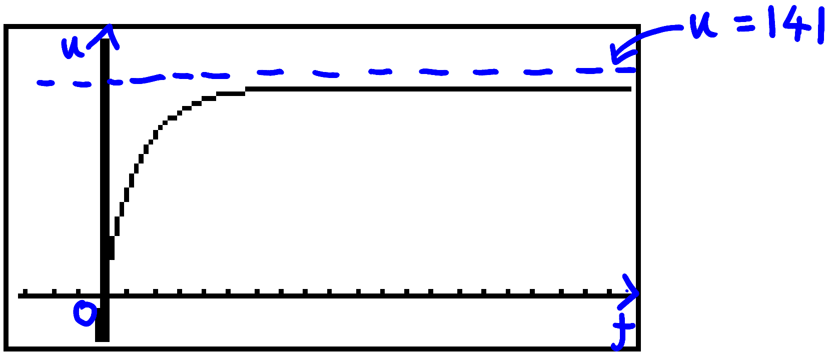
Anderson Junior College
Preliminary Examination 2011
H2 Mathematics Paper 2 (Solutions)

Section A: Pure Mathematics [40 marks]

1	(i)	$\frac{\mathrm{d}u}{\mathrm{d}t}=100-ku-3$ $=97-ku\ ,\ k>0$ $\int\frac{1}{97-ku}\,\mathrm{du}=\int 1\ \mathrm{d}t$ $\Rightarrow -\frac{1}{k}\ln\lvert 97-ku\rvert=t+c$ $\Rightarrow \ln\lvert 97-ku\rvert=-kt-kc$ $\Rightarrow 97-ku=A\mathrm{e}^{-kt}$ where $A=\pm e^{-kc}$ $\Rightarrow u=\frac{1}{k}\left(97-A\mathrm{e}^{-kt}\right)$ (shown)
	(ii)	For the processing tank, when t = 0, u = 0. Hence, $\frac{1}{k}\left(97-A\right)=0\Rightarrow A=97$. Therefore, $u=\frac{1}{k}\left(97-97\mathrm{e}^{-kt}\right)$. Since u = 70 when t = 1, we have $70=\frac{1}{k}\left(97-97\mathrm{e}^{-k}\right)\Rightarrow k=0.69220$ (5 s.f. from GC) Amount of water in the processing container when t = 2 $=\frac{1}{0.6922}\left(97-97\mathrm{e}^{-2(0.6922)}\right)$ = 105.03 litres Amount of water pumped in when t =2 = 200 litres Hence, amount of water processed = 200 – 105.03 - 6 litres = 89.0 litres (3 s.f.)

(iii)



As $t \to +\infty$, $u = \dfrac{1}{0.6922}\left(97 - 97e^{-0.6922t}\right) \to \dfrac{97}{0.6922} \approx 140.13$ litres

The minimum volume of the tank is 141 litres.

2

(a)

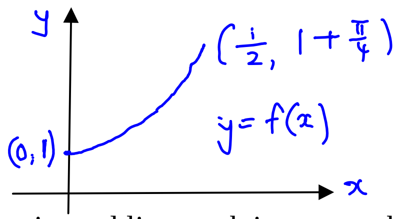
(i) <u>Method 1 (using differentiation)</u>

$$f'(x) = (x+1)\left(\frac{1}{\sqrt{1-x^2}}\right) + \sin^{-1} x$$

For $0 \le x \le \dfrac{1}{2}$, $f'(x) > 0 \quad \Rightarrow \quad f(x)$ is a strictly increasing function and therefore f is a 1 to 1 function.

Since f is 1-1, the inverse function f^{-1} exist.

<u>Method 2 (using graphical test):</u>



Any horizontal line y = k intersects the graph of f at **at most** one point $\Rightarrow$ f is a one-one function $\Rightarrow$ the inverse function f^{-1} exist.

(Or : any horizontal line $y = k$, $0 \le k \le 1 + \dfrac{\pi}{4}$, intersects the graph of f at **exactly one** point $\Rightarrow$ f is a one-one function $\Rightarrow$ the inverse function f^{-1} exist)

To find the value of $f^{-1}(1)$, solve $f(x) = 1$

$(x+1)\sin^{-1} x + 1 = 1$

$x = -1$ (rejected since $x \ge 0$) or $x = 0$

$\therefore f^{-1}(1) = 0$

(ii) At $x = 0$, $f'(0) = 1$ and $y = 1$

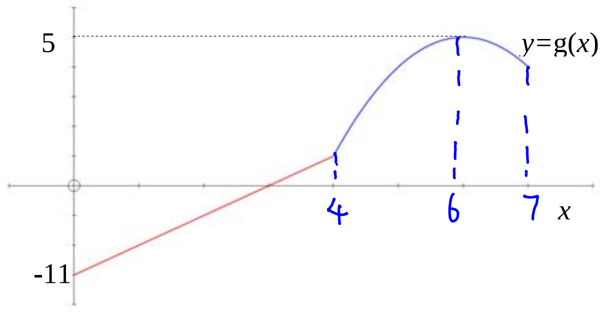
Equation of tangent to the curve $y = f(x)$ at $x = 0$: $y = x + 1$

(iii) At $x = 1$, we will use the symmetry properties of the curve of f and f^{-1} along $y = x \Rightarrow$

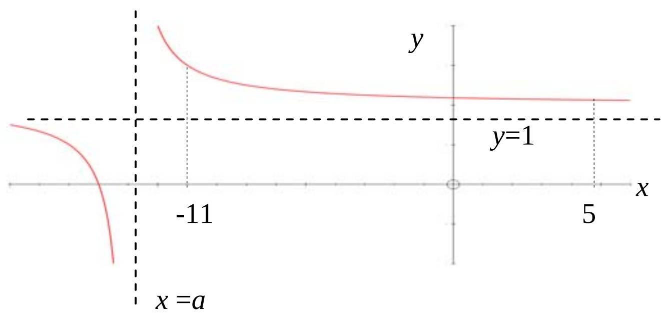
Equation of tangent to the curve $y = f^{-1}(x)$ at $x = 1$: $x = y + 1 \;\Rightarrow\; y = x - 1$

2 (b)

(b) (i) Range of g = [-11,5]



(ii) Different values of a will translate the curve $y=1+\dfrac{1}{x}$ along the x-axis by a units to give $y=1+\dfrac{1}{x-a}$



Domain of g $\xrightarrow{g}$ [-11, 5] $\xrightarrow{h}$ Range of hg

Since domain of h = [-11, 5] and we need range of hg to be subset of (1, ∞), therefore [-11, 5] must fall in the interval (a, ∞)

From the diagram, it can be observed that the greatest value of a is -12.

3

$$\text{(i)}\quad \begin{pmatrix}2\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}\lambda\\0\\2\end{pmatrix}=\sqrt{2^2+(-3)^2+1^2}\sqrt{\lambda^2+2^2}\cos\left(\cos^{-1}\left(-\frac{3}{\sqrt{70}}\right)\right)$$

$$2\lambda+2=-\frac{3}{\sqrt{70}}\sqrt{14}\sqrt{\lambda^2+2^2}$$

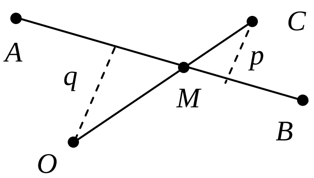
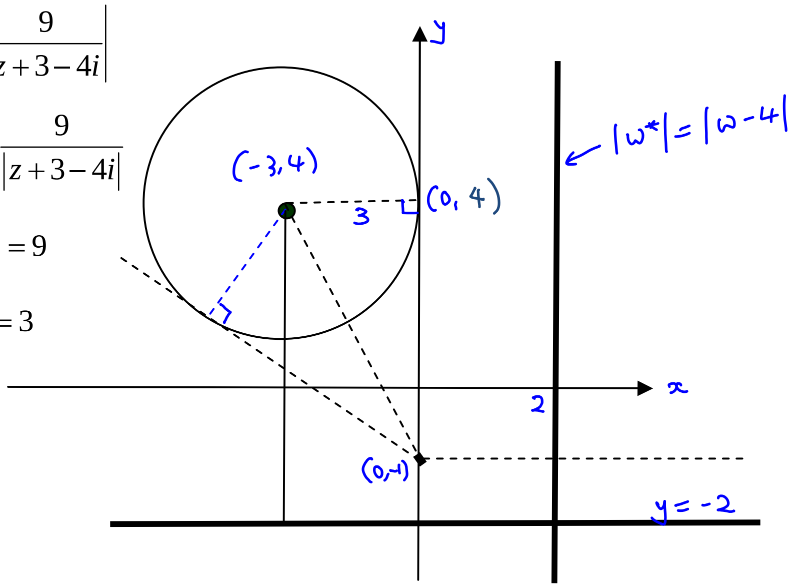
$$5\left(2\lambda+2\right)^2=9\left(\lambda^2+2^2\right)$$

$$20\left(\lambda^2+2\lambda+1\right)=9\left(\lambda^2+2^2\right)$$

$$11\lambda^2+40\lambda-16=0$$

$$(11\lambda-4)(\lambda+4)=0$$

$\lambda=-4$ as $\lambda=\dfrac{4}{11}$ (rejected as $\lambda<0$)

	(ii) Using ratio theorem, $\overrightarrow{OM}=\dfrac{3\overrightarrow{OB}+\overrightarrow{OA}}{4}=\dfrac{1}{4}\left(3\begin{pmatrix}-4\\0\\2\end{pmatrix}+\begin{pmatrix}2\\-3\\1\end{pmatrix}\right)=\dfrac{1}{4}\begin{pmatrix}-10\\-3\\7\end{pmatrix}$ $\overrightarrow{OC}=\dfrac{4}{3}\overrightarrow{OM}=\dfrac{4}{3}\left(\dfrac{1}{4}\begin{pmatrix}-10\\-3\\7\end{pmatrix}\right)=\dfrac{1}{3}\begin{pmatrix}-10\\-3\\7\end{pmatrix}$
	(iii) p represents the perpendicular distance of C from the line AB (or p is the height of the triangle ABC with AB as its base). $\overrightarrow{AC}=\dfrac{1}{3}\begin{pmatrix}-10\\-3\\7\end{pmatrix}-\begin{pmatrix}2\\-3\\1\end{pmatrix}=\dfrac{1}{3}\begin{pmatrix}-16\\6\\4\end{pmatrix}$, $\overrightarrow{AB}=\begin{pmatrix}-4\\0\\2\end{pmatrix}-\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}-6\\3\\1\end{pmatrix}$ $p=\dfrac{\left\|\overrightarrow{AC}\times\overrightarrow{AB}\right\|}{\left\|\overrightarrow{AB}\right\|}=\dfrac{\left\|\dfrac{1}{3}\begin{pmatrix}-16\\6\\4\end{pmatrix}\times\begin{pmatrix}-6\\3\\1\end{pmatrix}\right\|}{\sqrt{36+9+1}}=\dfrac{1}{3}\dfrac{\left\|\begin{pmatrix}-6\\-8\\-12\end{pmatrix}\right\|}{\sqrt{46}}=\dfrac{1}{3}\sqrt{\dfrac{122}{23}}$  $\dfrac{q}{p}=\dfrac{OM}{MC}=\dfrac{3}{1}$ $q=3p=\sqrt{\dfrac{122}{23}}$
4	(i) $\lvert iz+4+3i\rvert=\left\lvert\dfrac{9}{z+3-4i}\right\rvert$ $\Rightarrow \lvert i\rvert\lvert z-4i+3\rvert=\dfrac{9}{\lvert z+3-4i\rvert}$ $\Rightarrow \lvert z-(-3+4i)\rvert^2=9$ $\Rightarrow \lvert z-(-3+4i)\rvert=3$ 

	(ii) max arg (z + i) $= \dfrac{\pi}{2} + 2\sin^{-1}\left(\dfrac{3}{\sqrt{3^2+5^2}}\right)$ $= 2.65$ $\therefore \ \dfrac{\pi}{2} \le \arg\left(z+i\right) \le 2.65$
	(iii) $\left\|w^*\right\| = \left\|w-4\right\| \ \Rightarrow \left\|w\right\| = \left\|w-4\right\|$ - locus is the line x = 2 [1] $\left\|z+iw\right\| = \left\|z-(-iw)\right\|$ = distance between locus of z (circle) and the locus of points representing – iw . Locus of – iw is the line y = -2 Therefore least value of $\left\|z+iw\right\| = AB = 3$
5	(i) c = 0 as the line l is on π_2. [B1] (ii) Since the line l is on π_2, l is perpendicular to the normal of π_2. $\begin{pmatrix}1\\a\\-1\end{pmatrix} \cdot \begin{pmatrix}b\\5\\-1\end{pmatrix} = 0$ $b+5a+1=0$ $b+5a=-1$ (shown)
	(iii) Method 1: $\begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}1\\a\\-1\end{pmatrix} = \begin{pmatrix}-2\\1\\a-2\end{pmatrix}$ is a vector $\perp$ to π_1 $\begin{pmatrix}-2\\1\\a-2\end{pmatrix} \bullet \begin{pmatrix}b\\5\\-1\end{pmatrix} = 0 \Rightarrow -2b+5-a+2=0 \Rightarrow a+2b=7$ Method 2: $\begin{pmatrix}1\\a\\-1\end{pmatrix} \times \begin{pmatrix}b\\5\\-1\end{pmatrix} = \begin{pmatrix}5-a\\1-b\\5-ab\end{pmatrix}$ is a vector $\perp$ to π_1 $\begin{pmatrix}5-a\\1-b\\5-ab\end{pmatrix} \bullet \begin{pmatrix}1\\2\\0\end{pmatrix} = 0 \Rightarrow 5-a+2-2b=0 \Rightarrow a+2b=7$

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Method 3: $\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}b\\5\\-1\end{pmatrix}=\begin{pmatrix}-2\\1\\5-2b\end{pmatrix}$ is a vector $\perp$ to π_1

$$\begin{pmatrix}-2\\1\\5-2b\end{pmatrix}\bullet\begin{pmatrix}1\\a\\-1\end{pmatrix}=0\Rightarrow -2+a-5+2b=0\Rightarrow a+2b=7$$

Method 4:

$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}1\\a\\-1\end{pmatrix}=\begin{pmatrix}-2\\1\\a-2\end{pmatrix}$ is a vector $\perp$ to π_1

$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}b\\5\\-1\end{pmatrix}=\begin{pmatrix}-2\\1\\5-2b\end{pmatrix}$ is a vector $\perp$ to π_1

$$\therefore\begin{pmatrix}-2\\1\\a-2\end{pmatrix}=k\begin{pmatrix}-2\\1\\5-2b\end{pmatrix}$$

$\therefore k=1$ and $a-2=5-2b$

$\therefore a+2b=7$

Using $b=-5a-1$ from part (ii) and solving simultaneous equation,

we get a = -1 and b = 4.

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(iv) Possible answers are: $\mathbf{r}\cdot\begin{pmatrix}4\\5\\-1\end{pmatrix}=4\sqrt{42}$ or $\mathbf{r}\cdot\begin{pmatrix}4\\5\\-1\end{pmatrix}=-4\sqrt{42}$

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Section B (Statistics) [60 marks]

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There are 9 distinct letters: C, O (2), R (2), E, L, A, T, I, N

(i) Treat all the letters as distinct then arrange them according to the restriction. This will be followed by taking the repeated letters into account.

Number of arrangements $=\dfrac{5\times 9!\times 6}{2!2!}=2721600$

(ii) <u>Case 1: All distinct</u>

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No. of ways = $^{9}P_{6} = 60480$

<u>Case 2: 1 pair of repeated letters</u>

No. of ways = $^{2}C_{1} \times {}^{8}C_{4} \times \dfrac{6!}{2!} = 50400$

<u>Case 3: 2 pairs of repeated letters</u>

No. of ways = $^{7}C_{2} \times \dfrac{6!}{2!2!} = 3780$

Total no. of ways $= 60480 + 50400 + 3780 = 114660$
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(i) Let X be the number of lemon candies in a randomly selected packet of 20.
$X \sim B(20, 0.24)$

$E(X) = 20 \times 0.24 = 4.8$
$Var(X) = 20 \times 0.24 \times 0.76 = 3.648$

Since $n = 60$ (> 50), by Central Limit Theorem, $\overline{X} \sim N\left(4.8, \dfrac{3.648}{60}\right)$.

$P(\overline{X} \geq 5) = 0.20865208$ =0.209 (3sf)
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(ii) The sample is biased, as only students are surveyed. Not everyone in the population has an equal chance of being surveyed.

It will be difficult to get an exhaustive list of people of all age groups to do a proper stratification. (no sampling frame)
Use Quota Sampling
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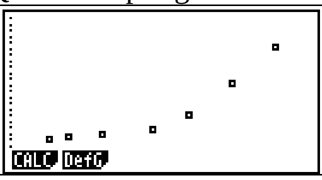
(i) 

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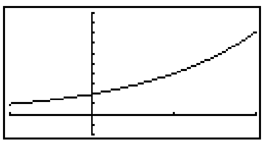
(ii) From GC, the product moment correlation coefficient = 0.940 (3 sig fig)



(iii) The points in the scatter plot fits closely to the curve $x = Ae^{Bt}$ (left) and the *r* value is close to 1. Thus this model is suitable.

Transformation of $x = Ae^{Bt}$ into a linear model: $\ln x = \ln A + Bt$
From GC, $\ln A = 1.71086 \Rightarrow A = 5.53, \quad B = 0.0476$
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(iv) If *t* is increased by 5, the increase in *x* is as follows:

$x_o = Ae^{Bt}$

$x_1 = Ae^{B(t+5)} = \left(e^{5B}\right)x_o = \left(e^{5(0.0476)}\right)x_o = (1.269)x_o$

The rate of chirps is estimated to increase by 26.9% from the rate before the temperature is increased.

The estimate is not reliable because we are extrapolating beyond the region where the
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	data is collected and analyzed. (v) The value of r remains the same since it is not affected by any translation or scaling.
9	$A \sim N(45000, 2000^2)$ $\quad\quad$ $B \sim N(30000, 1850^2)$ (i) $2A-(B_1+B_2) \sim N(30000, 22845000)$ $P\left(2A-(B_1+B_2) > 25000\right) = 0.85224 = 0.852$ (3 sig fig) Assumption: The distributions of the lifespans of all televisions are independent of each other. (ii) Let W denotes the number of plasma televisions out of 50 with a life span of more than 30000 hours. $W \sim B(50, 0.5)$ $P(14 < W < 22) = P(15 \le W \le 21)$ $= P(W \le 21) - P(W \le 14)$ $= 0.16112 - 0.0013011$ $= 0.15982$ $= 0.160$
	(iii) $\overline{X} \sim N\left(45000, \dfrac{2000^2}{n}\right)$ $P(\overline{X} \le 46500) \ge 0.99 \Rightarrow P\left(Z \le \dfrac{46500-45000}{\frac{2000}{\sqrt{n}}}\right) \ge 0.99$ $\Rightarrow P\left(Z \le \dfrac{3\sqrt{n}}{4}\right) \ge 0.99$ $\Rightarrow \dfrac{3\sqrt{n}}{4} \ge 2.32635$ $\Rightarrow n \ge 9.6212$ $\therefore$ Least $n = 10$ (iv) (I) will be greater as A > 25000 and B > 25000 is a subset of A+B > 50000.
10	$X \sim P_o(\lambda)$ (i) $P(X=2) = 10P(X=3) \Rightarrow \dfrac{\lambda^2 e^{-\lambda}}{2!} - \dfrac{10\lambda^3 e^{-\lambda}}{3!} = 0$ $\Rightarrow \dfrac{\lambda^2 e^{-\lambda}}{2!}\left(1-\dfrac{10\lambda}{3}\right) = 0$ $\Rightarrow \mathrm{E}(X) = \lambda = \dfrac{3}{10} = 0.3$ (shown)
	(ii) Let U denotes the total number of errors in a randomly selected textbook. $U \sim P_o(15)$ $P(U=10) = 0.04861075$
	(iii) Let A and B denotes the total number of errors in the pages 1 to 10 plus 41 to 50 and 11 to 40 respectively.

	$A \sim P_o(6)$ and $B \sim P_o(9)$, Probability $= \frac{P(A=1)P(B=9)+P(A=0)P(B=10)}{P(U=10)} = \frac{(0.01487251)(0.13175564)+(0.0024788)(0.11858007)}{0.04861075}$ = 0.046357=0.0464 (3 sig fig)
	(iv) $P(U>18)=1-P(U\le 18)=0.18053$ Let x be the minimum selling price, in order to make a profit, $x \ge 1.15(12+12P(U>18))$ $x \ge 16.291$ Minimum selling price is $16.30.
	(v) Let V be the r.v denoting the number of textbooks that has more than 18 errors out of 50 books. $V \sim B(50, 0.18053)$ Since n is large and np > 5 and nq > 5, $V \sim N(9.0265, 7.3969)$ approximately $\mathrm{P}\left(V<6\right)=\mathrm{P}(V\le 5)=\mathrm{P}(V<5.5)=0.0974$
11	(i) For the die, let x = P(1) = P(3) = P(5). $\Rightarrow x+2x+x+2x+x+2x=1$ $\Rightarrow x=\frac{1}{9} \Rightarrow \mathrm{P}(1)=\frac{1}{9}$ (shown)
	P(1) = P(3) = P(5) = $\frac{1}{9}$ and P(2) = P(4) = P(6) = $\frac{2}{9}$ (ii) For the coin, P('tail') = $\frac{1}{3}$ and P('head') = $\frac{2}{3}$ P(the coin to show 'tail' and the die to show a number that is at most 4) = P('tail',1) + P('tail',2) + P('tail',3) + P('tail',4) $=\frac{1}{3}.\frac{1}{9}+\frac{1}{3}.\frac{2}{9}+\frac{1}{3}.\frac{1}{9}+\frac{1}{3}.\frac{2}{9}$ $=\frac{2}{9}$ or independence P('tail') [P(1)+ P(2)+ P(3)+ P(4)]
	(iii) P(coin shows a 'tail' or the die shows a number that is at most 4, or both) = P(coin shows a 'tail')+ P(die shows a number that is at most 4) – P('tail' and at most 4) $=\frac{1}{3}+\frac{6}{9}-\frac{2}{9}$ $=\frac{7}{9}$
	(iv) P(coin shows a head if the score is at most 4) $= P(coin\ is\ \text{'tail'} \mid score\ is\ at\ \text{most } 4)$ $= \frac{P(coin\ is\ \text{'tail' and } score\ is\ at\ \text{most } 4)}{score\ is\ at\ \text{most } 4}$

	$= \dfrac{\frac{2}{9}}{P(\text{'tail'},1 \text{ or 'tail'},2 \text{ or 'tail'},3 \text{ or 'tail'},4 \text{ or 'head'},1 \text{ or 'head'},2)}$ $= \dfrac{\frac{2}{9}}{\frac{2}{9}+\frac{2}{3}.\frac{1}{9}+\frac{2}{3}.\frac{2}{9}}$ $= \dfrac{1}{2}$
	(v) For each throw, Mary is expected to gain $\$\left[-1.\left(\frac{4}{9}\right)+0.5\left(\frac{5}{9}\right)\right] = -\$\frac{1}{6}$ Therefore, Mary is not expected to make a profit as she is expected to lose \$10.00 in the game.
12	(i) Unbiased estimate of the population mean $= \bar{x} = \dfrac{44.8}{12} = 3.7333333$=3.73 (3sf) Unbiased estimate of the population variance $= \dfrac{\sum\left(x-\bar{x}\right)^2}{11} = \dfrac{1.9467}{11} = 0.1769727273$ = 0.177 (3sf)
	(ii) Let X be the time, in hours, that a randomly chosen customer spends shopping at Takayama Shopping Complex, and μ be the mean time. Test $H_o : \mu = 4$ (manager's claim) vs $H_1 : \mu \neq 4$ (superior's suspicion) Assume that ***X*** is normally distributed. Since $n = 12$ is small, and population variance is unknown, we use a T Test. Test Statistic: $T = \dfrac{\overline{X}-4}{s/\sqrt{n}} \sim t(11)$ Use a two-tailed test at 6%, and reject H_o if $p < 0.06$. Using GC, with $\bar{x} = \dfrac{56}{15}$, $s = \sqrt{0.1769727273}$, n=12 $p = 0.0504 < 0.06$ We reject H_o. There is sufficient evidence at 6% significance level to confirm the company superior's suspicion.
	(iii) Since the value of *p* in part (ii) is less than 6%, the new value of *p* would be half of the value in (ii), therefore it would still be less than 6%. Thus, the conclusion would remain the same. OR Since Tcalc < Tcritical in part (ii), and now Tcritical (new) > Tcrictical, this implies that Tcalc < Tcritical (new). Thus conclusion will remain the same.

(iv) Test $H_o : \mu = \mu_0$ (manager's claim)

vs $H_1 : \mu < \mu_0$ (manager overstated the average time)

Test Statistic:

Since X is normally distributed, and population variance is known, we use a Z Test.

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

Use a one-tailed test at 6%, and reject H_o if $p < 0.06$.

For manager to justify that he has not overstated the average time,

i.e. do not reject H_0 at 6% (z calc does not lie in critical region)

$$z_{calc} = \frac{4.2375 - \mu_0}{0.466/\sqrt{8}} > -1.5547736$$

$$4.2375 - \mu_0 > -1.5547736\left(\frac{0.466}{\sqrt{8}}\right)$$

$\mu_0 < 4.493658$ hrs

$\mu_0 < 4$ hrs 29.62 mins

Largest possible value of μ_0 is 4 hours 29 mins