\mathrm{d}t}=y=0$ when $t=0\Rightarrow 0=5-B\Rightarrow B=5$ $\therefore y=5-5\mathrm{e}^{-2t}$ Since $\frac{\mathrm{d}x}{\mathrm{d}t}=5-5\mathrm{e}^{-2t}$, $x=5t+\frac{5}{2}\mathrm{e}^{-2t}+C$ $x=0$ when $t=0\Rightarrow C=-\frac{5}{2}$ $\therefore x=5t+\frac{5}{2}\mathrm{e}^{-2t}-\frac{5}{2}$	

(ii)	$\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}=10-5\sin\dfrac{1}{2}t$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=10t+10\cos\dfrac{1}{2}t+A$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=0$ when $t=0\Rightarrow A=-10$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=10t+10\cos\dfrac{1}{2}t-10$ $x=5t^2+20\sin\dfrac{1}{2}t-10t+B$ $x=0$ when $t=0\Rightarrow B=0$ $\therefore x=5t^2+20\sin\dfrac{1}{2}t-10t$	
(iii)	$5=5t+\dfrac{5}{2}\mathrm{e}^{-2t}-\dfrac{5}{2}\Rightarrow t=1.47$ $5=5t^2+20\sin\dfrac{1}{2}t-10t\Rightarrow t=1.05$ By using the first and second model, the stone takes 1.47 seconds and 1.05 seconds respectively to reach the bottom of the pond.	Answers are gotten using GC.

Question 10

No.	Suggested Solution	Remarks for Student
(a)	$\mathrm{f} : x \mapsto 1+\sqrt{x}$, for $x \in \mathbb{R}$, $x \geq 0$	
(i)	$y = 1+\sqrt{x} \Rightarrow x = (y-1)^2$ $\mathrm{f}^{-1}(x) = (x-1)^2$ Domain of f^{-1} = Range of f $= [1, \infty)$	
(ii)	$\mathrm{ff}(x) = x$ $\mathrm{f}\left(1+\sqrt{x}\right) = x$ $1+\sqrt{1+\sqrt{x}} = x \quad \ldots(1)$ $\sqrt{1+\sqrt{x}} = x-1$ $1+\sqrt{x} = x^2-2x+1$ $\sqrt{x} = x^2-2x$ $x = x^4-4x^3+4x^2$ $x^4-4x^3+4x^2-x=0$ $x\left(x^3-4x^2+4x-1\right)=0$ Note that $\mathrm{ff}(0) = \mathrm{f}(1) = 2$, thus $x=0$ is not a solution to $\mathrm{ff}(x) = x$ Thus, $x^3-4x^2+4x-1=0$ From GC, $x = 1$, 0.382(3 sf) or 2.62 (3 sf) It can be observed from (1) that $x > 2$, so $x = 2.62$. Explanation: Since f^{-1} exists, $\mathrm{ff}(x) = x \Rightarrow \mathrm{f}^{-1}\mathrm{f}\left(\mathrm{f}(x)\right) = \mathrm{f}^{-1}(x)$ $\Rightarrow \mathrm{f}(x) = \mathrm{f}^{-1}(x) \quad$ since $\mathrm{f}^{-1}\mathrm{f}(a) = a$ Thus this value of x satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.	
(b)	$\mathrm{g}(n) = \begin{cases} 1 & \text{for } n = 0, \\ 2+\mathrm{g}\left(\frac{1}{2}n\right) & \text{for } n \text{ even}, \\ 1+\mathrm{g}(n-1) & \text{for } n \text{ odd}. \end{cases}$	

(i)	$\mathrm{g}(4)=2+\mathrm{g}(2)=2+2+\mathrm{g}(1)=4+1+\mathrm{g}(0)=6$ $\mathrm{g}(7)=1+\mathrm{g}(6)=1+2+\mathrm{g}(3)=3+1+\mathrm{g}(2)=4+4=8$ $\mathrm{g}(12)=2+\mathrm{g}(6)=2+7=9$	
(ii)	$\mathrm{g}(8)=2+\mathrm{g}(4)=2+6=8=\mathrm{g}(7)$ and $8\neq 7$ $\therefore$ g is not 1-1. Thus g does not have an inverse. Observe from (i) that the values of g is integral and seems to "increase" when *n* increases from 7 and 12. But g only increases from 8 to 9. So we suspect there should exist repeated numbers.	

Question 11

No.	Suggested Solution	Remarks for Student
	$p:\underset{\sim}{r}=\begin{pmatrix}1\\-3\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}a\\4\\-2\end{pmatrix}$ $l:\underset{\sim}{r}=\begin{pmatrix}a-1\\a\\a+1\end{pmatrix}+t\begin{pmatrix}-2\\1\\2\end{pmatrix}$	
(i)	$a=0$	
(a)	$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\4\\-2\end{pmatrix}=\begin{pmatrix}-4\\2\\4\end{pmatrix}=2\begin{pmatrix}-2\\1\\2\end{pmatrix}$ l is perpendicular to p $p:\underset{\sim}{r}=\begin{pmatrix}1\\-3\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}0\\4\\-2\end{pmatrix}$ $l:\underset{\sim}{r}=\begin{pmatrix}-1\\0\\1\end{pmatrix}+t\begin{pmatrix}-2\\1\\2\end{pmatrix}$ $1+\lambda=-1-2t \quad \Rightarrow \lambda+2t=-2$ $-3+2\lambda+4\mu=t \quad \Rightarrow 2\lambda+4\mu-t=3$ $2-2\mu=1+2t \quad \Rightarrow -2\mu-2t=-1$ $\lambda=-\frac{8}{9},\mu=\frac{19}{18},t=-\frac{5}{9}$	
(b)	$p:\underset{\sim}{r}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}=\begin{pmatrix}1\\-3\\2\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}=-2-3+4=-1$ Note that $A(0,-1,0)$ is a point on the plane p. Let l_2 be the line that passes through A and is perpendicular to p. Let H be a point on l_2 such that $AH=12$. $\overrightarrow{AH}$ is parallel to normal vector of p, and	

	$\Rightarrow \quad \overrightarrow{AH} = \alpha \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ for some $\alpha \in \mathbb{R}$. $\left\lvert\overrightarrow{AH}\right\rvert = 12 \Rightarrow \sqrt{(-2\alpha)^2 + \alpha^2 + (2\alpha)^2} = 12$ $\Rightarrow \sqrt{9\alpha^2} = \lvert 3\alpha \rvert = 12$ $\Rightarrow 3\alpha = \pm 12$ $\Rightarrow \alpha = \pm 4$ Hence, $\overrightarrow{AH} = \begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$ or $\begin{pmatrix} 8 \\ -4 \\ -8 \end{pmatrix}$. That is, $\overrightarrow{OH} = \begin{pmatrix} -8 \\ 3 \\ 8 \end{pmatrix}$ or $\begin{pmatrix} 8 \\ -5 \\ -8 \end{pmatrix}$ Scalar product equation of plane parallel to p : $\underset{\sim}{r} \bullet \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 3 \\ 8 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = 35$ or $\underset{\sim}{r} \bullet \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ -8 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = -37$ Cartesian equations of planes required : $-2x + y + 2z = 35$ or $-2x + y + 2z = -37$	
(ii)	$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} a \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4-2a \end{pmatrix}$ l and p do not meet $\Rightarrow$ normal of p and l to be perpendicular $\begin{pmatrix} -4 \\ 2 \\ 4-2a \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = 0 \Rightarrow 8 + 2 + 8 - 4a = 0 \Rightarrow a = \dfrac{9}{2}$	