ST ANDREW’S JUNIOR COLLEGE

General Certificate of Education Advanced Level

Higher 2

MATHEMATICS 9758/01

Paper 1 **4 October 2024**

FINAL EXAMINATION **3 hours**

Additional Materials: **Printed Answer Booklet**

List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.

Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.

You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

1 The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, ... u_n ...$ is given by

$$S_n = \frac{5}{24} - \frac{n+5}{(n+4)!}.$$

(i) Determine with a reason, if the series converges and write down the value of the sum to infinity if it exists. [2]

(ii) Find the exact value of $\sum_{r=1}^{\infty} u_{r+2}$, expressing your answer in the simplest form. [2]

(iii) Find a formula for u_n in the form $\frac{\mathrm{f}(n)}{(n+4)!}$ where $\mathrm{f}(n)$ is a quadratic polynomial. [4]

2 The parametric equation of a curve C are

$$x = 2\sqrt{t},\ y = 1 + \sqrt{1-t} \quad \text{where } 0 \le t \le 1.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. What can be said of the tangent to the curve as $t \to 1$? [2]

(ii) Sketch the curve C, showing clearly the coordinates of the endpoints. [3]

(iii) Show that the equation of the tangent to C at the point $T\left(2\sqrt{t}, 1+\sqrt{1-t}\right)$ is $2\sqrt{1-t}\,y = -\sqrt{t}x + 2 + 2\sqrt{1-t}$ [2]

(iv) The tangent to the curve at the point with x-coordinate $\sqrt{2}$ meets the x and y axis at the points P and Q respectively. Find the exact area of triangle OPQ. [4]

3 A curve C_1 has equation $\frac{y^2}{5} - \frac{x^2}{4} = 1, \quad x \ge 0$.

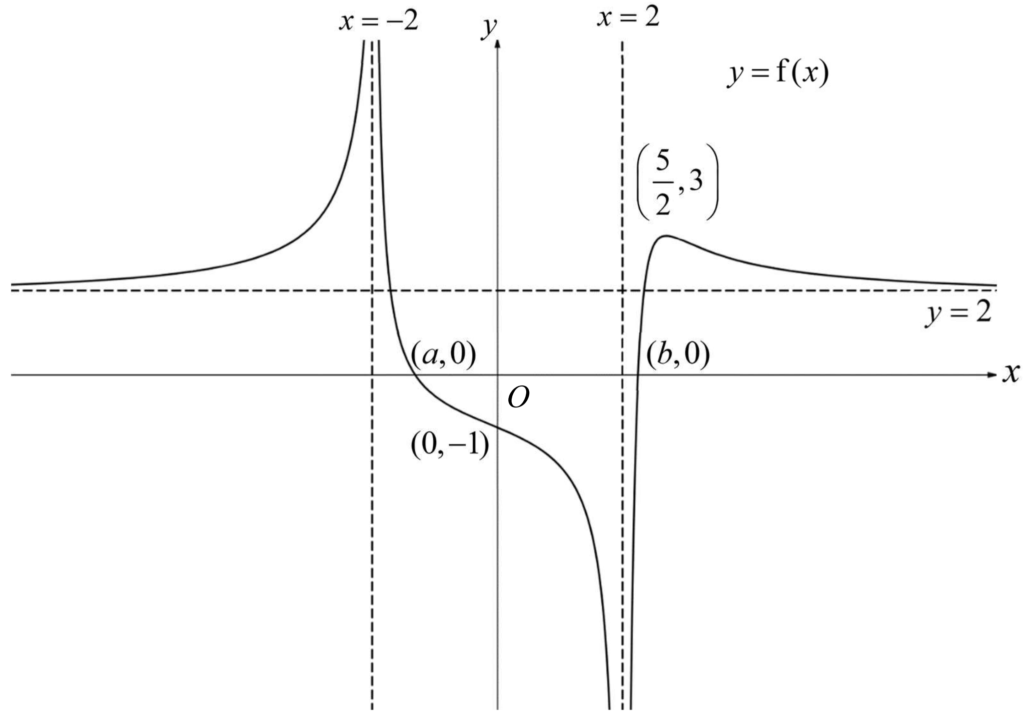
(i) Sketch the curve C_1, showing clearly the equations of any asymptotes and coordinates of any intersections with the axes. [3]

A curve C_2 has equation $y = |2x-3|$.

(ii) Sketch the curve C_2 on the same diagram as C_1, showing clearly the coordinates of any intersections with the axes. Find and label the x-coordinates of the point(s) of intersections of C_1 and C_2 on this diagram. [3]

(iii) Hence, state the range of values of x for which $|2x-3| \ge \sqrt{5\left(1+\frac{x^2}{4}\right)}$, where $x \ge 0$. [2]

4 **(a)**



The above diagram shows the graph of $y=\mathrm{f}(x)$. The curve has a maximum point at $\left(\frac{5}{2},3\right)$ and passes through the x-axis at $(a,0)$ and $(b,0)$ where a and b are real constants such that $-2<a<0$ and $2<b<\frac{5}{2}$. The curve cuts the y-axis at $y=-1$ with gradient $-\frac{1}{2}$. The lines $y=2$, $x=-2$ and $x=2$ are the asymptotes of the curve.

Sketch, on separate diagrams, the graph of

(i) $y=\mathrm{f}'(x)$,

(ii) $y=\mathrm{f}\left(\frac{x+1}{2}\right)$,

stating clearly in each case, the equations of the asymptotes, the coordinates of the turning points and the axial intercepts wherever possible. [6]

(b) A curve C has an equation $y=\frac{1}{4ax-x^2}$, where $a>0$. Describe a sequence of transformations that map the graph of C onto the graph of $y=\frac{1}{x^2-4a^2}$. [3]

5 **(a)** Find the range of values of θ in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ such that the sum to infinity of the geometric series $1 + 2\sin\theta + (2\sin\theta)^2 + \ldots$ exists.
Hence find the range of values of θ for which the sum to infinity of the geometric series is greater than 2. [5]

(b) The 5th, 9th and 11th terms of a geometric progression are also the 7th, 25th and 50th terms of an arithmetic progression with a non-zero common difference respectively.
Show that $18r^4 - 25r^2 - 25 = 0$, where r is the common ratio of the geometric progression and determine if the geometric progression is convergent. [5]

6 The function f is defined as

$$\mathrm{f} : x \mapsto \left|\frac{1}{2x-5}\right|, \text{ where } x \in \mathbb{R}, x \neq \frac{5}{2}.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. Hence explain why f does not have an inverse. [2]

(ii) If the domain of f is restricted to $x < k$, state the maximum value of k such that f^{-1} will exist. Hence find f^{-1} in a similar form. [4]

For the rest of the question, you are to use the domain of f found in part **(ii)**.
It is given that

$$\mathrm{g} : x \mapsto (x-3)^2 + 1, \text{ where } x \in \mathbb{R}, x \geq -2.$$

(iii) Explain why the composite function gf exists. [1]

(iv) Find the rule of gf, giving your answer in the form $\left(\frac{ax-14}{b-2x}\right)^2 + 1$, where a and b are constants to be determined.
Hence or otherwise, find the range of gf. [4]

7 The points A and B has coordinates $(-2, 0, 1)$ and $(1, 1, 2)$ respectively. The line L is parallel to $\mathbf{i}-2\mathbf{j}$ and passes through the point A.

(i) Find the coordinates of the point, C, on line L, which is closest to B. [3]

The plane, Π, contains the line L and the point D with coordinates $(-1, 4, 3)$.

(ii) Find a cartesian equation of Π. [3]

(iii) Find the shortest distance from B to Π. [2]

(iv) Hence or otherwise, find the acute angle that BC makes with Π. [2]

8 It is given that $y = \mathrm{e}^{\tan^{-1}(x)}$.

(i) Show that $(1+x^2)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = (1-2x)\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [3]

(ii) By further differentiation of the result in part **(i)**, find the Maclaurin's series for y, up to and including the term in x^3. [3]

(iii) Hence, by using a suitable value of x, show that $\mathrm{e}^{\frac{\pi}{6}} \approx \dfrac{p}{q} + \dfrac{r}{s}\sqrt{3}$, where p, q, r and s are integers to be determined. [4]

9 **(a)** Relative to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel. It is given that $|\mathbf{a}| = 9$, $|\mathbf{b}| = 1$ and $|2\mathbf{a}+\mathbf{b}| = 2\sqrt{74}$.

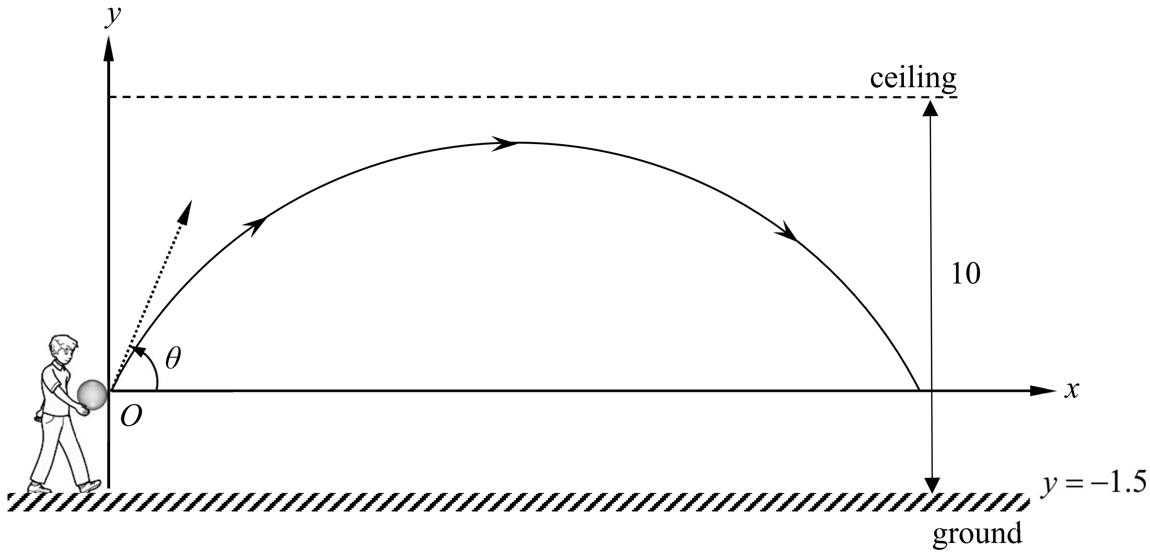
(i) By considering $(2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})$, show that $\mathbf{a}\cdot\mathbf{b} = -\dfrac{29}{4}$. [3]

(ii) Give a geometrical meaning of $|\mathbf{a}\cdot\mathbf{b}|$. [1]

(iii) The points P, Q and R have position vectors $7\mathbf{a}-5\mathbf{b}$, $6\mathbf{a}+5\mathbf{b}$ and $9\mathbf{a}+\lambda\mathbf{b}$ respectively. Given that these three points are collinear, find the value of λ. [3]

(b) In the triangle OUV where O is the origin, the position vectors of the points U and V are $\mathbf{u}$ and $\mathbf{v}$ respectively. The point W is the midpoint of OU, and the point X has position vector given by $\dfrac{1}{16}\mathbf{u} + \dfrac{3}{4}\mathbf{v}$. Show that the area of triangle UWX can be written as $m|\mathbf{u}\times\mathbf{v}|$ where m is a constant to be found. [3]

10



The diagram above shows the trajectory of a ball thrown in a sports hall with a ceiling height of 10 m. Nicholas throws a ball from the origin O, 1.5m above the ground with a fixed initial speed of v ms^{-1} and at a particular angle of θ made with the horizontal where $0<\theta<\frac{\pi}{2}$. At time t seconds, the position of the ball can be modelled by the parametric equations

$$x=(v\cos\theta)t,\ y=(v\sin\theta)t-5t^2,$$

where x m is the horizontal distance of the ball with respect to O and y m is the vertical distance of the ball with respect to O.

(i) Find $\frac{dy}{dx}$ in terms of v, θ and t. [3]

(ii) By using your **answer** to **part (i)**, determine the time taken for the ball to reach its maximum height and show that the corresponding height of the ball with respect to the ground is $\frac{v^2\sin^2\theta+A}{20}$ metres, where A is a constant to be determined.
[There is no need to prove that this height is the maximum height.] [3]

Use $v=20$ **to answer the remaining parts of the question.**

(iii) Hence, determine the range of θ that the ball should be thrown to avoid hitting the ceiling. [2]

(iv) Find the horizontal distance travelled by the ball when it reaches a height of 1.5m above the ground. Leave your answer in terms of θ. [2]

Nicholas throws the ball at an angle θ such that it does not hit the ceiling. Two seconds after the ball was thrown, his sister Isabelle runs from Nicholas' position along the x direction at a constant speed of 5 m/s to intercept the ball. Isabelle can only intercept the ball when it reaches a height of 1.5m above the ground exactly.

(v) Assuming that the trajectory of the ball does not change from that shown in the diagram, determine whether it is possible for Isabelle to intercept the ball thrown by Nicholas, justifying your answer. [3]

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