CHIJ ST. THERESA'S CONVENT
PRELIMINARY EXAMINATION 2023
SECONDARY 4 EXPRESS / 5 NORMAL (ACADEMIC)

CANDIDATE NAME	

CLASS		INDEX NUMBER	

ADDITIONAL MATHEMATICS **4049/2**

Paper 2 **25 Aug 2023**
2 hours 15 minutes

Candidates answer on the Question Paper as well as on the graph paper provided.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)...(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}ab\sin C$$

1 Given that $f(x) = 6x^3 + 7x^2 - x - 2$, show that $x + 1$ is a factor of $f(x)$ and hence factorise $f(x)$ completely. [3]

$f(-1) = 6(-1)^3 + 7(-1)^2 - (-1) - 2 = -6 + 7 + 1 - 2 = 0$

By the Factor Theorem, $x + 1$ is a factor of $f(x)$ (shown)

By inspection, $f(x) = (x+1)(6x^2 + x - 2)$

$= (x+1)(2x-1)(3x+2)$

2 The equation of a circle is $(x-3)^2 + (y+4)^2 = 26$.

Determine if the origin O lies inside or outside the circle. [3]

The coordinates of the centre of the circle are (3, –4).

Distance of centre from O is $\sqrt{3^2 + 4^2} = 5$ units

Radius of circle is $\sqrt{26}$ units.

Since $5 < \sqrt{26}$, the origin lies inside the circle.

3 Solve the equation $3e^x + 2 = e^{-x}$, giving your answer(s) correct to 3 significant figures. [5]

$$3e^x + 2 = e^{-x} \quad \Rightarrow \quad 3e^x + 2 = \frac{1}{e^x}$$

Let $u = e^x$. Then $3u + 2 = \frac{1}{u}$

$$\Rightarrow \quad 3u^2 + 2u = 1$$

$$\Rightarrow \quad 3u^2 + 2u - 1 = 0$$

$$\Rightarrow \quad (3u - 1)(u + 1) = 0$$

$$\Rightarrow \quad u = \frac{1}{3} \quad \text{or} \quad u = -1$$

$$\Rightarrow \quad e^x = \frac{1}{3} \quad \text{or} \quad e^x = -1$$

There is no solution for $e^x = -1$ since $e^x > 0$ for all real values of x.

$\therefore$ the only solution is $x = \ln\left(\frac{1}{3}\right) = -1.10$ (3 s.f.)

4 Given that $y = \dfrac{16}{\sqrt[3]{5-3x}}$, find

(i) the value(s) of x for which $\dfrac{dy}{dx} = 1$, [4]

$$\frac{dy}{dx} = 16\frac{d}{dx}(5-3x)^{-\frac{1}{3}}$$

$$= 16\left[-\frac{1}{3}(5-3x)^{-\frac{4}{3}}(-3)\right]$$

$$= 16(5-3x)^{-\frac{4}{3}}$$

$$\frac{dy}{dx} = 1 \quad \Rightarrow \quad 16(5-3x)^{-\frac{4}{3}} = 1$$

$$\Rightarrow \quad (5-3x)^{\frac{4}{3}} = 16$$

$$\Rightarrow \quad (5-3x)^{\frac{1}{3}} = \pm 2$$

$$\Rightarrow \quad 5-3x = \pm 8$$

$$\Rightarrow \quad x = -1 \quad \text{or} \quad x = \frac{13}{3}$$

(ii) the value of $\int_0^1 y\,dx$, giving your answer correct to 3 significant figures. [4]

$$\int_0^1 y\,dx = 16\int_0^1 (5-3x)^{-\frac{1}{3}}\,dx$$

$$= 16\left[\frac{(5-3x)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)(-3)}\right]_0^1$$

$$= -8\left[(5-3x)^{\frac{2}{3}}\right]_0^1$$

$$= -8\left[(5-3)^{\frac{2}{3}} - (5-0)^{\frac{2}{3}}\right]$$

$$= 10.7$$

5(a) **(i)** Using the substitution $z = 1 + x$, write down all the terms in the expansion of $[1+(1+x)]^6$, leaving each term in the form $k(1+x)^p$ where k and p are integers. [2]

$$[1+(1+x)]^6 = [1+z]^6$$
$$= 1+\binom{6}{1}z+\binom{6}{2}z^2+\binom{6}{3}z^3+\binom{6}{4}z^4+\binom{6}{5}z^5+\binom{6}{6}z^6$$
$$= 1+6z+15z^2+20z^3+15z^4+6z^5+z^6$$
$$= 1+6(1+x)+15(1+x)^2+20(1+x)^3+15(1+x)^4+6(1+x)^5+(1+x)^6$$

(ii) **Hence** show that the remainder when $(2+x)^6$ is divided by $1+x$ is 1. [1]

From **(i)**,

$$(2+x)^6 = 1+(1+x)\left[6+15(1+x)+20(1+x)^2+15(1+x)^3+6(1+x)^4+(1+x)^5\right]$$

so the remainder when $(2+x)^6$ is divided by $1+x$ is 1 (shown)

(b) Given that the coefficient of x^2 in the expansion of $(1+2x)(1-px)^5$ is 20, find the two possible values of the constant p. [5]

$$(1+2x)(1-px)^5 = (1+2x)\left[1+5(-px)+\binom{5}{2}(-px)^2+\ldots\right]$$
$$= (1+2x)\left[1-5px+10p^2x^2+\ldots\right]$$

The term in x^2 is $(1)(10p^2x^2)+(2x)(-5px) = 10p^2x^2-10px^2$

The coefficient of x^2 is $10p^2-10p = 20$ (given)

$$p^2-p-2 = 0$$
$$(p+1)(p-2) = 0$$
$$p=-1 \quad \text{or} \quad p=2$$

6 At time t seconds, the velocity, v m/s, of a metal ball falling through a very thick liquid is given by $v = 8(1-e^{-t/3})$.

(i) State the initial velocity of the ball. [1]

When $t = 0$, $v = 8(1-e^{0}) = 0$ m/s, the initial velocity of the ball

The depth of the ball from the surface of the liquid at time t seconds is s metres. The ball is initially on the surface of the liquid.

(ii) Find an expression for the depth of the ball in terms of t. [4]

Displacement, $s = \int v\,dt$

$$= \int 8\left(1-e^{-t/3}\right)dt$$

$$= 8t - 8\left[\frac{e^{-t/3}}{-1/3}\right] + C$$

$$= 8t + 24e^{-t/3} + C$$

When $t = 0$, $s = 0$ so $0 = 0 + 24e^{0} + C \Rightarrow C = -24$

$\therefore \quad s = 8t + 24e^{-t/3} - 24$

(iii) Show that the ball is accelerating throughout its motion. [3]

To show ball is accelerating throughout its motion, we show $a > 0$ for all values of t.

Acceleration, $a = \frac{dv}{dt}$

$$= 8\frac{d}{dt}\left(1-e^{-t/3}\right)$$

$$= 8\left[-e^{-t/3}\left(-\frac{1}{3}\right)\right]$$

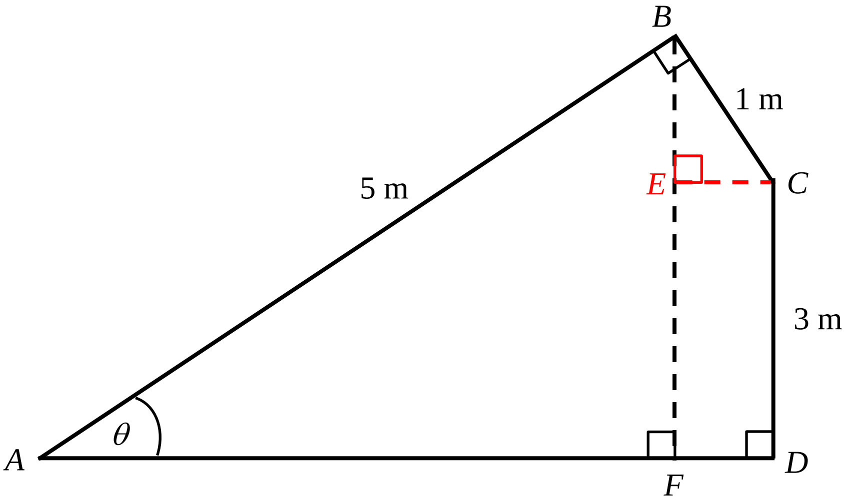
$$= \frac{8}{3}e^{-t/3}$$

Since $e^{-t/3} > 0$ for all real values of t, $a > 0$ for all real values of t so the ball is accelerating throughout its motion.

(iv) Deduce the speed of the ball for large values of t. [1]

When t is large, $e^{-t/3} \approx 0$ so $v \approx 8(1 - 0) = 8$.
For large values of t, the ball moves at an almost constant speed of 8 m/s.

7



The diagram shows a quadrilateral *ABCD* with angles *ABC* and *ADC* both equal to $90°$. Angle $BAD = \theta$. The lengths of *AB*, *BC* and *CD* are 5 m, 1 m and 3 m respectively. The point *F* on *AD* is such that *BF* is perpendicular to *AD*.

(i) Show that the length of *BF* is $\cos\theta + 3$. [2]

$\angle ABF = 90° - \theta$

so $\angle CBF = 90° - \angle ABF = 90° - (90° - \theta) = \theta$

Hence $BF = BE + EF$

$= BE + CD$

$= 1\cos\theta + 3$ (shown)

(ii) Hence show that $5\sin\theta - \cos\theta = 3$. [2]

In ΔABF, $BF = 5\sin\theta$

Hence $\cos\theta + 3 = 5\sin\theta$

$\Rightarrow 5\sin\theta - \cos\theta = 3$ (shown)

(iii) By expressing $5\sin\theta - \cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, find the value of θ. [4]

$$R = \sqrt{5^2 + 1^2} = \sqrt{26} \quad \text{and} \quad \alpha = \tan^{-1}\left(\frac{1}{5}\right) = 11.310° \quad \text{(5 s.f.)}$$

Hence $\sqrt{26}\sin(\theta - 11.310°) = 3$

$$\Rightarrow \quad \sin(\theta - 11.310°) = \frac{3}{\sqrt{26}}$$

$$\Rightarrow \quad \theta - 11.310° = \sin^{-1}\frac{3}{\sqrt{26}} = 36.040°$$

$$\Rightarrow \quad \theta = 47.3° \quad \text{(3 s.f.)}$$

8 The equation of a curve C_1 is $y = 6x - 3x^2$.

(i) Show that the coordinates of the maximum point of the curve C_1 are (1, 3). [2]

$$y = 6x - 3x^2 = -3(x^2 - 2x)$$
$$= -3[(x-1)^2 - (1)^2]$$
$$= -3(x-1)^2 + 3$$

The coordinates of the maximum point are (1, 3).

(ii) Using your answer in part **(i)**, state the range of values of k for which the curve C_1 lies entirely below the line $y = k$. [1]

Since the maximum value of y is 3, the line is entirely above the curve if $k > 3$.

The curve C_2 is the reflection of the curve C_1 in the line $y = k$, where k takes one of the values found in part **(ii)**.

(iii) Deduce the equation of the curve C_2, giving your answer in terms of k. [3]

Since k takes one of the values found in part **(ii)**, the curve lies entirely below $y = k$.

We just need the minimum point of C_2 to obtain the equation of C_2.

Let the maximum point of C_1 and the minimum point of C_2 be A and B respectively.

The coordinates of A are (1, 3) so the coordinates of the midpoint M of AB are (1, k).

Let the coordinates of B be (1, b).

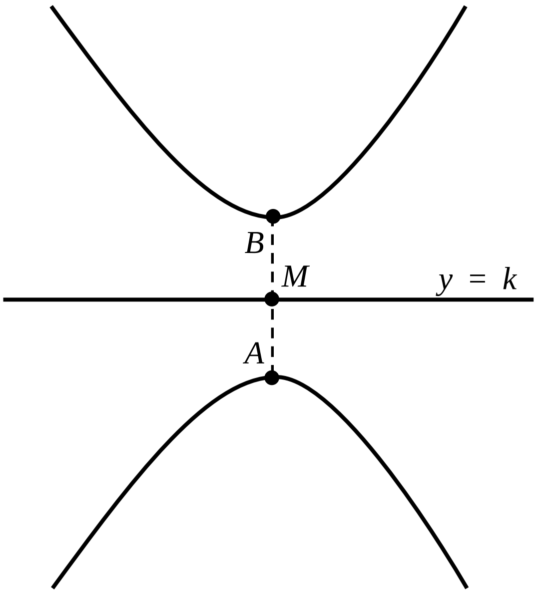
$$k = \frac{3+b}{2}$$

so $$b = 2k - 3$$

Hence the coordinates of B are (1, $2k - 3$).

The equation of C_2 is

$$y = 3(x-1)^2 + 2k - 3$$



Another line $y = x + a$ is a tangent to the curve C_1 at the point P.

(iv) Find the value of the constant a. [3]

$$6x - 3x^2 = x + a \quad \Rightarrow \quad 3x^2 - 5x + a = 0$$

Since the line is a tangent to the curve, this quadratic has real and equal roots.

$$\text{Discriminant,} \quad D = 0 \quad \Rightarrow \quad (-5)^2 - 4(3)(a) = 0$$

$$\Rightarrow \quad a = \frac{25}{12}$$

(v) Find the coordinates of P. [2]

When $a = \frac{25}{12}$, the quadratic equation becomes $3x^2 - 5x + \frac{25}{12} = 0$

The (equal) roots are $x = \frac{-(-5) \pm \sqrt{0}}{2(3)}$ so $x = \frac{5}{6}$

When $x = \frac{5}{6}$, $y = \frac{5}{6} + \frac{25}{12} = \frac{35}{12}$

$\therefore$ the coordinates of P are $\left(\frac{5}{6}, \frac{35}{12}\right)$.

9 The equation of a circle is $x^2 + y^2 - 4x - 12y + 36 = 0$. The equation of the line L is $y = kx$, where k is a constant.

(i) Find the radius of the circle and the coordinates of its centre. [4]

$$x^2 - 4x + y^2 - 12y + 36 = 0$$

$$\Rightarrow \quad (x-2)^2 - 2^2 + (y-6)^2 - 6^2 + 36 = 0$$

$$\Rightarrow \quad (x-2)^2 + (y-6)^2 = 4$$

Radius is 2 units and centre is at (2, 6).

(ii) State the equations of the horizontal tangents to the circle. [2]

The equations are: $y = 6 - 2 = 4$

and $y = 6 + 2 = 8$

(iii) Find the range of values of k for which L does not intersect the circle. [4]

Substitute $y = kx$ into the equation of the circle:

$$x^2 - 4x + (kx)^2 - 12(kx) + 36 = 0$$

$$(1 + k^2)x^2 - (4 + 12k)x + 36 = 0$$

Given that the line does not intersect the circle, equation has no real solution so

Discriminant, $D = [-(4+12k)]^2 - 4(1+k^2)(36) < 0$

$$16 + 96k + 144k^2 - 144 - 144k^2 < 0$$

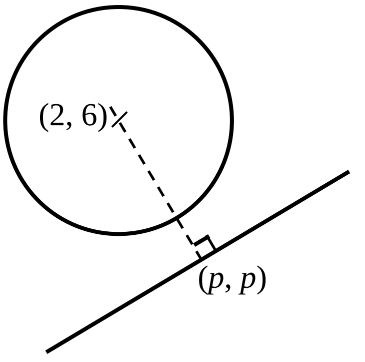
$$96k - 128 < 0$$

$$k < \frac{4}{3}$$

It is given that $k = 1$. The point on the line L that is closest to the circle is P.

(iv) Find the coordinates of P. [3]

Since $k = 1$ lies in the range of k found in part **(iii)**, the line L does not intersect the circle.



Let the coordinates of P be (p, p).

Since P is point on L closest to circle,

$CP \perp L$ where C is centre of circle.

Gradient of L is 1 so

$$\text{gradient of } CP \text{ is } -\frac{1}{1} = -1$$

$$\frac{p-6}{p-2} = -1 \quad \Rightarrow \quad p - 6 = 2 - p$$

$$\Rightarrow \quad 2p = 8$$

$$\Rightarrow \quad p = 4$$

$\therefore$ the coordinates of P are (4, 4).

10 **(i)** Show that $\frac{d}{dx}\left[x e^{-\frac{1}{4}x}\right] = \left(1-\frac{1}{4}x\right)e^{-\frac{1}{4}x}$. [2]

$$\frac{d}{dx}\left[x e^{-\frac{1}{4}x}\right] = x\frac{d}{dx}\left(e^{-\frac{1}{4}x}\right) + e^{-\frac{1}{4}x}\frac{d}{dx}(x)$$

$$= x\left(-\frac{1}{4}e^{-\frac{1}{4}x}\right) + e^{-\frac{1}{4}x}$$

$$= \left(1-\frac{1}{4}x\right)e^{-\frac{1}{4}x}$$

(ii) Hence show that $\int x e^{-\frac{1}{4}x}\,dx = -16e^{-\frac{1}{4}x} - 4x e^{-\frac{1}{4}x} + C$ where C is a constant. [3]

From **(i)**,

$$\int\left[\left(1-\frac{1}{4}x\right)e^{-\frac{1}{4}x}\right]dx = x e^{-\frac{1}{4}x} + C'$$

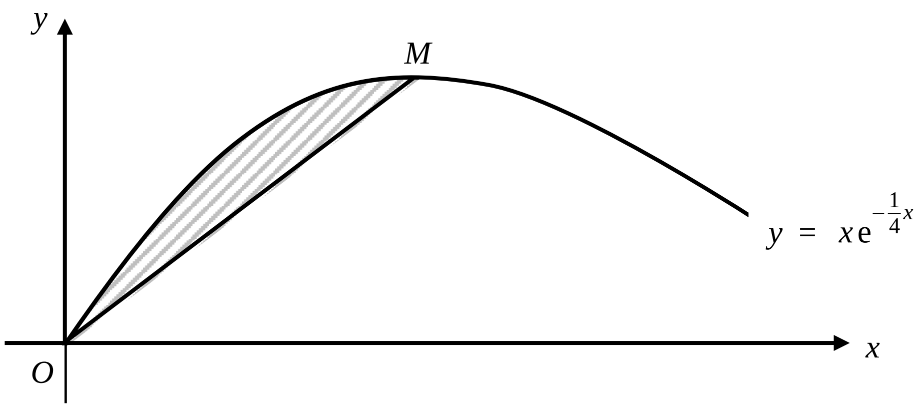
$$\int e^{-\frac{1}{4}x}\,dx - \frac{1}{4}\int x e^{-\frac{1}{4}x}\,dx = x e^{-\frac{1}{4}x} + C'$$

$$\frac{e^{-\frac{1}{4}x}}{-\frac{1}{4}} - \frac{1}{4}\int x e^{-\frac{1}{4}x}\,dx = x e^{-\frac{1}{4}x} + C'$$

$$\frac{1}{4}\int x e^{-\frac{1}{4}x}\,dx = -4e^{-\frac{1}{4}x} - x e^{-\frac{1}{4}x} + C''$$

$$\int x e^{-\frac{1}{4}x}\,dx = -16e^{-\frac{1}{4}x} - 4x e^{-\frac{1}{4}x} + C$$

(iii)



The diagram shows part of the curve $y = xe^{-\frac{1}{4}x}$. The point M is the maximum point of the curve and OM is a straight line.

Show that the area of the shaded region is $\left(16 - \frac{40}{e}\right)$ units^2. [7]

At point M, $\frac{dy}{dx} = 0$

From part **(i)**, $\frac{d}{dx}\left(xe^{-\frac{1}{4}x}\right) = \left(1 - \frac{1}{4}x\right)e^{-\frac{1}{4}x}$

$\left(1 - \frac{1}{4}x\right)e^{-\frac{1}{4}x} = 0 \quad \Rightarrow \quad 1 - \frac{1}{4}x = 0$ since $e^{-\frac{1}{4}x} \neq 0$ for all real x

$\Rightarrow \quad x = 4$

When $x = 4$, $\quad y = 4e^{-\frac{1}{4}(4)} = \frac{4}{e}$

Area bounded by curve and line $= \int_0^4 \left(xe^{-\frac{1}{4}x}\right) dx$ 1 – (area of triangle)

$= \left[-16e^{-\frac{1}{4}x} - 4xe^{-\frac{1}{4}x}\right]_0^4 - \frac{1}{2}(4)\left(\frac{4}{e}\right)$

$= \left[-16e^{-\frac{1}{4}(4)} - 4(4)e^{-\frac{1}{4}(4)}\right] - \left[-16e^{-\frac{1}{4}(0)}\right]$

$= \left(-\frac{16}{e} - \frac{16}{e} - \frac{8}{e}\right) + 16$

$= \left(16 - \frac{40}{e}\right)$ units^2 (shown)

Alternative method:

Gradient of line is $\frac{4/e - 0}{4 - 0} = \frac{1}{e}$ so equation of line is $y = \frac{1}{e}x$.

Area bounded by curve and line $= \int_0^4 \left(xe^{-\frac{1}{4}x} - \frac{1}{e}x\right) dx$

$= \left[-16e^{-\frac{1}{4}x} - 4xe^{-\frac{1}{4}x} - \frac{1}{2e}x^2\right]_0^4$

$= \ldots = \left(-\frac{16}{e} - \frac{16}{e} - \frac{8}{e}\right) + 16 = \left(16 - \frac{40}{e}\right)$ units^2 (shown)

11 The temperature, $\theta°C$, of a liquid placed in a container can be modelled by an equation of the form $\theta = 32 + ae^{-bt}$, where a and b are constants and t is the time in minutes that the liquid has been left in the container. The table below records the value of θ for various values of t.

t minutes	0	10	20	30
$\theta°C$	120.0	107.6	97.7	88.0

(i) On the grid given, plot $\ln(\theta - 32)$ against t and draw a straight line graph. [3]

(ii) Use the graph to estimate the value of each of the constants a and b. [5]

$\theta = 32 + ae^{-bt}$ $\Rightarrow$ $\theta - 32 = ae^{-bt}$

$\Rightarrow$ $\ln(\theta - 32) = \ln a + \ln e^{-bt} = (-b)t + \ln a$

From the graph,

Vertical intercept = 4.475 $\Rightarrow$ $\ln a = 4.475$

$\Rightarrow$ $a = 87.8$ (3 s.f.)

[accept $83.4 \le a \le 92.2$]

Gradient = $\frac{4.05 - 4.4}{27 - 5}$ $\Rightarrow$ $-b = -0.015909$

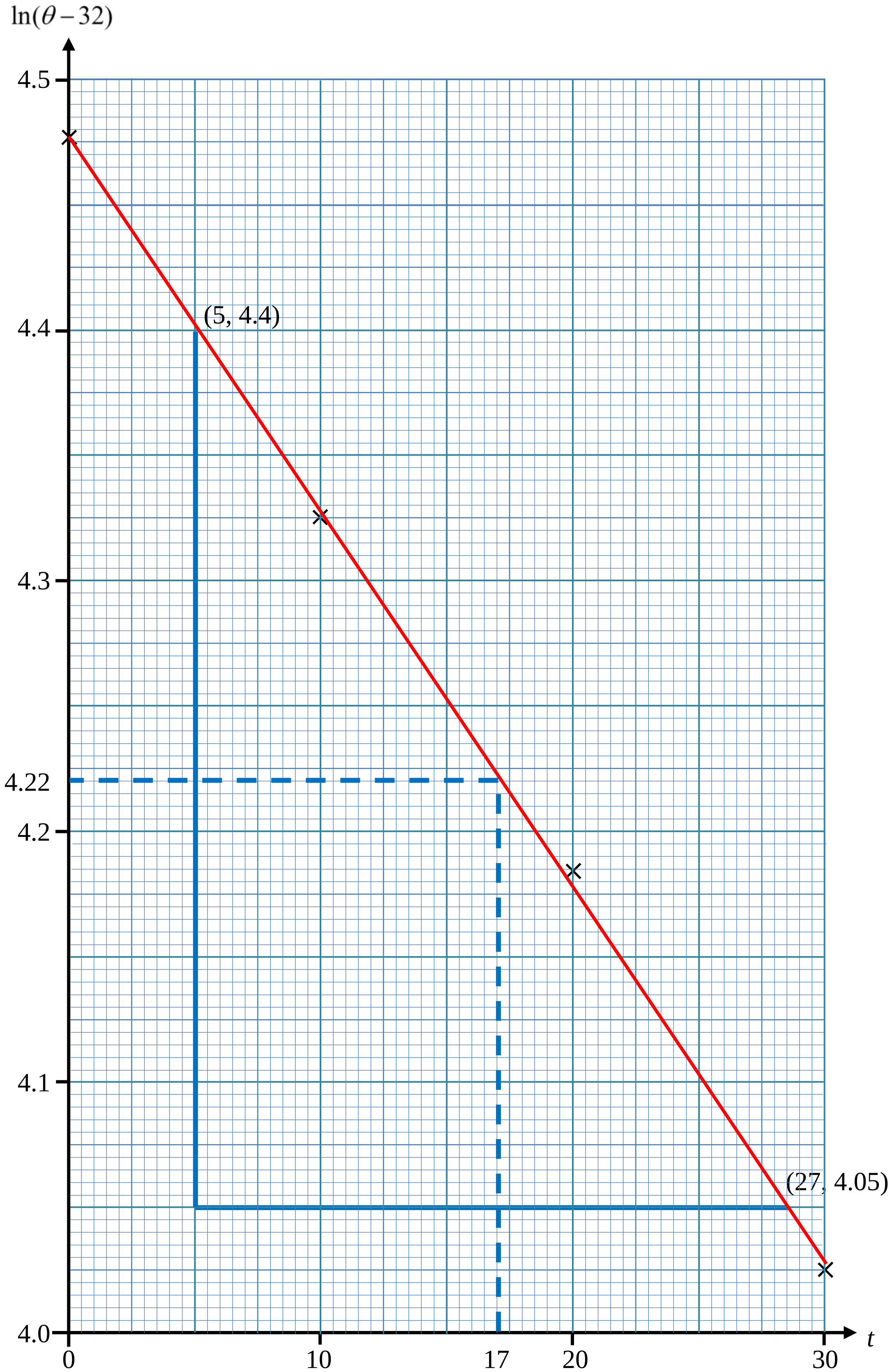
$\Rightarrow$ $b = 0.0159$ (3 s.f.)

[accept $0.015 \le b \le 0.017$]

(v) Use the graph to estimate when the temperature of the liquid drops to $100°C$. [2]

$\theta = 100°$ $\Rightarrow$ $\ln(\theta - 32) = \ln(100 - 32) = 4.219$

From the graph, the corresponding value of t is 17.



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