Chapter 2: Graphing Techniques I

Content requirement
Include: • use of a graphic calculator to graph a given function • important characteristics of graphs such as symmetry, intersections with the axes, turning points and asymptotes of the following: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ $y=\frac{ax+b}{cx+d}$ $y=\frac{ax^2+bx+c}{dx+e}$ • determining the equations of asymptotes, axes of symmetry, and restrictions on the possible values of x and/or y • simple parametric equations and their graphs

References
• http://www.h2maths.site [This website offers Applets which allow self-exploration of numerous types of graphs covered in this topic] • H2 Mathematics For 'A' Level, Federick Ho, David Khor, Yui-P'ng Lam, B.S. Ong Volume 1, Call No 510.76 HO • Mathematics The Core Course for A-level, L Bostock and Chandler, Call No 510 BOS • MEI Structured Mathematics (2nd Edition), Pure Mathematics 4, Terry Heard and David Martin, Call No 510 HEA • Pure Mathematics 4, Hugh Neill and Douglas Quadling, Call No 510 NEI

Prerequisites
• Long division of polynomials (O level Add Math knowledge) • Graphs (including trigonometry)(O level Elementary and Add Math knowledge) • Quadratic graphs and their properties (O level Add Math knowledge) • Differentiation techniques and applications (O level Add Math knowledge) • Coordinate geometry and Further Coordinate Geometry (O level Add Math knowledge)

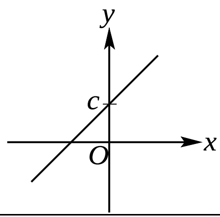
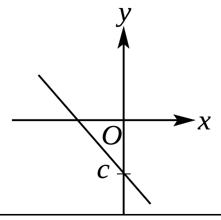
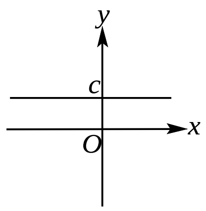
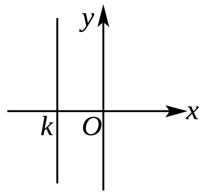
<u>1 Basic Graphs</u>

<u>1.1 Straight Lines</u>

A straight line that is <u>not</u> a vertical line has an equation in the form

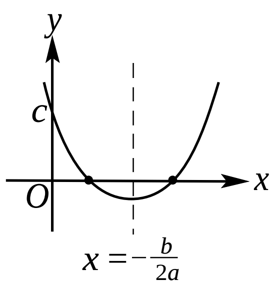
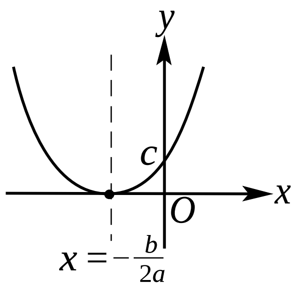
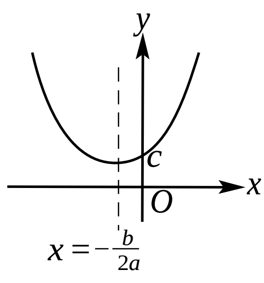
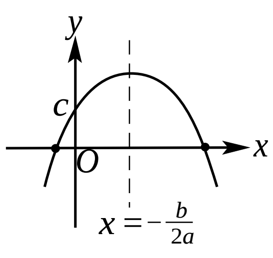
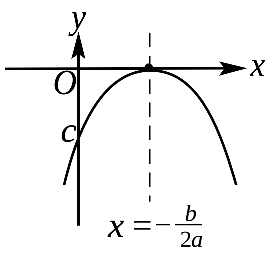
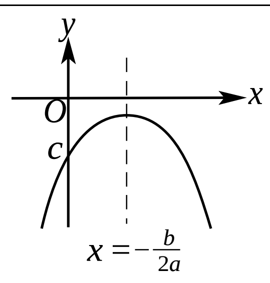
$y = mx + c$, where m is the gradient of the line and c is the y-intercept.

The equation in this form is called the gradient-intercept form.

Lines that are <u>not</u> vertical			Vertical line
			

<u>1.2 Quadratic Functions $y = ax^2 + bx + c,\ a \neq 0$</u>

A quadratic function has an equation in the form $y = ax^2 + bx + c$ where $a \neq 0$.
The graph of a quadratic function is a parabola and its concavity is determined by a.
The graph has a turning point at $x = -\dfrac{b}{2a}$.

	Concavity of graph			
$a > 0$	Concave upwards			
$a < 0$	Concave downwards			

Note: If the equation of a quadratic graph is expressed in the form $y = a\left(x-h\right)^2 + k$, then the coordinates of the turning point are $\left(h, k\right)$.

Recall: Find the turning point of $y = 2x^2 - 5x + 3$

Solution:

<u>Method 1:</u>

$$y = 2x^2 - 5x + 3$$
$$y = 2\left(x^2 - \frac{5}{2}x\right) + 3$$
$$y = 2\left(x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2\right) + 3 - 2\left(\frac{5}{4}\right)^2$$
$$y = 2\left(x - \frac{5}{4}\right)^2 + \frac{48}{16} - \frac{50}{16}$$
$$y = 2\left(x - \frac{5}{4}\right)^2 - \frac{1}{8}$$

The required turning point is $\left(\frac{5}{4}, -\frac{1}{8}\right)$

<u>Method 2:</u>

$$y = 2x^2 - 5x + 3$$
$$\frac{dy}{dx} = 4x - 5$$

Let $\frac{dy}{dx} = 0$

Then $x = \frac{5}{4}$

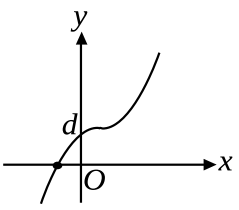
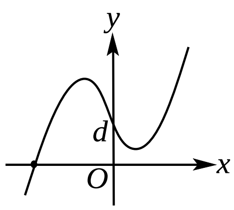
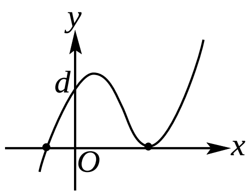
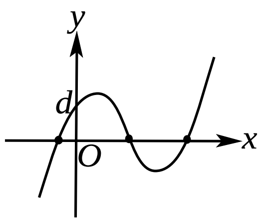
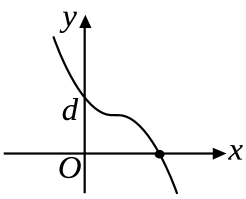
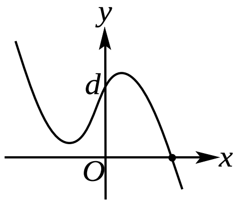
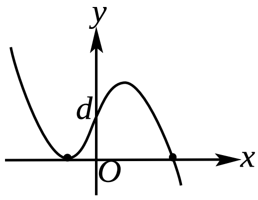
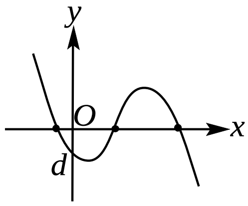
when $x = \frac{5}{4} \Rightarrow y = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) + 3 = \frac{50 - 100 + 48}{16} = -\frac{1}{8}$

The required turning point is $\left(\frac{5}{4}, -\frac{1}{8}\right)$

1.3 Cubic Functions, $y = ax^3 + bx^2 + cx + d,\ a \neq 0$

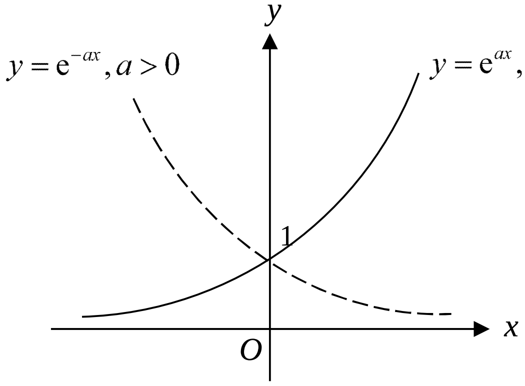
A cubic curve has an equation in the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

A cubic curve intersects the x-axis at least once and at most thrice.

	Intersection(s) with *x*-axis			
	once		**twice**	**thrice**
$a > 0$				
$a < 0$				

1.4 Exponential Functions

Recall:
The ***exponential constant***, e = 2.71828… , is an irrational number.



Note: For $y = \mathrm{e}^{ax}, a > 0$, the graph is as close as possible to, but <u>never touches</u> the x-axis while x approaches $-\infty$. We say that the x-axis is the **<u>horizontal asymptote</u>**.

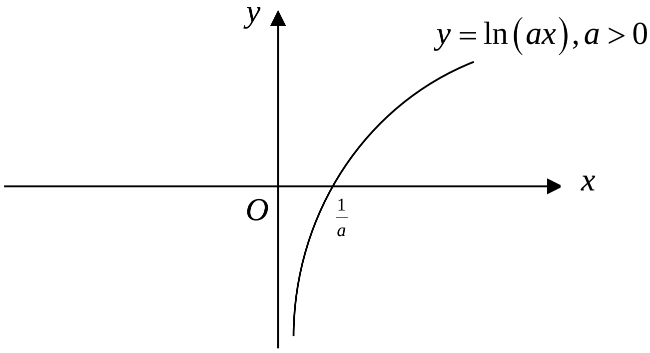
Similarly, for $y = \mathrm{e}^{-ax}, a > 0$, the graph is as close as possible to, but <u>never touches</u> the x-axis while x approaches ∞. We say that the x-axis is the **<u>horizontal asymptote</u>**.

1.5 Logarithmic Functions

Recall:

$$x = a^y \Leftrightarrow \log_a x = y$$

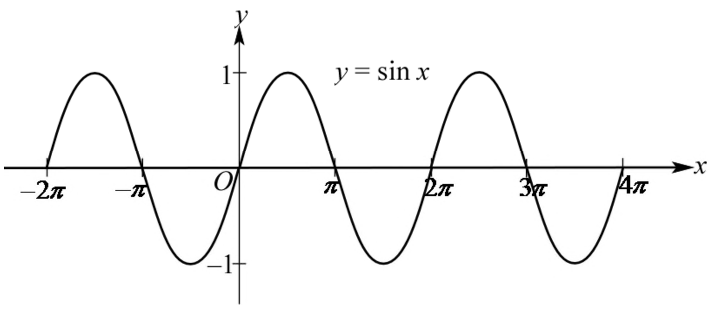
We denote $\log_e x$ as ln *x* (natural logarithm) where $x > 0$.



Note: For $y = \ln(ax), a > 0$, the graph is as close as possible to, but never touches the *y*-axis as *x* approaches 0 from the right. We say that the *y*-axis is the **vertical asymptote**.

1.6 Trigonometrical Functions

(a) $y = \sin x$



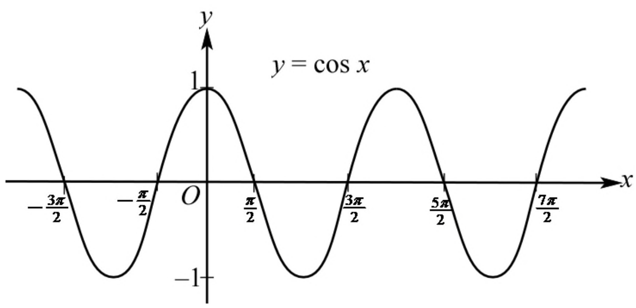
The sine curve has the following features:

(1) $-1 \le y \le 1$

(2) The sine curve has an ***amplitude*** equals to 1. (An ***amplitude*** is half the difference between the maximum and minimum values of *y* in a periodic function.)

(3) The sine curve has a ***period*** of 2π radians. (***Period*** is the interval the graph takes to go a complete cycle.)

(b) $y = \cos x$



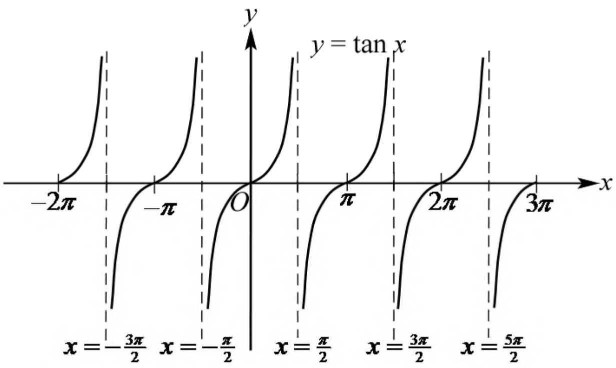
The cosine curve has the following features:

(1) $-1 \le y \le 1$

(2) The cosine curve has an amplitude equals to 1.

(3) The cosine curve has a period of 2π radians.

(c) $y = \tan x$



The tangent curve has the following features:

(1) y can take any real value.

(2) As x approaches $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$, the curve is closer and closer <u>but never touches</u> the lines $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ respectively. These lines are the vertical asymptotes.

<u>1.7 The use of Graphing Calculator (GC) in drawing graphs</u>

<u>Example 1</u>

Sketch the following graphs with the help of a GC. (You may refer to Annex A)

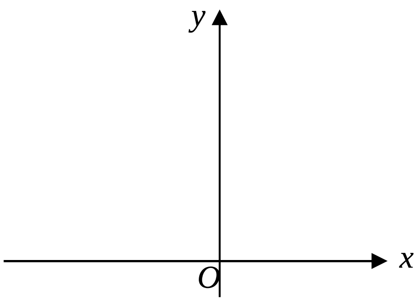
(a) $y = 2x^2 + x + 6$ (b) $y = e^x$

(c) $y = x(x+2)(x-4)$ (d) $y = \ln x$

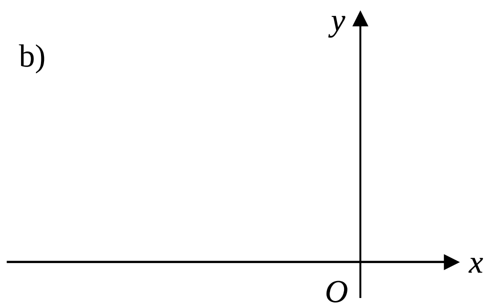
You should indicate the intersections with axes, turning points and asymptotes if possible.

Solution:

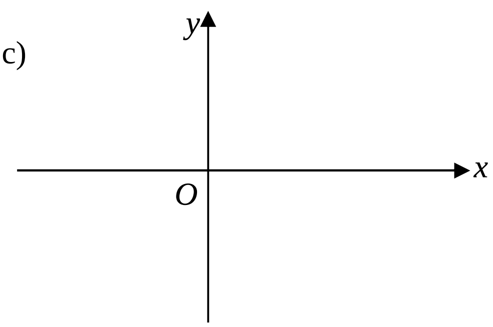
a)



b)

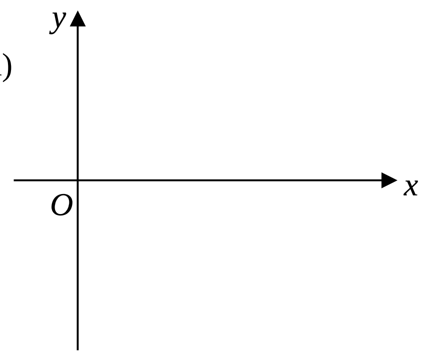


c)



(Note: This graph requires zooming out or adjusting the window settings in order to see the whole graph)

d)



Standard Window settings

Keystrokes	Screen shown on GC	Explanation
WINDOW	WINDOW Xmin=-10 Xmax=10 Xscl=1 Ymin=-10 Ymax=10 Yscl=1 Xres=1	Xmin: Extreme left value on *x*-axis Xmax: Extreme right value on *x*-axis Xscl: Distance between markings on the *x*-axis Ymin: Extreme bottom value on *y*-axis Ymax: Extreme top value on *y*-axis Yscl: Distance between markings on the *y*-axis Xres: Pixel resolution. You can always leave it as 1 Note: This setting is obtained by pressing **ZOOM** **6**. However, if the graph plotted is not fully captured in this setting, you may change the values of Xmin, Xmax, Xscl, Ymin, Ymax and Yscl to capture the graph as clearly as possible.

Lesson from Example 1

Despite the convenience of having the GC to assist us in curve sketching, it is not advisable to depend solely on the GC for doing so. As shown in Example 1, the shape of the curve that is shown on the GC may not always be correct due to *Window* settings or certain limitations in the GC. It is therefore still essential that we are familiar with the shape and characteristics of certain standard curves in order to sketch graphs quickly and accurately. **More importantly, we must understand the concepts behind curve sketching and use the GC only as a tool to assist us.**

<u>2 Essential Features of a graph</u>

(Refer to Annex A for GC Applications)

<u>(a) Axial Intercepts (Intersection with the axes)</u>

These are points where the curve crosses the y and x axes, obtained by substituting $x = 0$ and $y = 0$ into the equation of the graph respectively.

<u>(b) Stationary points</u>

These are points (maximum or minimum or stationary point of inflection, if any) for which the tangents are horizontal. It is possible to locate these points by solving the equation $\frac{dy}{dx} = 0$.

<u>(c) Linear Asymptotes</u>

In GCE 'A' Level H2 Mathematics syllabus, you are required to determine the equations of <u>linear</u> asymptotes only. There are 3 types of linear asymptotes: ***vertical, horizontal* and *oblique asymptotes*.**

(i) <u>Vertical Asymptote</u>

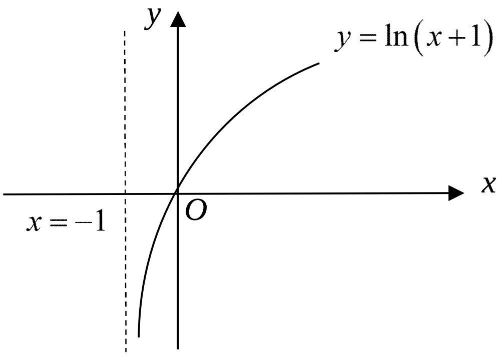
If there is a constant k such that

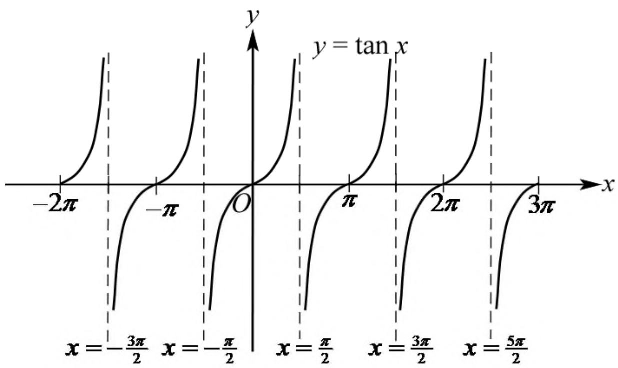
$$y \to \infty \text{ (read as "}y\text{ approaches } \infty\text{") or } y \to -\infty \text{ as } x \to k,$$

then the line $x = k$ is called a ***vertical asymptote*** of the curve.
A vertical asymptote can be found by deducing the value of x that causes $y \to \infty$ or $y \to -\infty$.

For example,

The line $x = -1$ is the vertical asymptote of the graph of $y = \ln(x + 1)$, and the line $x = \frac{\pi}{2}$ is a vertical asymptote of the graph of $y = \tan x$ (refer to Section 1.6 (c)).



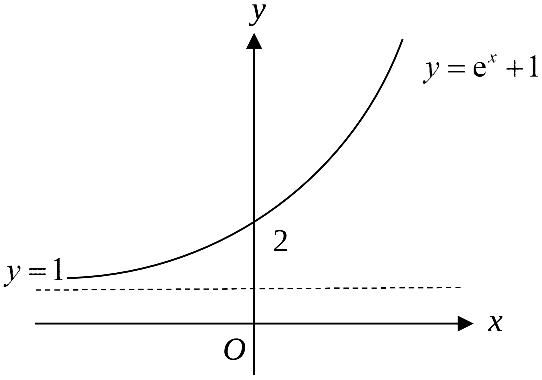


Note: The existence of a vertical asymptote $x = k$ implies that there is no value of y that corresponds to $x = k$. Hence, a curve will NEVER intersect the vertical asymptote $x = k$.

(ii) Horizontal Asymptote

If there is a constant h such that $$y \to h \text{ as } x \to \infty \text{ or } x \to -\infty,$$ then the line $y = h$ is called a ***horizontal asymptote*** of the curve. A horizontal asymptote can be found by deducing the tendency of y as $x \to \pm\infty$.

For example, the line $y = 1$ is the horizontal asymptote of the graph of $y = \mathrm{e}^x + 1$



Note: A horizontal asymptote $y = h$ applies only at the left and/or right extreme ends of the curve. It is possible for a curve to intersect the horizontal asymptote, $y = h$, at some other definite value(s) of x. [Refer to Example 2(a).]

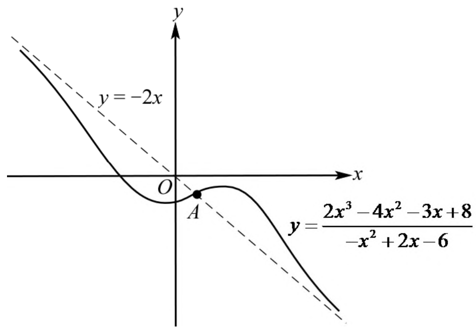
(iii) Oblique Asymptote

If there is a straight line $y = mx + c,\ m \neq 0$, such that $$y \to mx + c \text{ as } x \to \infty \text{ or } x \to -\infty,$$ then the line $y = mx + c$ is called an ***oblique asymptote*** of the curve. An oblique asymptote can be found by deducing the tendency of y as $x \to \pm\infty$.

As in the case of a horizontal asymptote, a curve may intersect an oblique asymptote $y = mx + c$ at some definite value(s) of x.

For example,

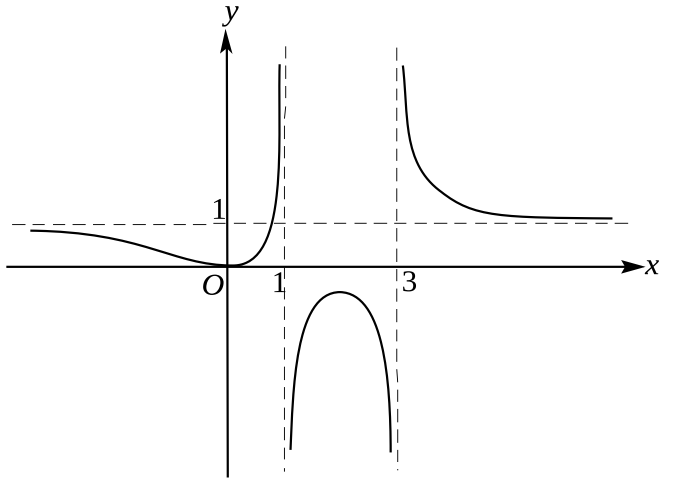
the curve $y = \dfrac{2x^3 - 4x^2 - 3x + 8}{-x^2 + 2x - 6}$ intersects the oblique asymptote $y = -2x$ at $A(0.533, -1.07)$ (3 sf).



Example 2

Write down the equations of the asymptotes of the following graphs.

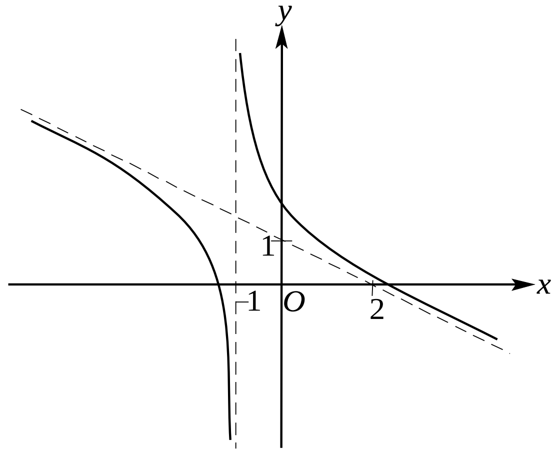
(a)



Vertical asymptotes:

Horizontal asymptote:

(b)



Vertical asymptote:

Oblique asymptote:

3 Graphs of Rational Functions

Recall: A ***polynomial*** in x is an expression of the form
$a_0 + a_1 x + a_2 x^2 + ... + a_{n-1}x^{n-1} + a_n x^n$, where $n \in \mathbb{Z}_0^+$ and
$a_n, a_{n-1}, ..., a_1, a_0$ are real constants with $a_n \neq 0$.
The **non-negative integer *n*** is called the **degree** or **order** of the polynomial.

For example, $4x^5 - 7x^3 + x + 2$ and $-x^3 + \frac{1}{3}x^2 + 5$ are polynomials but $x^2 + \frac{2}{x}$ and $\sqrt{x}$ are **NOT** polynomials (Why?)

A rational function is a function which can be represented as a quotient of polynomials i.e $\frac{P(x)}{D(x)}$, where $P(x)$ and $D(x)$ are polynomials.

A rational function $\frac{P(x)}{D(x)}$ is a **proper** rational function if degree of $P(x)$ < degree of $D(x)$, otherwise it is an **improper** rational function.

For example, $\frac{2x^3 + 3x - 1}{x - 4}$, $\frac{x^2 + 5x - 3}{x^3 - 3x^2 + 7x - 4}$, $\frac{2x + 3}{x + 4}$, $\frac{1}{x}$ are rational functions.

(Can you tell whether they are proper or improper rational functions?)

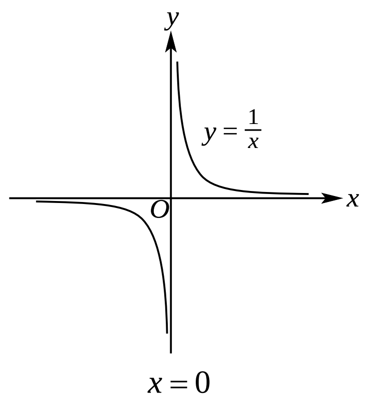
3.1 Finding asymptotes of the graph of a rational function, $y=\dfrac{P(x)}{D(x)}$

(a) Vertical Asymptotes

Since the rational function $y=\dfrac{P(x)}{D(x)}$ is undefined when $D(x)$ = 0, therefore a vertical asymptote (if any) is obtained by solving $D(x)=0$.

For example, $y=\dfrac{1}{x}$ is undefined when the denominator x equals 0.

Hence the vertical asymptote of the graph is $x=0$.
(Notice that $y\to\pm\infty$ as $x\to 0$)



(b) Horizontal or Oblique Asymptotes

As mentioned in Section 2c (ii) and (iii), horizontal or oblique asymptotes can be found by deducing the tendency of y as $x\to\pm\infty$.

To find the horizontal or oblique asymptotes of a rational function, we first rewrite

$$\frac{P(x)}{D(x)}=Q(x)+\frac{R(x)}{D(x)}, \text{ where } \frac{R(x)}{D(x)} \text{ is a proper rational function,}$$

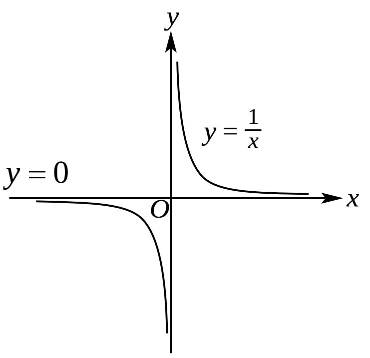
using long division if necessary.

Then, $y=Q(x)$ is the horizontal or oblique asymptote.

For example, $y=\dfrac{1}{x}$ can be re-written as $y=0+\dfrac{1}{x}$.

(There is no need to do long division in this case since $\dfrac{1}{x}$ is a proper rational function.)

Hence the horizontal asymptote of the graph is $y=0$.
(Notice also that $y\to 0$, as $x\to\pm\infty$.)



In another example, $y=\dfrac{x^2-7x+6}{x^2+1}$ can be re-written as $y=1-\dfrac{7x-5}{x^2+1}$ where $\dfrac{7x-5}{x^2+1}$ is a proper rational function. Hence the horizontal asymptote of the graph is $y=1$.

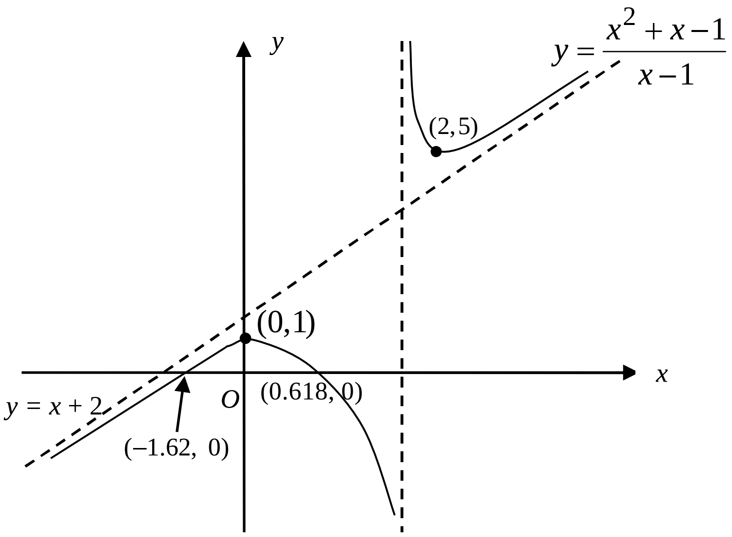
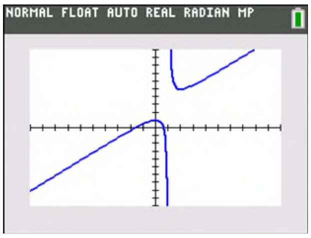
3.2 Sketching the graph of a rational function, $y=\dfrac{P(x)}{D(x)}$

In this section we will use Example 3 to illustrate the steps to follow when sketching the graph of a rational function.

Example 3

Sketch the curve, $y=\dfrac{x^2+x-1}{x-1}$, making clear the main relevant features of the curve.

Solution:

Step 1:	**Check whether rational function is a proper or improper rational function.** Since $\dfrac{x^2+x-1}{x-1}$ is improper, we do long division and re-write the equation as $y=x+2+\dfrac{1}{x-1}$ ... (*)
Step 2:	**Find asymptotes.** **(a) Vertical asymptote(s)** By equating the denominator to zero, **(b) Horizontal or oblique asymptote.** From (*), $y=x+2$ ____.
Step 3:	**Find axial intercepts.** You may use the GC to help you find the axial intercepts (refer to Annex A), unless otherwise stated. (How do we find axial intercepts without using the GC?)
Step 4:	**Find turning points.** You may use the GC to help you find the turning points (refer to Annex A), unless otherwise stated. (How do we find turning points without using the GC?)
Step 5:	**Sketch the curve.** Sketch and label the asymptotes first before filling in the shape of the curve and lastly, label all axial intercept(s), turning point(s) and the equation of your curve. 

Example 4

Sketch the following curves, showing clearly the asymptotes, stationary points and intersections with the axes (where applicable).

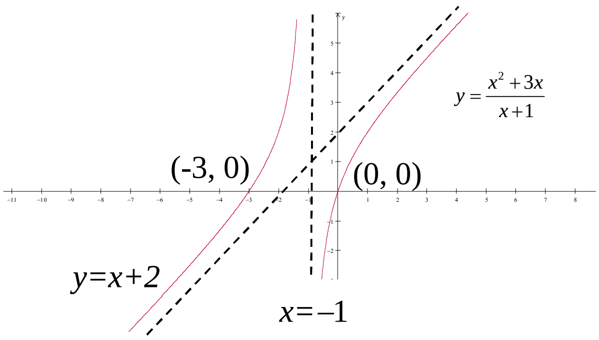
(a) $y=\dfrac{x^2+3x}{x+1}$ (b) $y=\dfrac{x^2-7x+6}{x^2+1}$

Solution:

(a) $y=\dfrac{x^2+3x}{x+1}=x+2-\dfrac{2}{x+1}$ by long division

Intercepts:
Asymptotes:

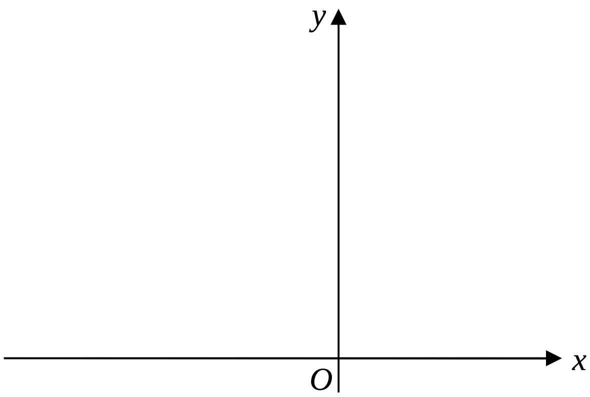
From GC (by graphing), there is no Max and Min.



(b) $y=\dfrac{x^2-7x+6}{x^2+1}=1-\dfrac{7x-5}{x^2+1}$

Intercepts: $(0,6)$,$(1,0)$,$(6,0)$
Asymptotes:

Use GC:
Maximum point: $(-0.515, 7.80)$
Minimum point: $(1.94, -0.801)$



Important Note:
Based on the graph obtained in part (b), observe that the curve actually meets the horizontal asymptote.

In general, since the horizontal and oblique asymptotes indicate the behaviours of the curve at the two ends of the sketch only, it is still possible for the curve to meet the horizontal asymptote.
This is unlike the vertical asymptote where the curve does not meet it at all.

Important:
Always check that you have labelled the following when sketching:

(1) axes (2) origin (3) intercepts (4) stationary points (5) asymptotes

3.3 Other curves involving asymptotes

Refer to sections 1.4, 1.5 and 1.6, graphs involving asymptotes are exp, log, tan, etc

Example 5

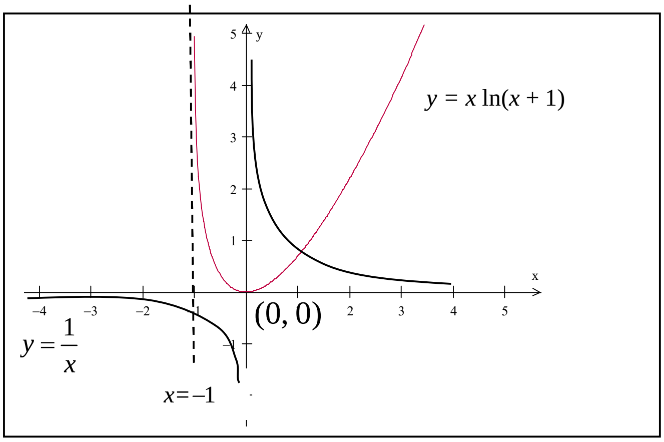
Sketch the graph $y = x\ln(x + 1)$, showing clearly the asymptotes, stationary points and intersections with the axes. Hence, by sketching another graph, solve the equation $x^2\ln(x+1)=1$.

Solution:

Step 1: As $x = -1$, y is undefined, then ____.

Step 2: $x = 0$, $y = 0$

Step 3: By GC, minimum point at (0,0) (3sf).



3.4 Finding the range of values of *y* using algebraic method, given that $y=\dfrac{P(x)}{D(x)}$

For the family of rational functions, $y=\dfrac{P(x)}{D(x)}$, which simplifies to a quadratic equation in *x* after cross multiplying, we can use the discriminant $\left(b^2-4ac\right)$ to find the range of values of *y*.

Example 6 [H2 Specimen Paper 2007/I/9(iii) modified]

Consider the curve $y=\dfrac{3x-6}{x(x+6)}$.

Find, using an algebraic method, the range of values of $y=\dfrac{3x-6}{x(x+6)}$.

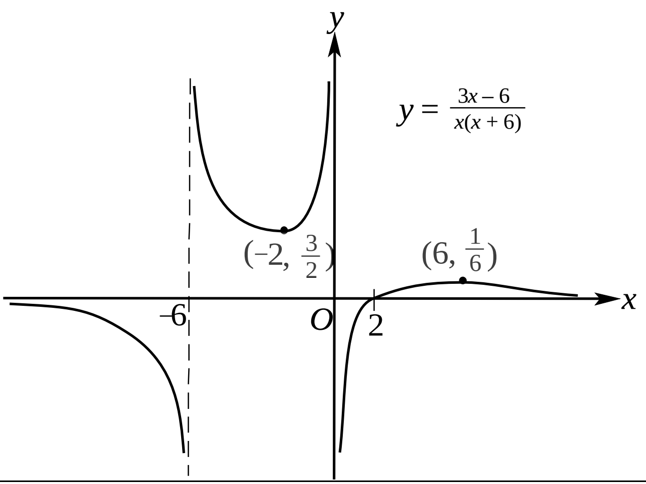
Solution:

$$y = \frac{3x-6}{x(x+6)}$$

$\Rightarrow$

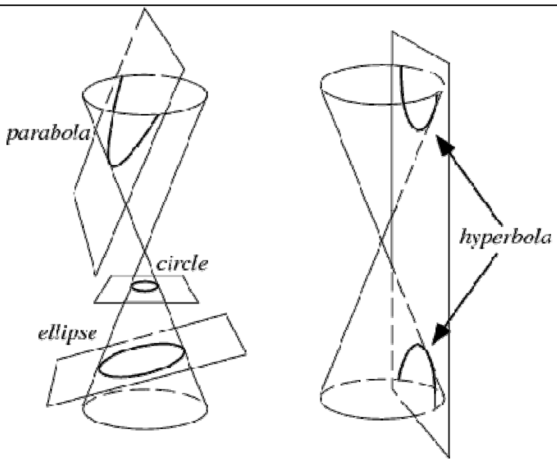
Rearrange terms to obtain quadratic equation in *x*:

To find range of *y* given by **real values of *x***, let this quadratic equation have **real roots**.

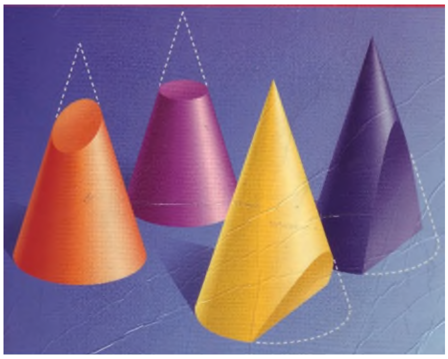
Note: The graph of $y=\dfrac{3x-6}{x(x+6)}$ is shown on the right. From the graph, we can see that for all **real values of *x***, the values that *y* can take are	

4 Conics (Refer to Annex B for GC Applications)

A ***conic section*** (or just ***conic***) is a curve obtained by intersecting one or two pieces of a double cone with a plane.



Source of images:
http://mathworld.wolfram.com/ConicSection.html



Source of images: Brian & Mark Gaulter. *Further Pure Mathematics* (2001) Cover page

http://en.wikipedia.org/wiki/Hyperbola
http://math2.org/math/algebra/conics.htm

Parabola: cutting plane parallel to side of cone
Hyperbola: plane intersects **both** halves of the double cone but does **not** pass through the apex of the cones
If the plane passes through the central apex a degenerate hyperbola results — two straight lines that cross at the apex point.

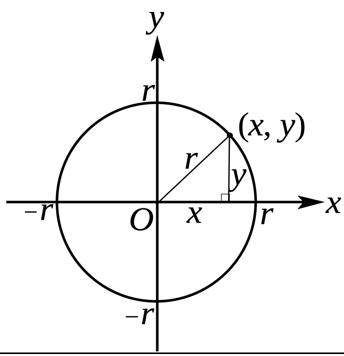
Conics is found to be related to studies on planetary orbits, sonic boom shockwaves and many other inventions such as satellites and headlamps.

4.1 Circle

(a) The equation

$$x^2 + y^2 = r^2$$

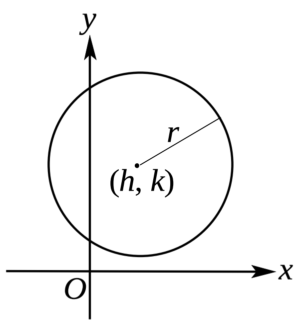
represents a ***circle*** with centre (0, 0) and radius r.



(b) The **standard form** of the equation of a circle with centre (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2 .$$

The equation $\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$ represents a **circle** with centre **(h, k)** and radius **r.**



Remarks

(1) Always label the **centre** and the **radius** of the circle in a sketch.

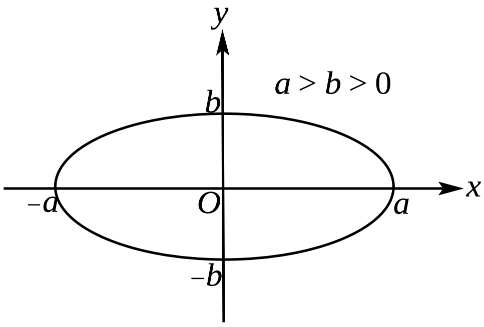
(2) In the **standard form** of the equation of a circle, the coefficients of x^2 and y^2 are positive and equal.

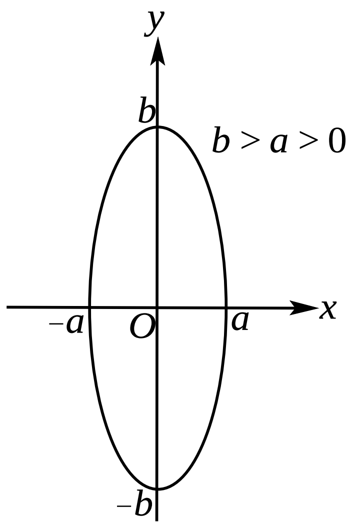
4.2 Ellipse

(a) The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{, where } a>0,\ b>0,$$

represents an ***ellipse*** with centre (0, 0) and with semi-major, *a*, and semi-minor, *b*, if $a>b>0$ (or with semi-major, *b*, and semi-minor, *a*, if $b>a>0$).

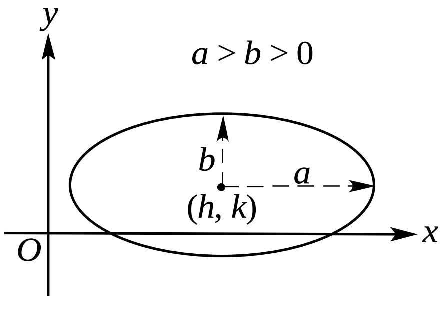


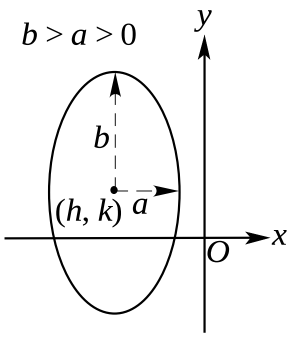


Note: The lines of symmetry are $x = 0$ and $y = 0$.

(b) The **standard form** of the equation of an ellipse with centre (h,k) with semi-major, *a*, and semi-minor, *b*, if $a>b>0$ (or with semi-major, *b*, and semi-minor, *a*, if $b>a>0$) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \text{, where } a>0,\ b>0.$$





Note: (1) The lines of symmetry are $x = h$ and $y = k$.
(2) Only the centre and semi-minor and semi-major need to be indicated.

(What will an ellipse become when $a = b$?)

Remarks

In the **standard form** of the equation of an ellipse,

(1) the RHS is 1.

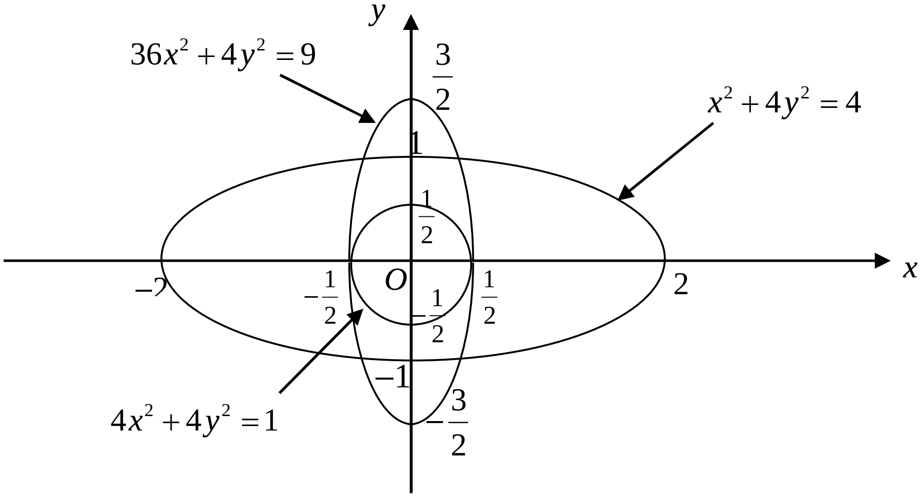
(2) the coefficients of x^2 and y^2 are positive and unequal.

Example 7

Sketch on the same diagram, the graphs of
(i) $4x^2 + 4y^2 = 1$, (ii) $x^2 + 4y^2 = 4$, (iii) $36x^2 + 4y^2 = 9$.

Solution:

(i) $4x^2 + 4y^2 = 1$	(ii) $x^2 + 4y^2 = 4$	(iii) $36x^2 + 4y^2 = 9$

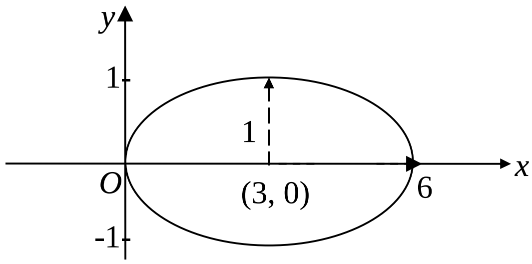


Example 8

Sketch the graph of $x^2 - 6x + 9y^2 = 0$.

Solution:

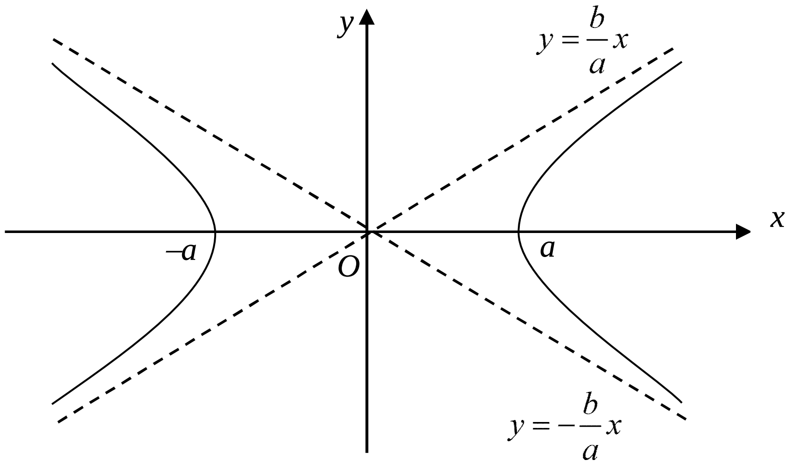
$x^2 - 6x + 9y^2 = 0$



4.3 Hyperbola

(a) The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>0$, $b>0$, represents a **hyperbola** with

- centre (0, 0)
- oblique asymptotes **:** $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$
- x- intercepts ($\pm a$, 0)
- lines of symmetry, $x = 0$ and $y = 0$.



Explanation on how the oblique asymptotes are obtained:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$$

$$y^2 = \frac{b^2x^2}{a^2}-b^2$$

$$\frac{y^2}{x^2} = \frac{b^2}{a^2}-\frac{b^2}{x^2}$$

As $x\to\pm\infty$, $\frac{b^2}{x^2}\to 0$

$$y^2 \to \frac{b^2x^2}{a^2}$$

$$\therefore\ y \to \pm\frac{b}{a}x$$

Trick to find equations of oblique asymptotes:
Replace RHS of equation with 0.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

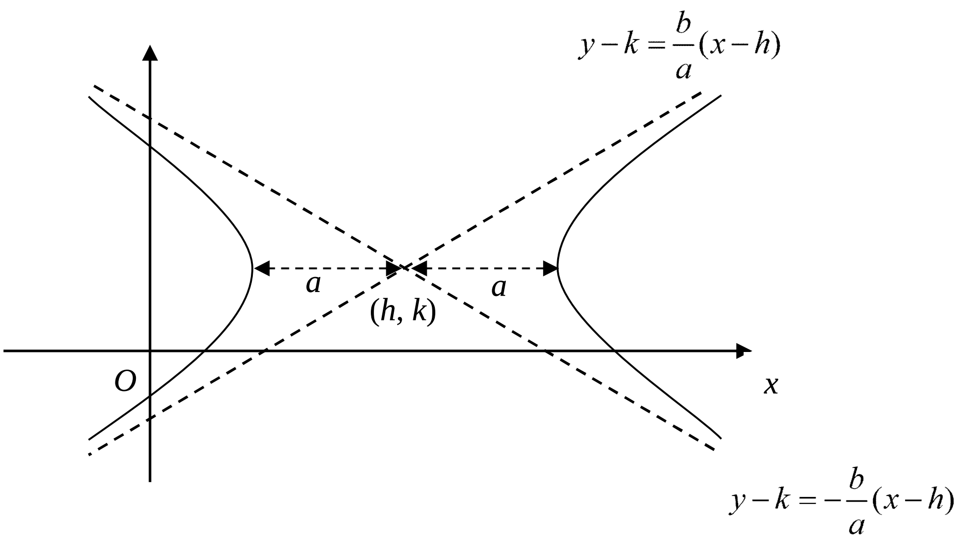
$$y^2=\frac{b^2x^2}{a^2}$$

$$y=\pm\frac{b}{a}x$$

(b) The **standard form** of the equation of a hyperbola is

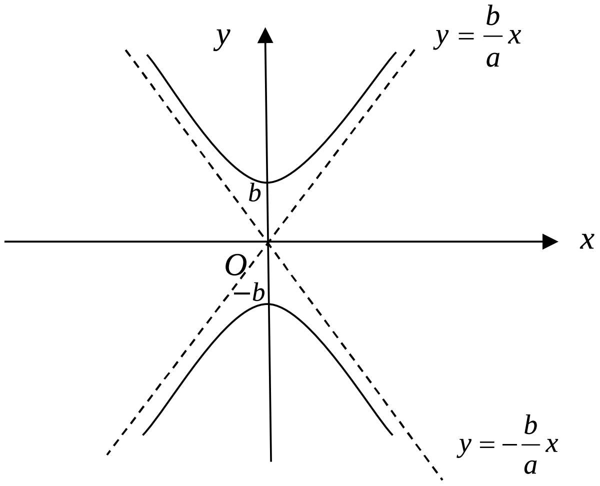
$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ , where } a>0\text{ , } b>0\text{ , with}$$

- centre (h, k)
- oblique asymptotes **:** $y-k=\frac{b}{a}(x-h)$ and $y-k=-\frac{b}{a}(x-h)$
  Note: Oblique asymptotes intersect at the centre, (h, k)
- distance between centre and vertex: a
- lines of symmetry, $x = h$ and $y = k$.



(c) The equation $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, where $a>0$, $b>0$, represents a **hyperbola** with

- centre (0, 0)
- Oblique asymptotes : $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$
- y- intercepts (0, $\pm b$)
- lines of symmetry, $x = 0$ and $y = 0$.

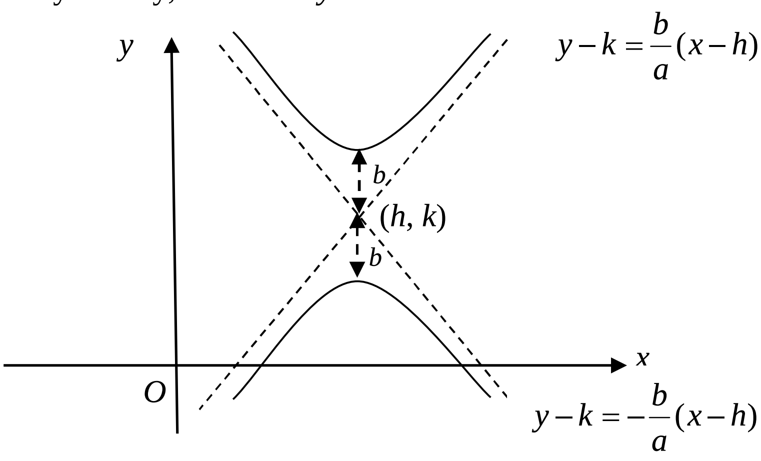


(d) The **standard form** of the equation of a hyperbola is

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$, where $a>0$, $b>0$, with

- centre (h, k)
- Oblique asymptotes **:** $y-k=\frac{b}{a}(x-h)$ and $y-k=-\frac{b}{a}(x-h)$

  Note: Oblique asymptotes intersect at the centre, (h, k)
- distance between centre and vertex: b
- lines of symmetry, $x = h$ and $y = k$.



<u>Remarks</u>

In the **standard form** of the equation of a hyperbola,

(1) the RHS is 1.

(2) the coefficients of x^2 and y^2 have <u>opposite</u> signs.

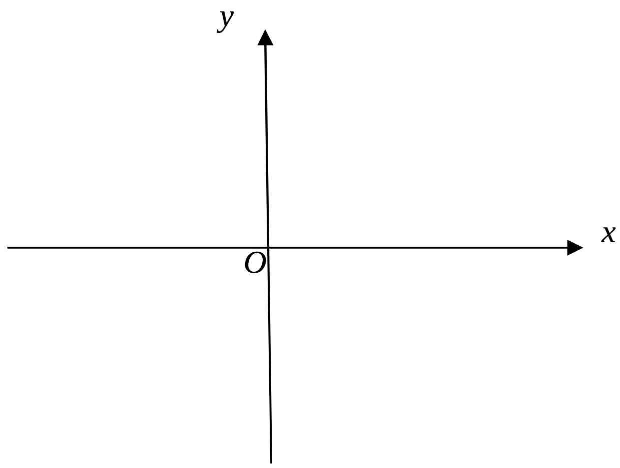
(Recall: What about in the equation of an ellipse?)

Example 9

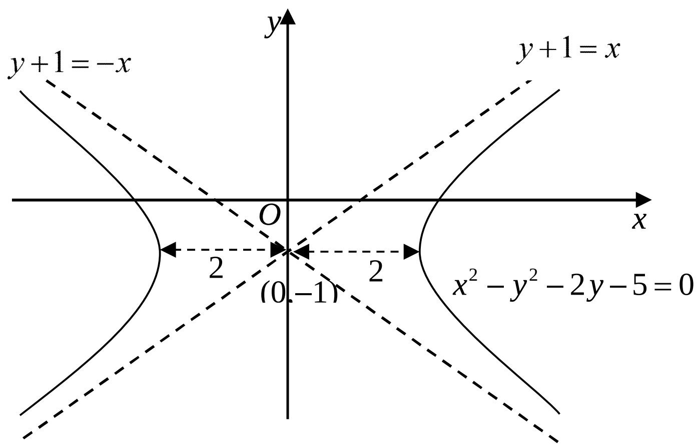
Sketch on separate diagrams, the graphs of
(i) $3y^2-4x^2=12$, (ii) $x^2-y^2-2y-5=0$.

Solution:

(i) $3y^2-4x^2=12$



(ii) $x^2-y^2-2y-5=0$



<u>Example 10</u>

On the same diagram, sketch the graphs of

(a) $y = x^2$ (b) $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$.

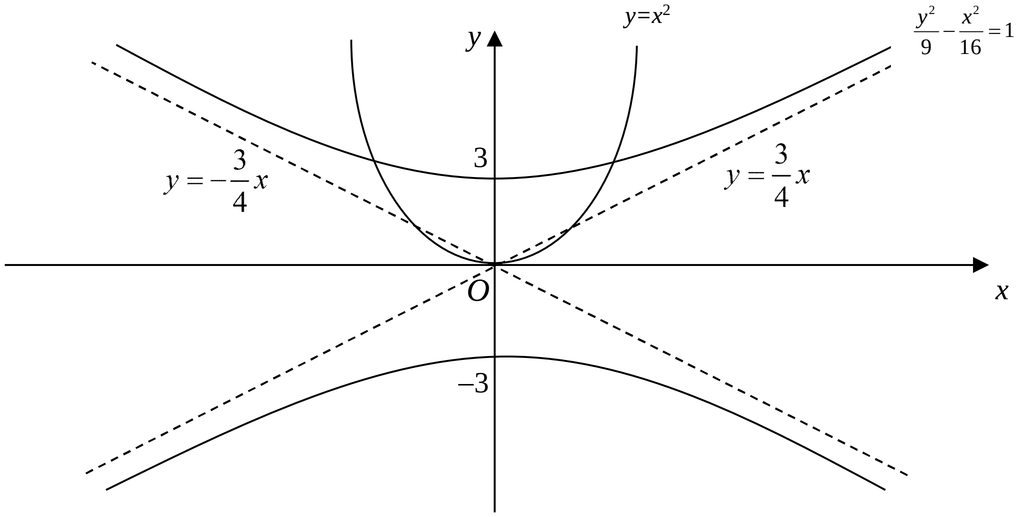
Find the points of intersection of the two graphs.

Solution:

At $x = 0$, $y = 3$ and -3

At $y = 0$, x is undefined

Asymptotes: $y = -\dfrac{3}{4}x$ and $y = \dfrac{3}{4}x$



From GC, the points of intersection are (–1.82, 3.29) and (1.82, 3.29)

<u>NOTE:</u>

Using GC, set

we can see the graphs in GC but without the asymptotes.

To find the points of intersection, use GC to find intersect between Y_1 and Y_2 (refer to Annex A).

For use of GC to draw ellipses, circles and hyperbola, refer to Annex B. (Limitation of Apps – cannot draw two conics or graphs in Apps – Refer to Example 10)

5 Parametric Equations

5.1. (Refer to Annex C for GC Applications)

So far we have considered graphs defined by equations involving two variables *x* and *y*. It is common for *x* and *y* to be related with a third variable called a ***parameter***. For example, *x* can be the weight of a man, *y* is his height and *t* (the **parameter**) is the time in years.
We can have ***parametric equations*** $x = \text{f}(t), y = \text{g}(t)$ and from there, form the Cartesian equation $y = \text{h}(x)$.

For instance, the Cartesian equation of curve $5y = 2x^2 - x$ can be formed by a pair of ***parametric equations***:

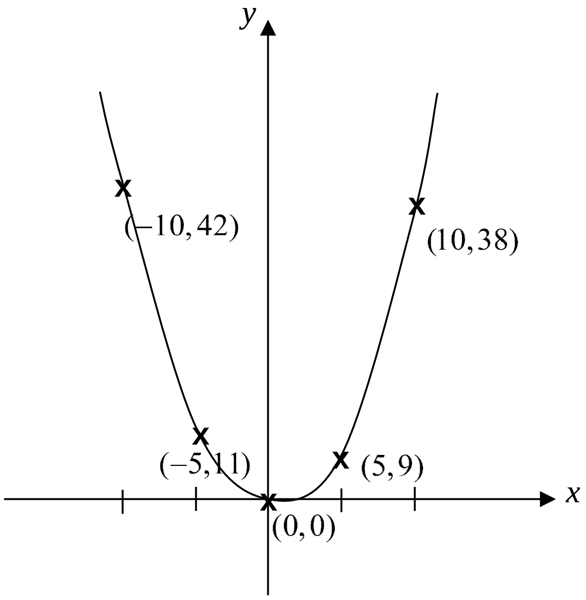
$$x = 5t$$
$$y = 10t^2 - t$$

where *t* is known as the ***parameter***.

The table below shows the points on the curve corresponding to each value of *t*.

t	–2	–1	0	1	2
x	–10	–5	0	5	10
y	42	11	0	9	38

We can use these points to plot the graph of *y* against *x*:



Note: We will usually use the GC to help us sketch the curve given its parametric equations.
[Refer to Annex C for GC Applications]

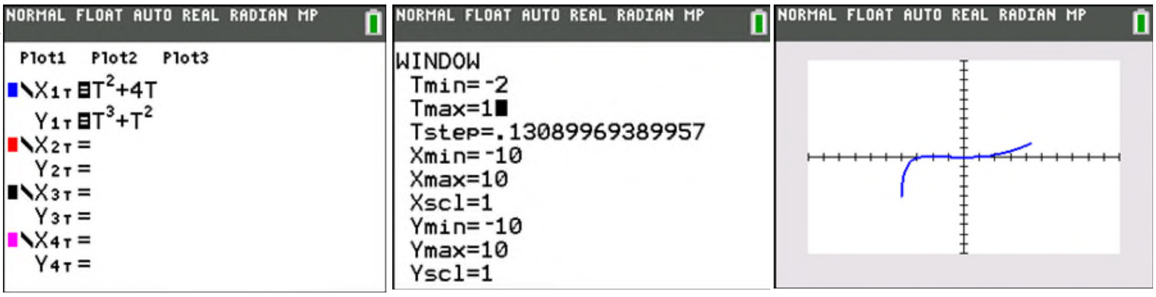
Example 11

The curve C has parametric equations

$$x = t^2 + 4t\ , \quad y = t^3 + t^2 .$$

Sketch the curve for $-2 \le t \le 1$

Solution:



Finding *y* intercepts:

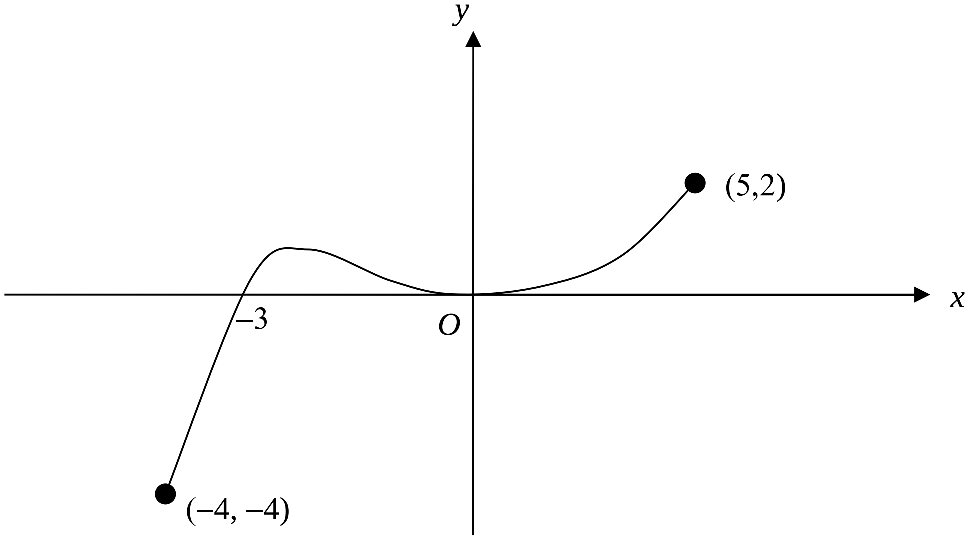
When $x = 0$,

Finding *x* intercepts:

When $y = 0$,

Hence the sketch of the graph is as shown.

Note that the end coordinates of the curve should be labelled, if applicable.



<u>Annex A</u>
<u>Using the TI-84 Plus C Graphic Calculator for Graphing Techniques</u>

To graph a function, the calculator has to be in **FUNCTION** Graphing Mode.

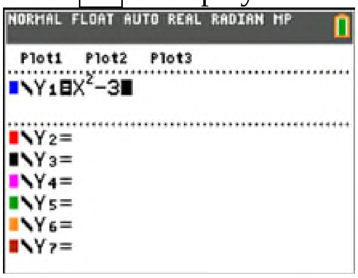
1. Press **MODE** to go to the Mode Menu.

2. Go down to line 5 by pressing the arrow key down repeatedly. **FUNCTION** will be highlighted. Press **ENTER** to accept.

3. To go to the Home Screen, press **CLEAR** or **2ND** **QUIT**.

<u>**To key the function into the calculator**</u>,

1. Press **Y=** to display the Function Definition Screen.

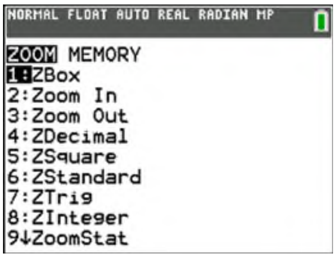


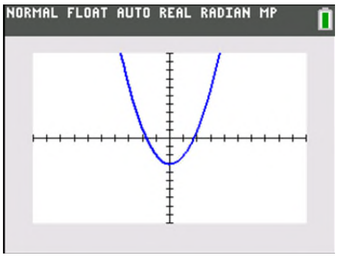
2. Use Scroll Buttons to move to $\mathbf{Y_1} =$

3. Press **CLEAR** to delete previous equations, if needed.

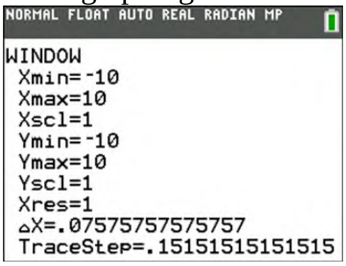
4. Key in the function into the calculator using **X,T,θ,n,** for the variable *x*. For example, key in $\mathrm{Y}_1 = x^2 - 3$. Use brackets appropriately as far as possible. Then press **ENTER** to accept and confirm the equation.

5. To display the graph on the Viewing Window using the predefined window, press **ZOOM** **6** to choose **ZStandard.** Otherwise, just press **GRAPH**.





6. Press **WINDOW** to display the Window Editor. It defines the ranges of values of x and y on the Viewing Window. Take note of the setting. You may choose to change the settings when graphing functions.



Xmin and **Xmax** are the starting and end values of the x-axis.
Xscl is the scale of the markings on the x-axis.
Ymin and **Ymax** are the starting and end values of the y-axis.
Yscl is the scale of the markings on the y-axis.
Xres is the resolution. There is no need to change this value.

7. Alternatively, you may also use other functions like **ZBox**, **ZSquare**, **ZoomFit** by pressing **ZOOM** **1**, **ZOOM** **5** or **ZOOM** **0** respectively.

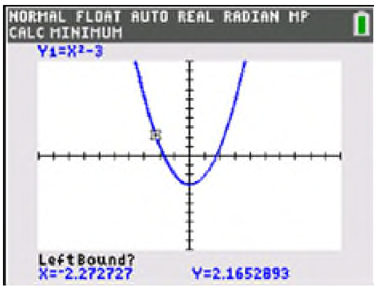
- **Zbox** could be used to zoom in a particular section of the graph that you are interested in and magnify it for viewing.
- **ZSquare** is used to ensure that the scales for the x and y axes are equal (i.e. the circle that you draw will look like a circle and not an ellipse).
- **ZoomFit** will instead include the maximum and the minimum y values of the selected functions.

Determining the maximum point and the minimum point of a curve

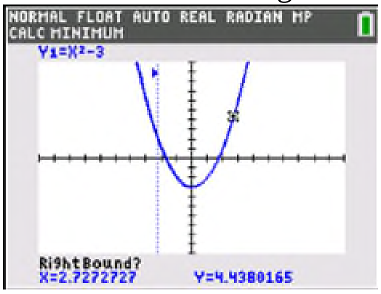
1) Keeping $Y_1 = x^2 - 3$ and scroll down to the = of Y_2. Press **ENTER**. You will notice that the = will no longer be highlighted. The Y_2 graph will no longer be displayed when you press **GRAPH**

2) Press **2ND** **TRACE** and select **3:minimum** or **4:maximum** from the **CALCULATE** menu.

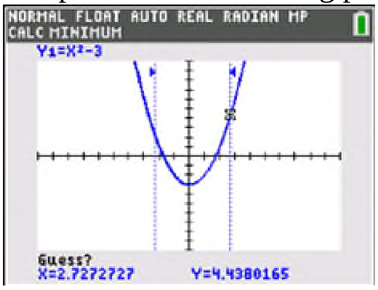
3) The current graph is displayed with **Left Bound?** in the bottom left corner of the Viewing Window. Move cursor to the left of the required turning point and press **ENTER**.



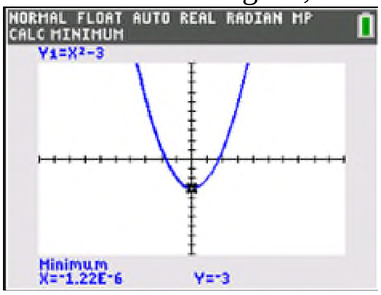
4) **Right Bound?** is then displayed in the bottom left hand corner of the Viewing Window. Move cursor to the right of the required turning point and press **ENTER**.



5) **Guess?** is then displayed in the bottom left corner of the Viewing Window. Move cursor to a point near the turning point and press **ENTER**.

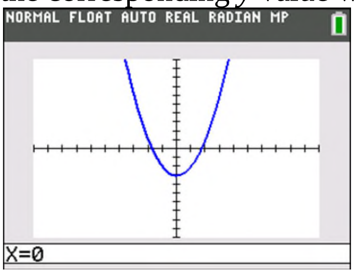


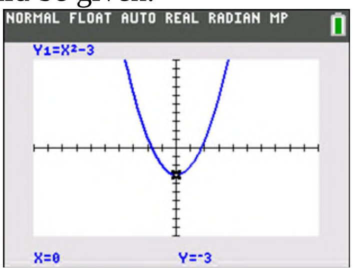
6) The turning point will be displayed at the bottom of the Viewing Window. Note that due to error in rounding off, the x-value of the GC should be taken as 0.



<u>Determining the *y*-intercept of a curve</u>

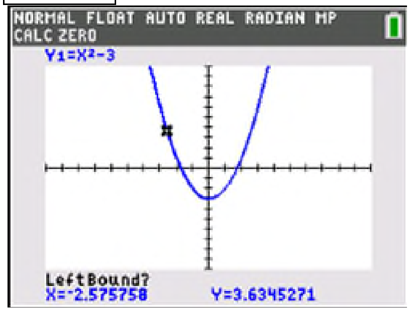
1) To find the ***y*-intercept**, press **2ND** **TRACE** and select **1:value** from the **CALCULATE** menu. **X=** is seen at the bottom of the Viewing Window. Key in **0** for the value of x and the corresponding y value would be given.



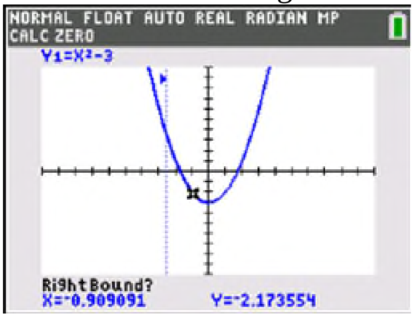


Determining the *x*-intercepts of a curve (roots of an equation)

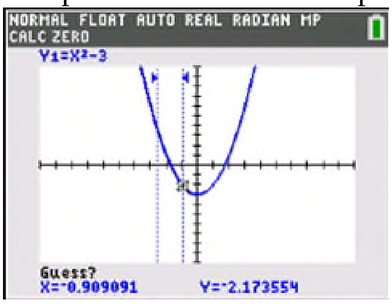
1) To find the ***x*-intercepts**, press **2ND** **TRACE** and select **2:zero** from the **CALCULATE** menu. The current graph is displayed with **Left Bound?** in the bottom left corner of the Viewing Window. Move the cursor to the left of one of the *x*-intercepts and then press **ENTER**.



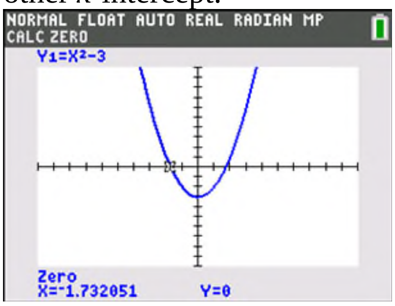
2) **Right Bound?** is then displayed in the bottom left hand corner of the Viewing Window. Move cursor to the right of the *x*-intercepts and press **ENTER**.



3) **Guess?** is then displayed in the bottom left corner of the Viewing Window. Move cursor to a point near the *x*-intercepts and press **ENTER**.

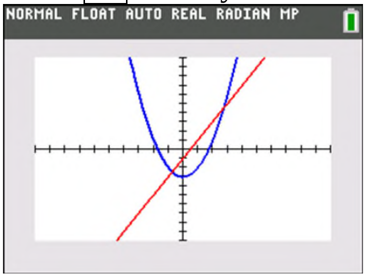


4) The *x*-intercepts will be displayed at the bottom of the Viewing Window. Try again on the other *x*-intercept.



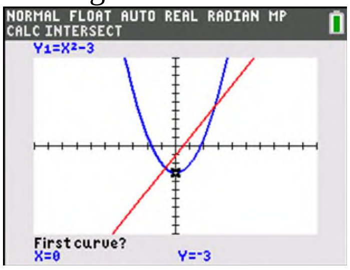
<u>Determining the points of intersection of 2 curves:</u>

1. Go to **Y=** and key in $Y_1 = x^2 - 3$ and $Y_2 = 2x - 1$. Press **GRAPH**.

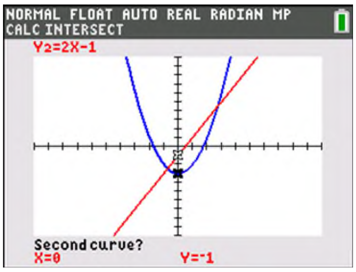


2. Press **2ND** **TRACE** and select **5:intersect** from the **CALCULATE** menu.

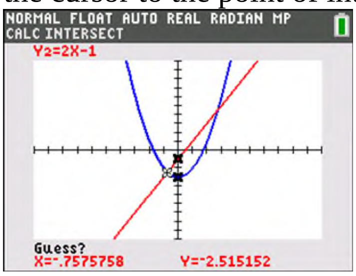
3. The current graph is displayed with **First curve?** in the bottom left hand corner of the Viewing Window. Make sure the cursor is on the first graph and press **ENTER**.



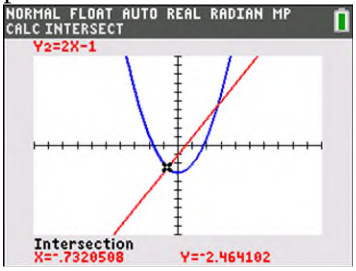
4. **Second curve?** is then displayed in the bottom left hand corner of the Viewing Window. Make sure the cursor is now on the 2nd graph and press **ENTER**.



5. **Guess?** is then displayed in the bottom left hand corner of the Viewing Window. Move the cursor to the point of intersection that you want to find and press **ENTER**.



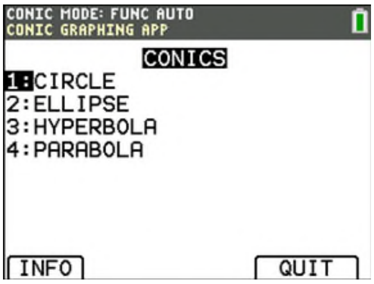
6. Intersection is displayed in the bottom left corner. Try again on the other intersection point.



For more information, please refer to the Manual.

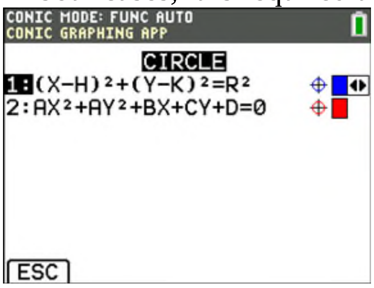
Annex B
Using the Conics Application of TI-84 Plus C Graphic Calculator

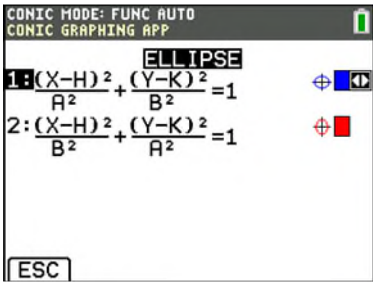
1. Press **APPS** and select **2:Conics,** followed by the required shape .



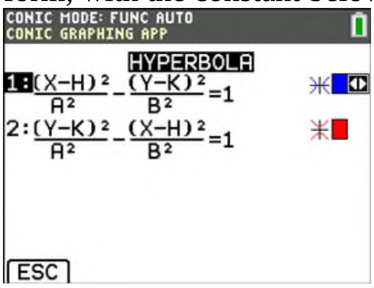
Note: To select the option at the lower left and right corners of the screen, press **Y=** or **Graph** .

2. For each shape, there are 2 available formulas. Note that for ellipse, the first formula $\frac{(\boldsymbol{X}-\boldsymbol{H})^2}{\boldsymbol{A}^2}+\frac{(\boldsymbol{Y}-\boldsymbol{K})^2}{\boldsymbol{B}^2}=1$ is used when the major axis is horizontal and $\frac{(\boldsymbol{X}-\boldsymbol{H})^2}{\boldsymbol{B}^2}+\frac{(\boldsymbol{Y}-\boldsymbol{K})^2}{\boldsymbol{A}^2}=1$ is used when the major axis is vertical.
In both cases, it is required that $A > B$.



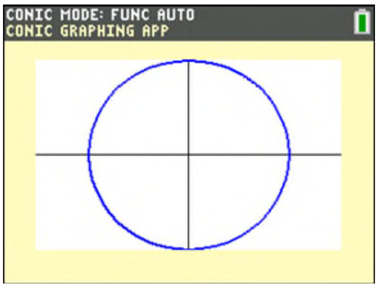


Note that for hyperbola, the second formula for vertical hyperbola is written in a different form, with the constant below Y as A^2 , instead of B^2 .



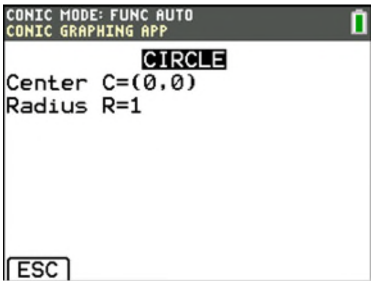
3. Key in the values of the respective parameters.

4. Press **GRAPH** to view the graph.

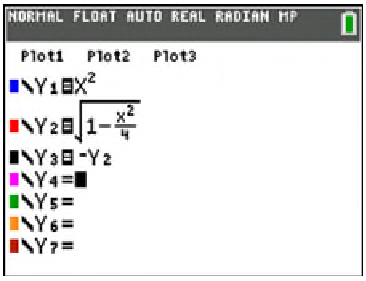


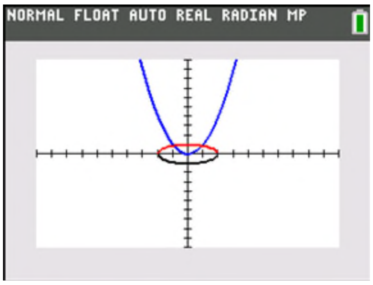
5. To change the window settings, press **MODE** and select **MAN** under **WINDOW SETTINGS**. Then press **ESC** (the button **Y=**), followed by **WINDOW** or **ZOOM** as per usual.

6. To find the centre of the conics, press **ALPHA** **ENTER**. Note that you cannot do so when the graph is being displayed. In this case, select **ESC** (the button **Y=**) then press **ALPHA** **ENTER**.



7. However, this application does not allow you to plot conics with other general graphs together on the same axes. If you want to draw a conic and another graph on the same axes, then you have to use the **FUNCTION** graphing mode and key in the equations of the conics in the form *y* in terms of *x*. For example, we wish to draw the graphs $\frac{x^2}{2^2}+y^2=1$ and $y=x^2$ on the same axes in the G.C. Then we have to key in three equations in the G.C: $y=x^2$, $y=\sqrt{1-\frac{x^2}{4}}$ and $y=-\sqrt{1-\frac{x^2}{4}}$.





Annex C
Using the TI 84 Plus C Graphic Calculator for parametric equations

(Refer to Example 11) The curve C has parametric equations $$x = t^2 + 4t\ ,\quad y = t^3 + t^2 .$$ Sketch the curve for $-2 \le t \le 1$

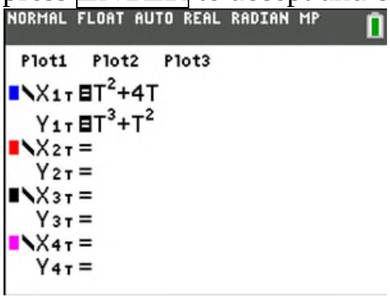
1. Press **MODE** to go to the Mode Menu.

2. Go down to line 5 and move the cursor to **PARAMETRIC**. Press **ENTER** to accept.

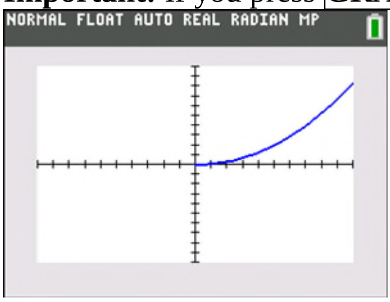
3. To go to the Home Screen, press **CLEAR** or **2ND** **QUIT**.

4. Press **Y=** to display the Function Definition Screen. Use Scroll Buttons to move to **X1T =** and **Y1T =**. Press **CLEAR** to delete previous equations, if needed.

5. Key in the function into the calculator using **X,T,θ,n,** for the variable *x*. For example, key in $\mathrm{X_{1T}} = \mathrm{T}^2 + 4\mathrm{T}$ and $\mathrm{Y_{1T}} = \mathrm{T}^3 + \mathrm{T}^2$. Use brackets appropriately as far as possible. Then press **ENTER** to accept and confirm the equation.

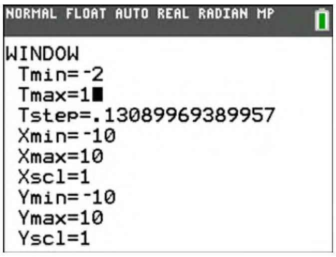


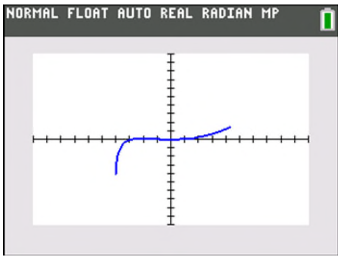
6. **Important:** If you press **GRAPH** now, an incomplete graph will be displayed!!!



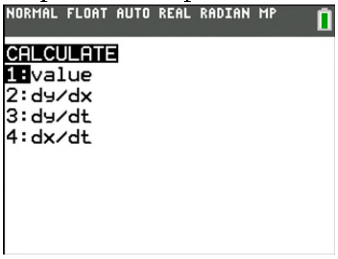
(This is due to the default **TMIN** and **TMAX** settings. You need to change this every time to sketch the graph of a set of parametric equations)

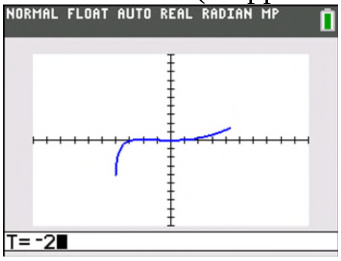
7. Press **WINDOW** and change the **TMIN** and **TMAX** to the range of values of the parameter as given by the question, before you press **GRAPH**.

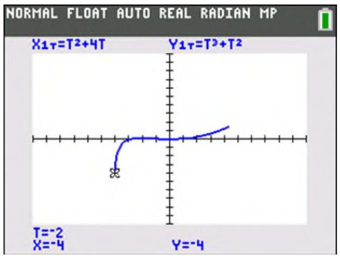




8. To find the end points of the graph, press **2ND** **TRACE** and select **1:value** from the **CALCULATE** menu. **T=** is seen at the bottom of the Viewing Window. Key in the minimum value for the value of *t* (if applicable) and the corresponding *x* and *y* values would be given.
Repeat the steps for the maximum value of *t* (if applicable).







9. If the question did not give a range for the parameter, it means the parameter can take all real values. In this case, choose appropriate values for **TMIN** and **TMAX**, usually -10 and 10 respectively, so that the whole curve will be displayed when you press **GRAPH**.

10. **Note:** Do not press **ZOOM** **6** to choose **ZStandard** now as the settings for **TMIN** and **TMAX** will be changed back to the wrong default settings.

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