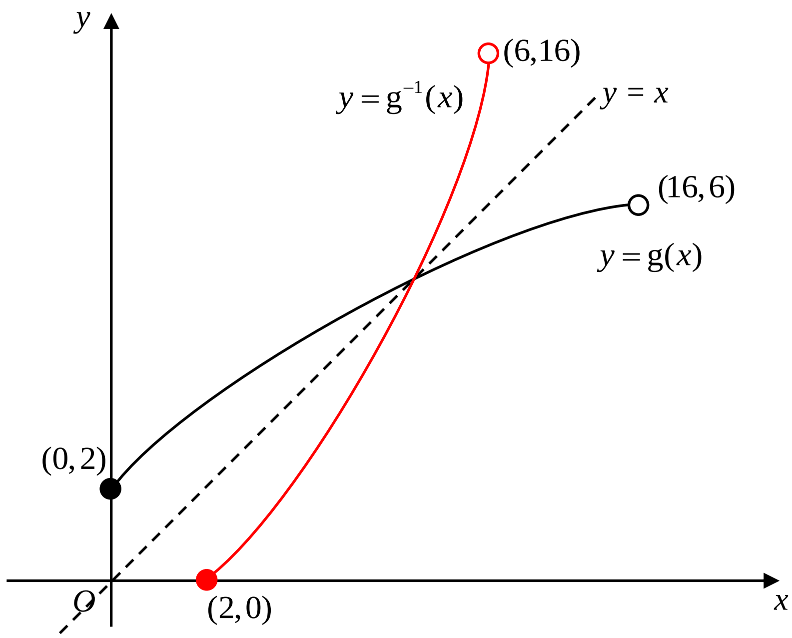
2015 NYJC JC2 Prelim Exam 9740/2 Solutions

Qn	Solution
1	Let P_n be the statement $\sum_{r=1}^{n}\cos(2r\theta)=\frac{\sin(2n+1)\theta-\sin\theta}{2\sin\theta}$ for $n\in\mathbb{Z}^+$ When n = 1, LHS $=\sum_{r=1}^{1}\cos(2r\theta)=\cos 2\theta$ RHS $=\frac{\sin3\theta-\sin\theta}{2\sin\theta}=\frac{2\cos2\theta\sin\theta}{2\sin\theta}=\cos2\theta$ $\therefore P_1$ is true. Assume that P_k true for some $k\in Z^+$, $k\geq 1$. i.e. $\sum_{r=1}^{k}\cos(2r\theta)=\frac{\sin(2k+1)\theta-\sin\theta}{2\sin\theta}$ For $n=k+1$, we want to prove $\sum_{r=1}^{k+1}\cos(2r\theta)=\frac{\sin(2k+3)\theta-\sin\theta}{2\sin\theta}$ LHS $=\sum_{r=1}^{k+1}\cos(2r\theta)=\sum_{r=1}^{k}\cos(2r\theta)+\cos(2k+2)\theta$ $=\frac{\sin(2k+1)\theta-\sin\theta}{2\sin\theta}+\cos(2k+2)\theta$ $=\frac{\sin(2k+1)\theta-\sin\theta+2\cos(2k+2)\theta\sin\theta}{2\sin\theta}$ $=\frac{\sin(2k+1)\theta-\sin\theta+\sin(2k+3)\theta-\sin(2k+1)\theta}{2\sin\theta}$ $=\frac{\sin(2k+3)\theta-\sin\theta}{2\sin\theta}$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by the principle of Mathematical Induction, P_n is true for all $n\in\boldsymbol{Z}^+$. $\sum_{r=1}^{n}\cos(r\pi)=\sum_{r=1}^{n}\cos\left(2r\cdot\frac{\pi}{2}\right)=\frac{\sin(2n+1)\frac{\pi}{2}-\sin\frac{\pi}{2}}{2\sin\frac{\pi}{2}}=\frac{(-1)^n-1}{2}$

Qn	Solution
2(a)	Let $y = \dfrac{2x-1}{2x-2}$ $y(2x-2)=2x-1$ $x(2y-2)=2y-1$ $x=\dfrac{2y-1}{2y-2}$ $\therefore \mathrm{f}^{-1}: x \to \dfrac{2x-1}{2x-2}, \quad x<1$ $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ $\Rightarrow \mathrm{f}^{2}(x) = x$ $\mathrm{f}^{1965}(0.5) = 0$
(b)	**(i), (ii)**  **(iii)** For equation $\mathrm{g}^{-1}\mathrm{g}(x) = g\mathrm{g}^{-1}(x)$, $2 \le x < 6$.

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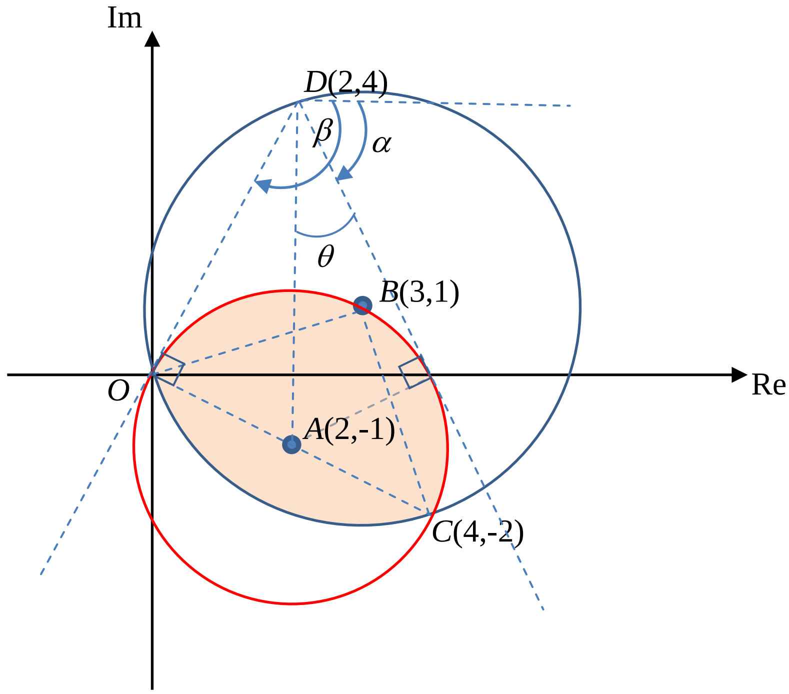
3(i)	
(ii)	Since $\|z_1-2+\mathrm{i}\|=\sqrt{5}$ and $\|z_1-3-\mathrm{i}\|=\sqrt{10}$, thus $z_1=4-2\mathrm{i}$ satisfies the equation $\|z-2+\mathrm{i}\|=\sqrt{5}$ and $\|z-3-\mathrm{i}\|=\sqrt{10}$.
(iii)	Note that triangle *OBC* is a right angle triangle. Further *B* lies on the locus of $\|z-2+\mathrm{i}\|=\sqrt{5}$. Thus $area=\frac{1}{2}\pi\left(\sqrt{5}\right)^2+\left[\frac{1}{4}\pi\left(\sqrt{10}\right)^2-\frac{1}{2}\left(\sqrt{10}\right)^2\right]$ $=\frac{5\pi}{2}+\frac{5\pi}{2}-5$ $=5(\pi-1)$
(iv)	$\theta=\sin^{-1}\frac{\sqrt{5}}{5}$ Thus $\alpha=-\frac{\pi}{2}+\theta$ and $\beta=-\frac{\pi}{2}-\theta$. Hence the required range is $-2.03\le\arg(z-2-4\mathrm{i})\le-1.11$.

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(i) Let $y=\tan^{-1}x$.

Then $\tan y = x$

Differentiate with respect to x,

$$\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=1$$

$$\left(1+\tan^2 y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x^2}\text{ (shown)}$$

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(ii) From graph,

area of shaded rectangle < area under curve for $n-1\le x\le n$.

$$\frac{1}{1+n^2}\left[n-(n-1)\right]<\int_{n-1}^{n}\frac{1}{1+x^2}\,\mathrm{d}x$$

$$\frac{1}{1+n^2}<\left[\tan^{-1}(x)\right]_{n-1}^{n}$$

$$\frac{1}{1+n^2}<\tan^{-1}(n)-\tan^{-1}(n-1)\quad\text{(show)}$$

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(iii) Comparing area of rectangles with area under curve for $0\le x\le n$,

$$\frac{1}{1+1^2}+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\ldots+\frac{1}{1+n^2}<\int_0^n\frac{1}{1+x^2}\,\mathrm{d}x$$

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\ldots+\frac{1}{1+n^2}<\left[\tan^{-1}(x)\right]_0^n$$

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\ldots+\frac{1}{1+n^2}<\tan^{-1}(n)$$

Alternative: Use Method of Difference using (i) result

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$$y=\frac{1}{1+(x+3)^2}$$

$$(x+3)^2=\frac{1}{y}-1$$

$$x+3=-\sqrt{\frac{1}{y}-1}\text{ (rejected) or } x+3=\sqrt{\frac{1}{y}-1}$$

$$x=-3+\sqrt{\frac{1}{y}-1}$$

Required volume $=\pi\int_{\frac{1}{10}}^{\frac{1}{2}}x^2\,\mathrm{d}y$

$$=\pi\int_{\frac{1}{10}}^{\frac{1}{2}}\left(-3+\sqrt{\frac{1}{y}-1}\right)^2\mathrm{d}y$$

$$\approx 2.5998$$

$$=2.60\text{ units}^3\text{ or }0.828\pi$$

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Qn	Solution

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i) A random sample refers to the sample is obtained by selecting 50 students from 500 students in such a way that each of the 500 students will have an equal chance of being selected. Each selection is independent.

ii) To obtain a sample of 50 students using stratified random sampling, we will determine the sample size of the method of transport such that the sample size of each method of travel is proportional to the size of each method of travel in the school.

	Car	Public transport	On foot
Sample Size	30% of 50 (=15)	50% of 50 (= 25)	20% of 50 (= 10)

Then **conduct simple random sampling** from each method of travel. These simple random samples from the different method of travel groups are combined to form the overall stratified random sample of 50 students.

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Let X be the weight of the chicken sold in the supermarket. $X \sim \mathrm{N}(\mu, \sigma^2)$

$\mathrm{P}(X < 1) = 0.2$

$\mathrm{P}\left(\frac{X-\mu}{\sigma} < \frac{1-\mu}{\sigma}\right) = 0.20$

$\mathrm{P}\left(Z < \frac{1-\mu}{\sigma}\right) = 0.20$

Using GC, $\frac{1-\mu}{\sigma}$ =-0.84162 -----(1)

$\mathrm{P}\left(\frac{X-\mu}{\sigma} > \frac{1.8-\mu}{\sigma}\right) = 0.15$

$1-\mathrm{P}\left(Z < \frac{1.8-\mu}{\sigma}\right) = 0.15$

$\mathrm{P}\left(Z < \frac{1.8-\mu}{\sigma}\right) = 0.85$

Using GC, $\frac{1.8-\mu}{\sigma}$ =1.0364 ------(2)

Solving equation (1) and (2),

$\mu = 1.36$ and $\sigma = 0.426$

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(i) The calls may not be independent as people may be calling in to lodge a complaint on bad service rendered by the company.

(ii) Since $n = 60 > 50$ and $np = 60(0.08) = 4.8 < 5$,

$C \sim \mathrm{Po}(4.8)$.

$\mathrm{P}(C \geq 6) = 1 - \mathrm{P}(C \leq 5) = 0.349$ (3 s.f.)

Qn	Solution
	(iii) $C \sim B(60, 0.08)$ Since $n = 80$ is large, $\overline{C} \sim N\left(4.8, \frac{4.416}{80}\right)$ approximately by CLT. $P(\overline{C} > 5) = 0.197$ (3 s.f.)
8	(i) required probability = $\frac{45}{140} = \frac{9}{28}$ (ii) P(father born in Asia | mother born in UK) $= \frac{\text{P(father born in Asia} \cap \text{mother born in UK)}}{\text{P(mother born in UK)}}$ $= \frac{\frac{8}{140}}{\frac{60}{140}} = \frac{2}{15}$ (iii) P(parent born in either UK or Europe) $= 1 -$ P(both parents born in Asia) $= 1 - \frac{45}{140} = \frac{19}{28}$ **OR** Let *A* be the event at least one parent in UK Let *B* be the event at least one parent in Europe Req'd prob. = $\text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$ $= \frac{80}{140} + \frac{38}{140} - \frac{23}{140} = \frac{19}{28}$
9(a)	Let μ be the mean of *X*. To test $H_0 : \mu = 455$ $H_1 : \mu \neq 455$ Level of Significance: $\frac{\alpha}{100}$ Perform Z test Calculation: p-value = 0.096875 For sufficient evidence to reject H_0, p – value < $\frac{\alpha}{100}$. Thus $\frac{\alpha}{100} > 0.096875$ $\Rightarrow \alpha > 9.6875$ Thus the least value of level of significance is 9.69%.

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$s^2 = \frac{10}{9}(0.56921)^2 = 0.360$
$H_0 : \mu = 455$
$H_1 : \mu \neq 455$
Level of Significance: 0.05
Perform t test
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Reject H_0 if p-value < 0.05
If $\bar{x} \geq 455$
$2P(\bar{X} > \bar{x}) < 0.05$
$\Rightarrow P\left(T > \frac{\bar{x} - 455}{s/\sqrt{10}}\right) < 0.025$
Since $P(T > 2.2622) = 0.025$, thus
$\frac{\bar{x} - 455}{s/\sqrt{10}} > 2.2622$
$\Rightarrow \bar{x} > 455.4292$
If $\bar{x} < 455$
$2P(\bar{X} < \bar{x}) < 0.05$
$\Rightarrow P\left(T < \frac{\bar{x} - 455}{s/\sqrt{10}}\right) < 0.025$
Since $P(T < -2.2622) = 0.025$, thus
$\frac{\bar{x} - 455}{s/\sqrt{10}} < -2.2622$
$\Rightarrow \bar{x} < 454.5707$
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Reject H_0 if $|T_{\text{calc}}| > t_{0.025}(9) = 2.2622$
Since H_0 is rejected,
$\left|\frac{\bar{x} - 455}{s/\sqrt{10}}\right| > 2.2622$
$\Rightarrow \bar{x} < 455 - 2.2622\frac{s}{\sqrt{10}}$
or $\bar{x} > 455 + 2.2622\frac{s}{\sqrt{10}}$
$\Rightarrow \bar{x} < 454.5707$ or $\bar{x} > 455.4292$
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Thus the required set of values of $\bar{x}$ is $(0, 454.57] \cup [455.43, \infty)$.
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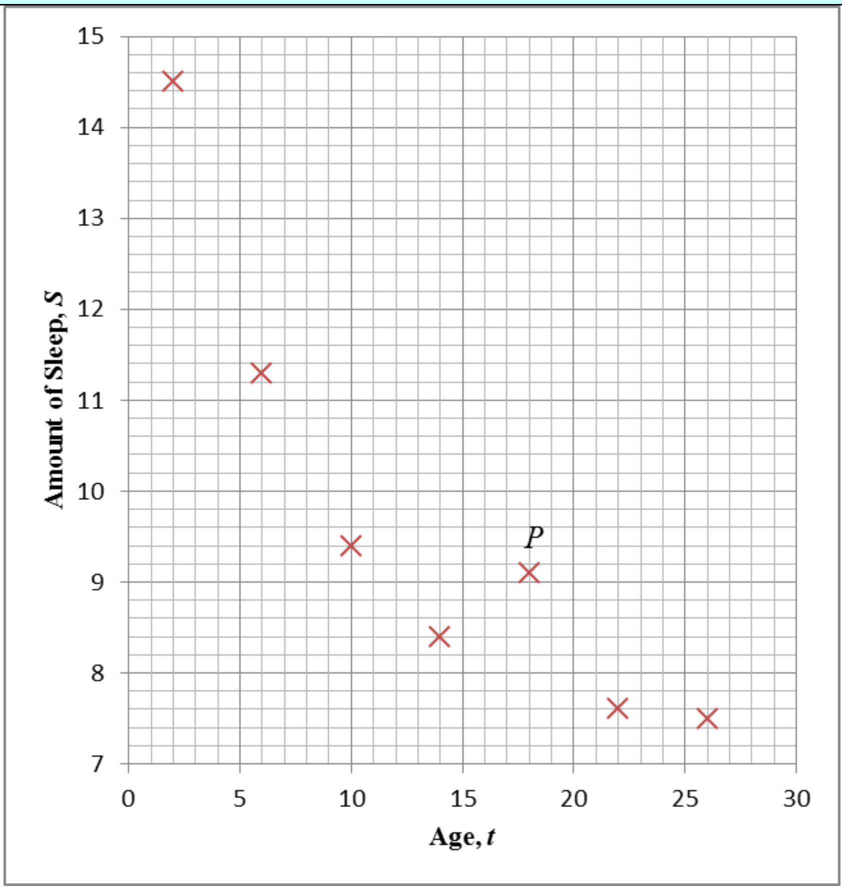
Qn	**Solution**
10(i)	
10(ii)	The scatter diagram shows that S is decreasing at a decreasing rate as t increase. Further, if a linear model is used, there will be a negative amount of sleep after a certain age, which is impossible. Thus a linear model is inappropriate.
10(iii)	Since S is decreasing at a decreasing rate as t increase and approaching a value, the proposed model may be appropriate.
10(iv)	Using GC, $\hat{a} = 7.51$, $\hat{b} = 14.8$
10(v)	Since t =50 is out of the given data range, it will be inappropriate to use the model due to extrapolation.

Qn	Solution
11	(i) No of ways = $10^6 = 1000000$ (ii) No of possible 3 identical digits on the left = 10 No of possible 3 identical digits on the right and is different from the left = 9 Total no of ways = 10 x 9 = 90 (iii)No of possible ways in the other 2 digits = $\binom{6}{2} = 15$ Total no of ways = 15 x 6! = 10800 (iv)Case 1: LHS – all 3 identical No of ways = 10 x 9 x 9 x 9 = 7290 Case 2 : LHS – 1pair identical No of ways = $\binom{10}{2}\frac{3!}{2!}\cdot 2!\cdot 8^3 = 138240$ Case 3 : LHS – all different No of ways = $\binom{10}{3}\cdot 3!\cdot 7^3 = 246960$ Total no of ways = 392490
12	**(i)** Incidences of people joining the self-checkout queue occur *randomly* and *independently*. The *average rate* of people joining the self-checkout queue is *constant over the chosen time interval*.
	(ii) Let X be the number of people joining the self-checkout queue in 4 minutes. Then $X \sim \text{Po}(6)$. $\text{P}(5 \le J \le 10) = \text{P}(J \le 10) - \text{P}(J \le 4)$ $= 0.672$ (3 s.f.)
	(iii) Let J be the number of people joining the self-checkout queue in an hour. Then $J \sim \text{Po}(90)$. Since $\lambda = 90 > 10$, $J \sim N(90,90)$ approximately. $\text{P}(60 < J < 80) \xrightarrow{c.c.} \text{P}(60.5 < J < 79.5) = 0.133$ (3 s.f.)

Qn	Solution
	(iv) Let L be the number of people leaving the self-checkout queue in an hour. Then $L \sim \text{Po}(78)$. Since $\lambda = 78 > 10$, $L \sim N(78, 78)$ approximately. Assume that J and L are *independent* Poisson random variables. Then $\text{Var}(J - L) = \text{Var}(J) + \text{Var}(L) = 168$ Hence $J - L \sim N(12, 168)$ approximately. $\text{P}(J - L \geq 20) \xrightarrow{c.c.} \text{P}(J - L > 19.5) = 0.281$ (3 s.f.)