1 A function f is defined by $\mathrm{f}(x)=ax^3+bx^2+cx+d$. The graph of $y=\mathrm{f}(x)$ passes through the points $(-3,4)$ and $(1,8)$. Given that the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2,\frac{1}{4})$ find the values of a,b,c and d. [4]

[4]	$\mathrm{f}(x)=ax^3+bx^2+cx+d$ $\mathrm{f}(-3)=4\Rightarrow -27a+9b-3c+d=4$ ----- (1) $\mathrm{f}(1)=8\Rightarrow a+b+c+d=8$ ----- (2) Since the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2,\frac{1}{4})$, the graph of $y=\mathrm{f}(x)$ has a turning point at $(2,4)$. $\therefore \mathrm{f}(2)=4$ and $\mathrm{f}'(2)=0$. $\mathrm{f}(2)=4\Rightarrow 8a+4b+2c+d=4$ ------ (3) $\mathrm{f}'(x)=3ax^2+2bx+c$ $\mathrm{f}'(2)=0\Rightarrow 12a+4b+c=0$ ------ (4) From the GC, $a=1,\ b=-1,\ c=-8$ and $d=16$.	Question was well done by most students. Some students had difficulty with understanding the statement: “$y=\dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2,\frac{1}{4})$” with some working out $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{f}'(x)}{\left[\mathrm{f}(x)\right]^2}=0$.

2 [The volume of a sphere with radius r is given by $\frac{4}{3}\pi r^3$ and the surface area of a sphere with radius r is given by $4\pi r^2$.]

(a) The volume of an expanding sphere is increasing at a constant rate of $5\ \text{cm}^3\text{s}^{-1}$. Show that, at any instant, the rate of increase of the surface area is $\frac{k}{r}\ \text{cm}^2\text{s}^{-1}$, where r is the radius of the sphere and k is a constant to be determined. [3]

(b) Find the exact rate of change of surface area of the expanding sphere when the surface area is $20\ \text{cm}^2$. [2]

(a) **[3]**	Volume, $V=\frac{4}{3}\pi r^3$ and Surface area, $A=4\pi r^2$ $\frac{\mathrm{d}V}{\mathrm{d}r}=4\pi r^2$ $\qquad$ $\frac{\mathrm{d}A}{\mathrm{d}r}=8\pi r$ $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $=\frac{\mathrm{d}A}{\mathrm{d}r}\times\left(\frac{\mathrm{d}r}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}\right)$ $=8\pi r\times\frac{1}{4\pi r^2}\times 5$ $=\frac{10}{r}$, where $k=10$. (shown)	Question was well done by most students. **Common error:** Some students related but then treated A or V as constants. So, by writing $V=\frac{Ar}{3}$ or $A=\frac{3V}{r}$, one should actually get $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{1}{3}\left(A+r\frac{\mathrm{d}A}{\mathrm{d}r}\right)$ or $\frac{\mathrm{d}A}{\mathrm{d}r}=3\left(-\frac{V}{r^2}+\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\right)$ and not $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{A}{3}$ or $\frac{\mathrm{d}A}{\mathrm{d}r}=-\frac{3V}{r^2}$.
(b) **[2]**	When $A=20$, $4\pi r^2=20$ $r=\sqrt{\frac{5}{\pi}}$ (since $r>0$) $\therefore \frac{\mathrm{d}A}{\mathrm{d}t}=10\sqrt{\frac{\pi}{5}}=2\sqrt{5\pi}\ \text{cm}^2\text{s}^{-1}$.	

3 **(a)** Without using a calculator, solve exactly $\dfrac{x^2-x-1}{x+1}\le 1$. [4]

(b) Hence solve exactly $\dfrac{x^2+|x|-1}{1-|x|}\le 1$. [4]

<table>
<tr>
<td>(a)
[4]</td>
<td>

$$\frac{x^2-x-1}{x+1}\le 1$$

$$\frac{x^2-x-1}{x+1}-1\le 0$$

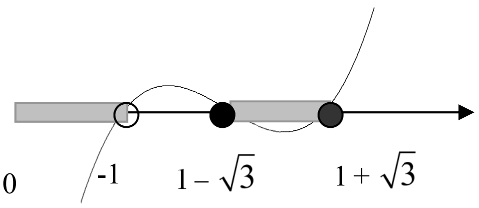
$$\frac{x^2-x-1-(x+1)}{x+1}\le 0$$

$$\frac{x^2-2x-2}{x+1}\le 0$$

Consider $x^2-2x-2=0$:

$$x=\frac{2\pm\sqrt{(-2)^2-4(1)(-2)}}{2(1)}=1\pm\sqrt{3}$$

So $\dfrac{\left(x-1-\sqrt{3}\right)\left(x-1+\sqrt{3}\right)}{x+1}\le 0$



$\left(x-1-\sqrt{3}\right)\left(x-1+\sqrt{3}\right)(x+1)\le 0$

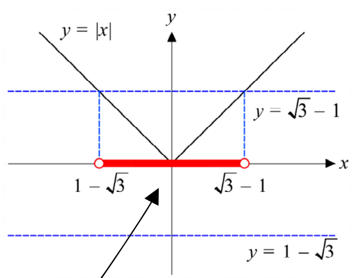
$x<-1$ or $1-\sqrt{3}\le x\le 1+\sqrt{3}$

</td>
<td>Most students are able to manipulate the inequality and solve for the roots. However, some students are unable to identify the regions correctly and not able to get the correct final answer.</td>
</tr>
<tr>
<td>(b)
[4]</td>
<td>

To solve $\dfrac{x^2+|x|-1}{1-|x|}\le 1$

$$\frac{\left(-|x|\right)^2-\left(-|x|\right)-1}{\left(-|x|\right)+1}\le 1$$

Replace x with $-|x|$ from part (a)



Hence, $-|x|<-1$ or $1-\sqrt{3}\le -|x|\le 1+\sqrt{3}$

i.e. $|x|>1$ or $-1-\sqrt{3}\le |x|\le -1+\sqrt{3}$

$x<-1$ or $x>1$ or $1-\sqrt{3}\le x\le -1+\sqrt{3}$

</td>
<td>Majority of the students are able to identify the replacement correctly. However, many are not able to open up the modulus and get the correct answer.

Similar Question: Tut 2 Qn 8(c)</td>
</tr>
</table>

4 **(a)** Find $\displaystyle\int \frac{\cos x}{\cos 3x + \cos x}\, \mathrm{d}x$. [3]

(b) Find $\displaystyle\int x \tan^{-1} x^2 \ \mathrm{d}x$. Hence find the exact value of $\displaystyle\int_{-1}^{1} \left| x \tan^{-1} x^2 \right| \mathrm{d}x$. [5]

<table>
<tr>
<td>(a)
[3]</td>
<td>

$$\int \frac{\cos x}{\cos 3x + \cos x}\, \mathrm{d}x = \int \frac{\cos x}{2\cos 2x \cos x}\, \mathrm{d}x$$
$$= \frac{1}{2}\int \sec 2x \ \mathrm{d}x$$
$$= \frac{1}{4}\ln\left|\sec 2x + \tan 2x\right| + c,$$

where c is an arbitrary constant.

</td>
<td>Students are able to use the factor formula to simplify the integration to $\frac{1}{2}\int \sec 2x \ \mathrm{d}x$.
However, quite a number of students are not able to integrate $\sec 2x$.</td>
</tr>
<tr>
<td>(b)
[5]</td>
<td>

$\int x \tan^{-1} x^2 \,\mathrm{d}x$

$u = \tan^{-1} x^2 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = x$

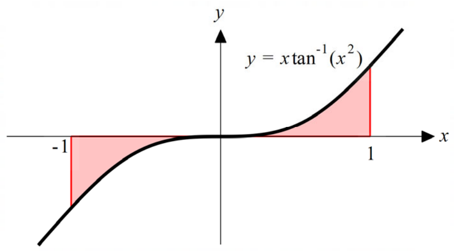
$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+\left(x^2\right)^2}(2x) \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\tan^{-1} x^2 - \int \frac{x^2}{2}\left(\frac{2x}{1+x^4}\right)\mathrm{d}x$$
$$= \frac{x^2}{2}\tan^{-1} x^2 - \int \frac{x^3}{1+x^4}\ \mathrm{d}x$$
$$= \frac{x^2}{2}\tan^{-1} x^2 - \frac{1}{4}\ln\left(1+x^4\right) + c, \text{ where } c \text{ is an arbitrary constant.}$$

Method 1

$$\int_{-1}^{1}\left|x\tan^{-1}x^2\right|\ \mathrm{d}x = 2\int_0^1 x\tan^{-1}x^2\,\mathrm{d}x \quad (\because \text{ of symmetry})$$
$$= \left[x^2\tan^{-1}x^2 - \frac{1}{2}\ln\left(1+x^4\right)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

[Note that $y = x\tan^{-1}x^2$ is an odd function since $(-x)\tan^{-1}(-x)^2 = -x\tan^{-1}x^2 = -\left(x\tan^{-1}x^2\right)$]



Method 2

$$\int_{-1}^{1}\left|x\tan^{-1}x^2\right|\ \mathrm{d}x$$
$$= -\int_{-1}^{0} x\tan^{-1}x^2\,\mathrm{d}x + \int_0^1 x\tan^{-1}x^2\,\mathrm{d}x$$
$$= -\left[\frac{1}{2}x^2\tan^{-1}x^2 - \frac{1}{4}\ln\left(1+x^4\right)\right]_{-1}^{0} + \left[\frac{1}{2}x^2\tan^{-1}x^2 - \frac{1}{4}\ln\left(1+x^4\right)\right]_0^1$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

</td>
<td>Most students are able to use the integration by parts to solve this problem. There was quite a number of students who are unable to open up the modulus function.

Similar Question:
Assignment 8B Qn 4,
Mock Paper 1 Qn 5(b)</td>
</tr>
</table>

5 **(a)** Using the formulae for $\sin(A \pm B)$, prove that

$$\sin(2r+1)\theta - \sin(2r-1)\theta = 2\cos 2r\theta \sin\theta .$$ [1]

(b) Hence find a formula for $\sum_{r=1}^{n} \cos 2r\theta$, where $0 < \theta < \pi$, in terms of $\sin(2n+1)\theta$ and $\sin\theta$. [3]

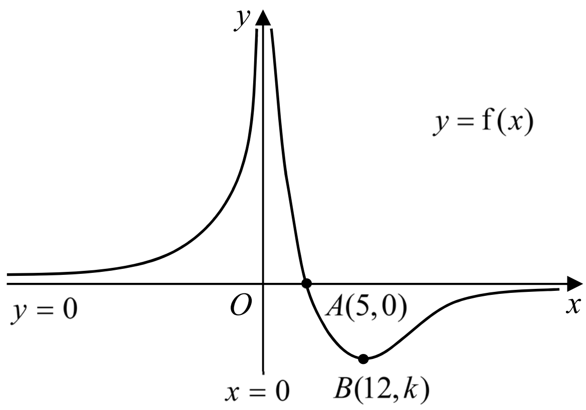
(c) Using the formula found in part **(b)**, show that the sum of the series

$$\sin^2 10\theta + \sin^2 11\theta + \sin^2 12\theta + \ldots + \sin^2 20\theta, \text{ for } 0 < \theta < \pi$$

is $k - \dfrac{\sin(41\theta) - \sin(19\theta)}{4\sin\theta}$, where k is a constant to be determined. [4]

(a) **[1]**	$\begin{aligned} &\sin(2r+1)\theta - \sin(2r-1)\theta \\ &= \sin 2r\theta\cos\theta + \cos 2r\theta \sin\theta - (\sin 2r\theta \cos\theta - \cos 2r\theta \sin\theta) \\ &= 2\cos 2r\theta \sin\theta \text{ (proven)} \end{aligned}$	The question indicated the use of addition formulae, so it must be used instead of factor formulae Similar Question: Assignment 6B Qn 2
(b) **[3]**	$\begin{aligned} 2\sum_{r=1}^{n}\cos 2r\theta\sin\theta &= \sum_{r=1}^{n}\left[\sin(2r+1)\theta - \sin(2r-1)\theta\right] \\ &= \cancel{\sin 3\theta} - \sin\theta \\ &\quad + \cancel{\sin 5\theta} - \cancel{\sin 3\theta} \\ &\quad + \cancel{\sin 7\theta} - \cancel{\sin 5\theta} \\ &\quad + \ldots \\ &\quad + \cancel{\sin(2n-1)\theta} - \cancel{\sin(2n-3)\theta} \\ &\quad + \sin(2n+1)\theta - \cancel{\sin(2n-1)\theta} \\ &= \sin(2n+1)\theta - \sin\theta \end{aligned}$ $\therefore \sum_{r=1}^{n}\cos 2r\theta = \dfrac{\sin(2n+1)\theta - \sin\theta}{2\sin\theta}$	Most did well on this part, but do remember that cancellations must be clearly shown, and that the final two lines are in terms of *n*, not *r*, because *r* is the running index. Similar Question: Assignment 6B Qn 2
(c) **[4]**	$\begin{aligned} &\sin^2 10\theta + \sin^2 11\theta + \sin^2 12\theta + \ldots + \sin^2 20\theta \\ &= \sum_{r=10}^{20}\sin^2 r\theta \\ &= \sum_{r=10}^{20}\frac{1-\cos(2r\theta)}{2} \\ &= \frac{1}{2}(20-10+1) - \frac{1}{2}\left[\sum_{r=1}^{20}\cos(2r\theta) - \sum_{r=1}^{9}\cos(2r\theta)\right] \\ &= \frac{11}{2} - \frac{1}{2}\left[\frac{\sin(41\theta)-\sin\theta}{2\sin\theta} - \frac{\sin(19\theta)-\sin\theta}{2\sin\theta}\right] \\ &= \frac{11}{2} - \frac{\sin(41\theta)-\sin(19\theta)}{4\sin\theta}\text{, where } k = \frac{11}{2} \text{ (shown)} \end{aligned}$	As this is a show question, the use of the answer in part (b) must be explicitly shown, especially the subtraction of the first 9 terms from the first 20 terms. Also, students should note that the number of terms from $r = 10$ to 20 is eleven.

6 **(a)** The diagram below shows the graph of $y = f(x)$.



The graph cuts the x-axis at point $A\,(5,0)$. It has a turning point at $B(12,k)$, where $k < 0$ and asymptotes with equations $x = 0$ and $y = 0$. On separate diagrams, sketch the graph of

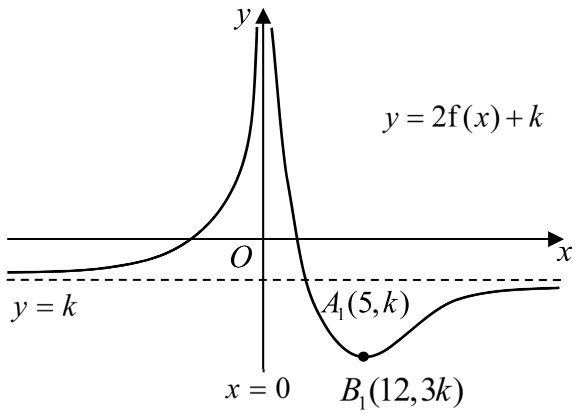
(i) $y = 2\mathrm{f}(x) + k$, stating the equations of any asymptotes and the coordinates of any turning point(s). [2]

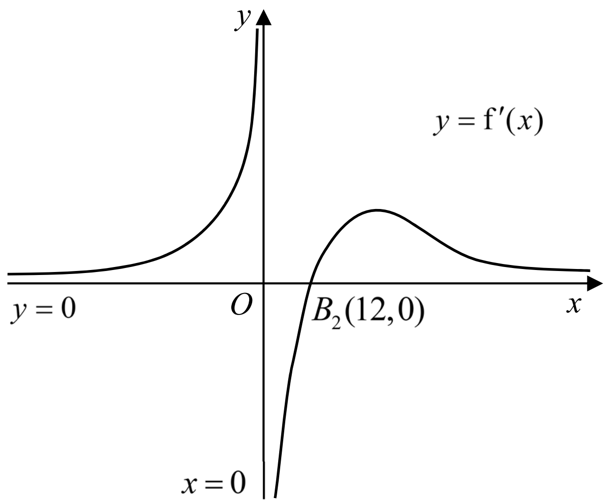
(ii) $y = \mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of any point(s) where the graph crosses the axes. [2]

(b) The graph with equation $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = x(x-1)^2$ undergoes a single transformation and the equation of the resultant graph is $y = \mathrm{h}(x)$. Describe the transformation if

(i) $\mathrm{h}(x) = -x(x+1)^2$, [1]

(ii) $\mathrm{h}(x) = \dfrac{1}{8}x(x-2)^2$. [2]

<table>
<tr>
<td>(a)(i)
[2]</td>
<td>Note that $k < 0$.

</td>
<td>One possible way to do this part is:
$y = \mathrm{f}(x) \xrightarrow{\text{Replace } y \text{ by } y/2}$
$y = 2\mathrm{f}(x) \xrightarrow{\text{Replace } y \text{ by } y-k}$
$y = 2\mathrm{f}(x) + k$
which means that there is a stretch of factor 2 parallel to y-axis, and a translation $-k$ units in the negative y direction.
Note that (5, k) was not asked in the question and so no marks were allocated to it.</td>
</tr>
</table>

(a)(ii) **[2]**		Most students did well on this part. Students are reminded to indicate the equations of the asymptotes as well as the *x*-intercept on the graph.
b(i) **[1]**	$\mathrm{g}(x) = x(x-1)^2$ Consider $\mathrm{g}(-x) = -x(-x-1)^2 = -x(x+1)^2 = \mathrm{h}(x)$ Reflection of the graph of g in the *y*-axis to obtain the graph of h.	You are required to describe, not just indicate "replace *x* by -*x*".
b(ii) **[2]**	$\mathrm{g}(x) = x(x-1)^2$ Consider $\mathrm{g}\left(\frac{x}{2}\right) = \frac{x}{2}\left(\frac{x}{2}-1\right)^2 = \frac{x}{2}\left(\frac{x-2}{2}\right)^2 = \frac{1}{8}x\left(x-2\right)^2 = \mathrm{h}(x)$ Scaling the graph of g parallel to the *x*-axis by factor of 2 to obtain the graph of h.	The question asked for a <u>single</u> transformation.

7 An arithmetic series has first term a and common difference d, where $a > 0$ and $d \neq 0$. The first, sixth and ninth terms of the arithmetic series are consecutive terms of a geometric series.

(a) Show that $25d = -2a$. [2]

(b) The sum of the first n terms of the arithmetic series is denoted by S. Find the set of possible values of n for which S exceeds $6a$. [3]

(c) Find the common ratio of the geometric series, and deduce that the geometric series is convergent. [2]

(d) Hence find the smallest value of m such that the sum of the terms of the geometric series after, but not including, the mth term, is less than 1% of the sum to infinity. [3]

(a) **[2]**	Given $a,\ a+5d,\ a+8d$ are consecutive terms of a GP. Then $\dfrac{a+5d}{a} = \dfrac{a+8d}{a+5d}$ $(a+5d)^2 = a(a+8d)$ $a^2 + 10ad + 25d^2 = a^2 + 8ad$ $25d^2 = -2ad$ Since $d \neq 0$, $25d = -2a$ (shown).	Common mistakes: • $\dfrac{a+5d}{a} = \dfrac{a+8d}{a+5d} \Rightarrow \dfrac{1}{a} = \dfrac{a+8d}{1}$ Similar question: Tut 6A Qn 2, Assignment 6A Qn 1
(b) **[3]**	$\dfrac{n}{2}\left[2a + (n-1)\left(-\dfrac{2}{25}a\right)\right] > 6a$ Since $a > 0$, $n - \dfrac{n^2 - n}{25} > 6 \quad \Rightarrow \quad n^2 - 26n + 150 < 0$ Using GC, $8.64 < n < 17.36$ Set of values of n $= \{n \in \mathbb{Z} : 9 \leq n \leq 17\}$ or $\{9, 10, 11, 12, 13, 14, 15, 16, 17\}$.	Common mistakes: • $\dfrac{n}{2}[2a + (n-1)d] = na + (n-1)d$ • $d\left(n^2 - 26n + 150\right) > 0$ $\Rightarrow n^2 - 26n + 150 > 0$ Since $a > 0$, from **(a)**, $d < 0$. So, actually should be $n^2 - 26n + 150 < 0$ • $n \in [9, 17]$ There is a need to specify that the values of n are integers.
(c) **[2]**	Common ratio, $r = \dfrac{a+5d}{a} = \dfrac{a + (-\frac{2}{5}a)}{a} = \dfrac{3}{5}$ **Method 1** Since $\lvert r \rvert = \dfrac{3}{5} < 1$, the geometric series is convergent. **Method 2** $S_n = \dfrac{a\left[1 - \left(\frac{3}{5}\right)^n\right]}{1 - \frac{3}{5}} = \dfrac{5}{2}a\left[1 - \left(\dfrac{3}{5}\right)^n\right]$ As $n \to \infty$, $\left(\dfrac{3}{5}\right)^n \to 0$. Then, $S_n \to \dfrac{5}{2}a$. So, the geometric series is convergent.	Common mistakes: • $r < 1$ for the series to be convergent $r < 1$ would include values less than -1 which will not make the series convergent • $u_n = a\left(\frac{3}{5}\right)^{n-1} \to 0$ as $n \to \infty$ so, the series is convergent Note that $u_n \to 0$ does not imply series is convergent.

<table>
<tr>
<td>(d)
[3]</td>
<td>Let b be the first term of the geometric series. It can be deduced that $b > 0$ as a is a term ($a > 0$) and $r > 0$.

The sum of the terms of the series after, but not including, the mth term
$= u_{m+1} + u_{m+2} + u_{m+3} + \cdots$
$= S_\infty - S_m$
$= \dfrac{b}{1-r} - \dfrac{b(1-r^m)}{1-r}$
$= \dfrac{b}{1-r}\left[1-(1-r^m)\right]$
$= \dfrac{br^m}{1-r}$

$$\frac{b\left(\frac{3}{5}\right)^m}{1-\frac{3}{5}} < 0.01\left(\frac{b}{1-\frac{3}{5}}\right)$$
Since $b > 0$, $\left(\frac{3}{5}\right)^m < 0.01$
$$m \ln\frac{3}{5} < \ln 0.01$$
$$m > \frac{\ln 0.01}{\ln\frac{3}{5}} = 9.02$$
Smallest value of $m = 10$</td>
<td>Common mistakes:
• $\dfrac{b\left(\frac{3}{5}\right)^m}{1-\frac{3}{5}} < 0.1\left(\dfrac{b}{1-\frac{3}{5}}\right)$
1% is equivalent to 0.01, and not 0.1
• $\dfrac{3}{5}^m < 0.01$
Quite a few students wrote the above. This means that the power m is meant for 3 only. The proper way of writing should be $\left(\frac{3}{5}\right)^m < 0.01$
• $m \ln\frac{3}{5} < \ln 0.01$
$\Rightarrow m < \dfrac{\ln 0.01}{\ln\frac{3}{5}}$
Note that $\ln\left(\frac{3}{5}\right) < 0$</td>
</tr>
</table>

8 **Do not use a calculator in answering this question.**

(a) The complex number w is such that where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that

$$ww^* = 4 - 2\text{i} + 2\text{i}w^*,$$

find the two possible values of w. [4]

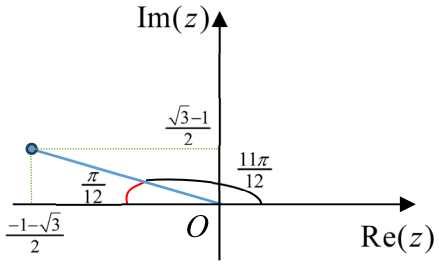
(b) The complex number z is given by $z = \dfrac{1-\sqrt{3}\,\text{i}}{-1+\text{i}}$.

(i) Find arg(z). [3]

(ii) Find z in cartesian form $x + \text{i}y$. [2]

(iii) Hence find the value of $\tan\dfrac{\pi}{12}$ in the form $c + d\sqrt{3}$, where c and d are exact integers to be determined. [3]

(a) **[4]**	$ww^* = 4 - 2\text{i} + 2\text{i}w^*$ Let $w = a + \text{i}b$, then $a^2 + b^2 = 4 - 2\text{i} + 2\text{i}(a - \text{i}b)$ $a^2 + b^2 = 4 + 2b + 2\text{i}(a-1)$ Comparing real and imaginary parts, $a^2 + b^2 = 4 + 2b$ and $0 = 2(a-1) \Rightarrow a = 1$ $1 + b^2 = 4 + 2b$ $b^2 - 2b - 3 = 0$ $(b-3)(b+1) = 0$ $b = 3$ or $b = -1$ $\therefore$ The 2 roots are $1 + 3\text{i}$ and $1 - \text{i}$.	Overall, this part was well-done by most students, demonstrating a clear understanding and application of the method for comparing real and imaginary parts. However, a few instances of carelessness in manipulation were noted.
(b)(i) **[3]**	$\arg\left(\dfrac{1-\sqrt{3}\text{i}}{-1+\text{i}}\right)$ $= \arg\left(1-\sqrt{3}\text{i}\right) - \arg\left(-1+\text{i}\right) + 2\pi$ $= -\dfrac{\pi}{3} - \dfrac{3\pi}{4} + 2\pi$ $= -\dfrac{13\pi}{12} + 2\pi$ $= \dfrac{11\pi}{12}$	Many students failed to connect this part to the application of argument properties. Among those who did, several were careless in identifying the correct quadrant for the complex number. Additionally, do note that no credit will be awarded for answers derived solely from a calculator.

(ii) **[2]**	$z=\dfrac{1-\sqrt{3}\mathrm{i}}{-1+\mathrm{i}}$ $=\dfrac{1-\sqrt{3}\mathrm{i}}{-1+\mathrm{i}}\times\dfrac{-1-\mathrm{i}}{-1-\mathrm{i}}$ $=\dfrac{-1-\mathrm{i}+\sqrt{3}\mathrm{i}-\sqrt{3}}{1+1}$ $=\dfrac{-1-\sqrt{3}}{2}+\dfrac{\sqrt{3}-1}{2}\mathrm{i}$	Quite many rationalize by multiplying (i + 1). While this worked, this is not the complex conjugate of $-1+\mathrm{i}$. Do revise the concept of complex conjugate.
(iii) **[3]**	 From the diagram, $\tan\dfrac{\pi}{12}=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ $=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\times\dfrac{\sqrt{3}-1}{\sqrt{3}-1}$ $=\dfrac{3-2\sqrt{3}+1}{3-1}$ $=2-\sqrt{3}$ where $c=2$ and $d=-1$	Full credit will not be given if the argument in **(b) (i)** was incorrectly determined. Many gave the final answer as $-2+\sqrt{3}$, which is clearly incorrect as $\tan\dfrac{\pi}{12}$ should give a positive value. Do try to double-check your answer whenever possible.

9 A curve C has equation $y = 2ax + \dfrac{b}{x}$ where a and b are non-zero real constants and $x \neq 0$.

(a) Using differentiation, determine whether C has any stationary points if $ab < 0$. [2]

It is now given that $b = \dfrac{1}{2}a$ where $a > 0$.

(b) Sketch C, stating the equations of any asymptotes, and the coordinates of any stationary points and points where C crosses the axes (if any). [3]

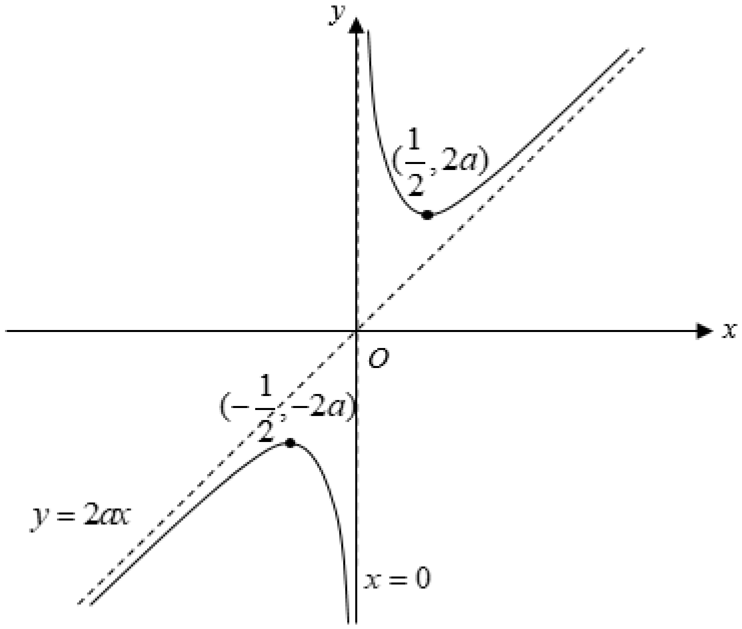
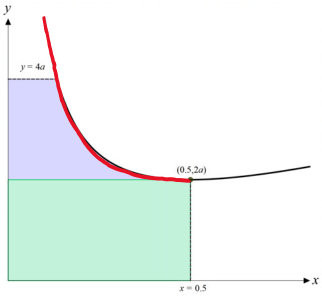
(c) State the range of values of k, in terms of a, for which the equation $2ax + \dfrac{b}{x} = kx$ has no real roots. [1]

(d) The region bounded by C, the axes, the lines $x = \dfrac{1}{2}$ and $y = 4a$ is rotated about the y-axis through 2π radians. Show that the volume generated is given by

$$\frac{1}{2}a\pi + \frac{\pi}{8a^2}\int_{2a}^{4a}\left(y^2 - 2a^2 - y\sqrt{y^2 - 4a^2}\right)\mathrm{d}y .$$

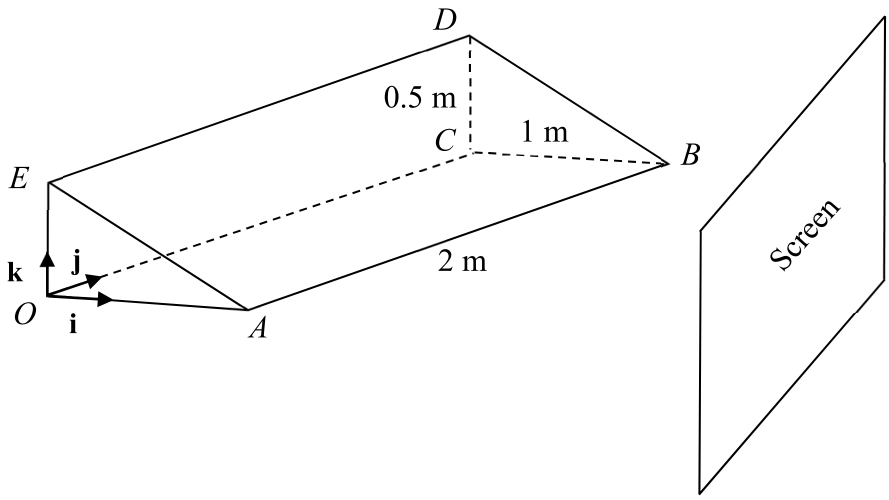
Hence, find, in terms of a and π, the exact volume generated. [6]

(a) **[2]**	$y = 2ax + \dfrac{b}{x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2a - \dfrac{b}{x^2}$ If C has stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2a - \dfrac{b}{x^2} = 0$ $x^2 = \dfrac{b}{2a}$ However, if $ab < 0$, then $\dfrac{b}{a} < 0$. So $x^2 = \dfrac{b}{2a} < 0$ has no solutions, thus C has no stationary points if $ab < 0$.	Most students are able to differentiate and relate the given condition to deduce non-existence of stationary points.
(b) **[3]**	Given that $b = \dfrac{1}{2}a$, $y = 2ax + \dfrac{a}{2x} = a\left(2x + \dfrac{1}{2x}\right)$. Equations of asymptotes are $y = 2ax$ and $x = 0$. Since $b = \dfrac{1}{2}a$, thus $ab > 0$ and C has stationary points. Coordinates of stationary points = $\left(\dfrac{1}{2}, 2a\right)$ and $\left(-\dfrac{1}{2}, -2a\right)$.	While most students are able to deduce the shape of graph correctly, some students missed out the oblique asymptote which may caused by referring to GC without knowing the existence of 2 asymptotes. The drawing of stationary points and curve should be smooth

		and approaching the respective asymptotes. The graph should be fully labelled as stated in the question.
(c) **[1]**	Range of values of k is $k \le 2a$.	The answer can be deduced by referring to the gradient of oblique asymptote and graph in part **(b)**.
(d) **[6]**	$y = 2ax + \frac{a}{2x}$ $4ax^2 - 2yx + a = 0$ $x = \frac{2y \pm \sqrt{4y^2 - 4(4a)(a)}}{2(4a)} = \frac{2y \pm \sqrt{4y^2 - 16a^2}}{8a} = \frac{y \pm \sqrt{y^2 - 4a^2}}{4a}$ *Note for students :* *Coord. of turning point is* $(0.5, 2a)$. *We are interested in the portion of the curve that's highlighted in red, where* $x \le 0.5$. *Hence, equation of the required curve is* $x = \frac{y - \sqrt{y^2 - 4a^2}}{4a}$.  Volume generated	While students attempted to find x in term of y, some did not identity the correct x which disallow them to show the volume generated. Students were able to use the shown result to integrate correctly. Quite a number of students did not manage to simplify and obtain correct final answer.

	$=\pi\left(\frac{1}{2}\right)^2(2a)+\pi\int_{2a}^{4a}\left(\frac{y-\sqrt{y^2-4a^2}}{4a}\right)^2\ \mathrm{d}y$ $=\frac{1}{2}a\pi+\frac{\pi}{16a^2}\int_{2a}^{4a}\left(y^2+y^2-4a^2-2y\sqrt{y^2-4a^2}\right)\mathrm{d}y$ $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\int_{2a}^{4a}\left(y^2-2a^2-y\sqrt{y^2-4a^2}\right)\mathrm{d}y$ (shown) $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\left[\frac{1}{3}y^3-2a^2y-\frac{1}{2}\frac{\left(y^2-4a^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{2a}^{4a}$ $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\left[\frac{64}{3}a^3-8a^3-\frac{1}{3}\left(24\sqrt{3}a^3\right)-\frac{8}{3}a^3+4a^3+\frac{1}{3}(0)\right]$ $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\left(\frac{44}{3}a^3-8\sqrt{3}a^3\right)$ $=\left(\frac{7}{3}-\sqrt{3}\right)a\pi$	

10 For an upcoming motor-car race, a spectators' gallery is to be set up near the racing track. As part of the preparations, a model of this gallery, shaped as a prism, is constructed.



The diagram above shows the model of the gallery with O as the origin and the unit vectors **i**, **j** and **k** are parallel to OA, OC and OE respectively. Points $(x,\ y,\ z)$ are defined relative to O, where units are in metres.

It is given that $OA = CB = 1$ m, $OC = AB = ED = 2$ m and $OE = CD = 0.5$ m.

(a) Find a cartesian equation of the plane $ABDE$. [2]

A shelter is to be constructed above the plane $ABDE$. On the model, this shelter is a rectangular plane that intersects plane $ABDE$ in the line ED.

(b) Given that the equation of the shelter is $-0.5x + z = h$, show that $h = 0.5$. [1]

(c) Find the acute angle between the plane $ABDE$ and the shelter. [2]

A spotlight is shone towards the gallery and the beam of light is in the form of a line l with cartesian equation $x - a = z,\ y = 1$ for some real number a. The beam lands on a point M on the rectangular surface $ABDE$.

(d) Find in terms of a, the position vector of M, and find the range of values of a. [5]

(e) A large rectangular flat screen is to be placed in front of the gallery at a distance away. The screen can be taken to be part of a vertical plane with equation $12x + 5y = d$, where $d > 50$. Using $a = \frac{1}{2}$, find the value of d so that the shortest distance between point M and the screen is 4 m. [4]

(a) **[2]**	$\overrightarrow{OA} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $\overrightarrow{OE} = \begin{pmatrix}0\\0\\0.5\end{pmatrix}$ Normal to $ABDE$, $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AE}$ $= \begin{pmatrix}0\\2\\0\end{pmatrix} \times \begin{pmatrix}-1\\0\\0.5\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$	This part should have been done better. The normal vector to the plane can be obtained by taking the cross product of 2 vectors **parallel** to the plane, and not the position vectors of 2 points on the plane. Furthermore, the question asked for cartesian equation form of the plane as

	Equation of plane $ABDE$: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=1$ Cartesian equation: $x+2z=1$	the final answer which many students are not familiar with.
(b) **[1]**	Since the shelter plane contains line ED, point E $(0,0,0.5)$ lies on the shelter plane $-0.5x+z=h$ $\therefore -0.5(0)+0.5=h$ $h=0.5$ (shown)	This part was generally well done, but students should note that for a "show" question, the working should be clear and without doubt. Effort should be taken to explain which point is the position vector used to substitute into the equation of the plane.
(c) **[2]**	Acute angle between plane $ABDE$ and the shelter $=\cos^{-1}\dfrac{\left\|\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}-0.5\\0\\1\end{pmatrix}\right\|}{\sqrt{5}\sqrt{1.25}}$ $=53.1^\circ$ (1 d.p)	This part should have been done better. A few students chose to take 90° to subtract the answer. Perhaps a confusion with the formula between line and plane. Some others did not get the correct normal vector from part **(a)**, or copied the vector of the shelter wrongly from the question.
(d) **[5]**	line l : $\mathbf{r}=\begin{pmatrix}a\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ At M, $\begin{pmatrix}a+\lambda\\1\\\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=1$ $a+3\lambda=1 \quad\Rightarrow\quad \lambda=\dfrac{1-a}{3}$ $\overrightarrow{OM}=\begin{pmatrix}a+\dfrac{1-a}{3}\\1\\\dfrac{1-a}{3}\end{pmatrix}=\begin{pmatrix}\dfrac{2a+1}{3}\\1\\\dfrac{1-a}{3}\end{pmatrix}$ From the diagram, $0\le\dfrac{2a+1}{3}\le1$ and $0\le\dfrac{1-a}{3}\le\dfrac{1}{2}$ $-\dfrac{1}{2}\le a\le1$ and $-\dfrac{1}{2}\le a\le1$ $\therefore -\dfrac{1}{2}\le a\le1$	The first part of this question was well done. Care should be taken to simplify the answer. Many students either left out the part on range of values or did not put the range fully for both the lower and upper bound.

(e) **[4]**	When $a=\frac{1}{2}$, $\overrightarrow{OM}=\begin{pmatrix}\frac{2}{3}\\1\\\frac{1}{6}\end{pmatrix}$. Let $S\left(\frac{d}{12},0,0\right)$ be a point on the screen. Then $\overrightarrow{SM}=\begin{pmatrix}\frac{2}{3}-\frac{d}{12}\\1\\\frac{1}{6}\end{pmatrix}$ Shortest distance from M to the screen $=\frac{\left\|\overrightarrow{SM}.\begin{pmatrix}12\\5\\0\end{pmatrix}\right\|}{\sqrt{12^2+5^2}}=4$ $\Rightarrow \frac{\|8-d+5\|}{13}=4 \qquad \Rightarrow \|13-d\|=52$ $13-d=52$ or $13-d=-52$ $d=-39$ (reject as $d>50$) $\quad d=65$	A few students who did well for this question simply imagined a vertical plane that M lies on, and used the distance between the 2 planes: $\frac{\left\|\overrightarrow{OM}.\begin{pmatrix}12\\5\\0\end{pmatrix}-d\right\|}{\sqrt{12^2+5^2}}=4$ thus not needing to find a point on the screen. Many students who did not do the shortest method started well by letting F be the foot of perpendicular from M to the screen. Following which, many faltered by either: 1) Using $\overrightarrow{MF}$ as the position vector of the F. 2) Putting $\left\|\overrightarrow{OF}\right\|=4$. 3) Not putting a modulus around the parameter, thus not having another value to reject. 4) Using a wrong $\overrightarrow{OM}$ from earlier part.

11 A wafer fabrication company uses the floating-zone method to purify polysilicon ingots, each having a uniform cross-sectional area and a length of 200 cm. The method involves placing a polysilicon ingot with impurity concentration C_0 atoms/cm^3 on top of a single seed crystal. The polysilicon ingot is then heated externally by an RF coil, which locally melts the ingot. The impurities prefer to stay in the molten state than in the solid state and thus as the coil and the melt zone move upwards, a single crystal, that is purer, solidifies on top of the seed crystal. A schematic illustration of the method is shown below in Fig. 1 and Fig. 2.

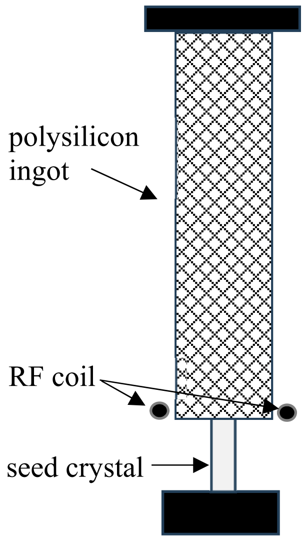


Fig. 1 Initial set-up

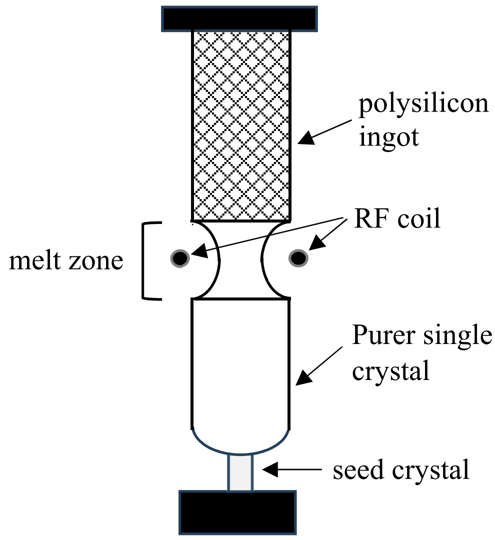


Fig. 2 During purification process

For a 'floating' melt zone of length L cm, the concentration of impurities in the melt zone, C atoms/cm^3, and the distance moved by the RF coil, x cm, are related by the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{1}{L}\left(C_0 - kC\right),$$

where k is a constant such that $0 < k < 1$.

The length of the "floating" melt zone, L cm, adopted by the company is 2 cm and $0 \le x \le 198$. It is also given that when $x = 0$, $C = C_0$.

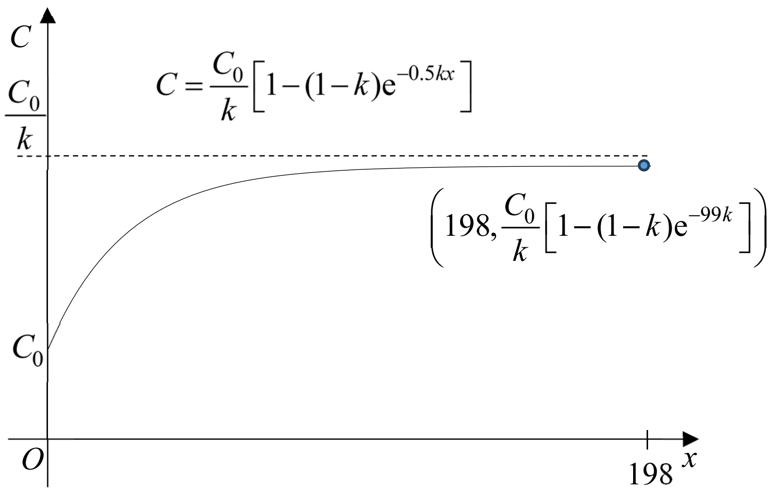
(a) Solve the differential equation to find an expression for C in terms of C_0, k and x. [4]

(b) Sketch the graph of C against x. [2]

(c) Assume that $k = 0.3$ and that the RF coil moves upwards at a constant speed of 8 mm per hour. Find the time taken for the concentration of impurities in the melt zone to reach $2C_0$ and the rate of change of the concentration of impurities, in terms of C_0 at this instant. [5]

The company decides to change the length of the "floating" melt zone.

(d) Explain, with a reason, whether a shorter length is preferable over a longer one. [1]

(a) **[4]**	$$\frac{\mathrm{d}C}{\mathrm{d}x}=\frac{1}{2}\left(C_0-kC\right)$$ $$\int \frac{1}{C_0-kC}\ \mathrm{d}C=\int \frac{1}{2}\ \mathrm{d}x$$ Since $\frac{\mathrm{d}C}{\mathrm{d}x}=\frac{1}{L}\left(C_0-kC\right)>0,\ \left(C_0-kC\right)>0.$ $$-\frac{1}{k}\ln\left(C_0-kC\right)=0.5x+A$$ $$C_0-kC=B\mathrm{e}^{-0.5kx}$$ $$C=\frac{1}{k}\left(C_0-B\mathrm{e}^{-0.5kx}\right)$$ When $x=0,\ C=C_0$ and hence $B=C_0(1-k)$. Thus $C=\frac{C_0}{k}\left[1-(1-k)\mathrm{e}^{-0.5kx}\right]$.	Avoid using C for the arbitrary constant as C is already used to represent the concentration of impurities in this question. When solving for the arbitrary constant, the answer should be in terms of C_0.
(b) **[2]**		Note: It is not necessary to indicate the horizontal asymptote as we are sketching for $0\le x\le 198$.
(c) **[5]**	Assuming $k=0.3$, $C=\frac{C_0}{0.3}\left(1-0.7\mathrm{e}^{-0.15x}\right)$. When $C=2C_0$, $2C_0=\frac{C_0}{0.3}\left(1-0.7\mathrm{e}^{-0.15x}\right)$ $$\mathrm{e}^{-0.15x}=\frac{4}{7}$$ $$x=\frac{\ln\left(\frac{4}{7}\right)}{-0.15}=3.73077\text{ cm}$$ Hence the time taken $=\frac{37.3077}{8}=4.66\text{ h}$.	Give your answers with the units used. Some converted mm to cm incorrectly. You are expected to know the metric system.

	$\frac{\mathrm{d}C}{\mathrm{d}t}=\frac{\mathrm{d}C}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$ $=0.8\times\frac{1}{2}\left(C_0-0.3(2C_0)\right)$ $=0.16C_0$ atoms/cm^3 per hour	Rate of change of the concentration of impurities refers to $\frac{\mathrm{d}C}{\mathrm{d}t}$.
(d) **[1]**	A shorter length for the floating melt zone is preferable since $\frac{\mathrm{d}C}{\mathrm{d}x}=\frac{1}{L}\left(C_0-kC\right)$ will be larger for this case. Thus for the same distance moved, the shorter melt zone has higher concentration of impurities *which means that the single crystal that solidifies has higher purity.*	It is insufficient to say that the concentration of impurities increases as that occurs regardless of length used. The key idea is the increased rate of change of concentration, with respect to x, when a shorter length is used.