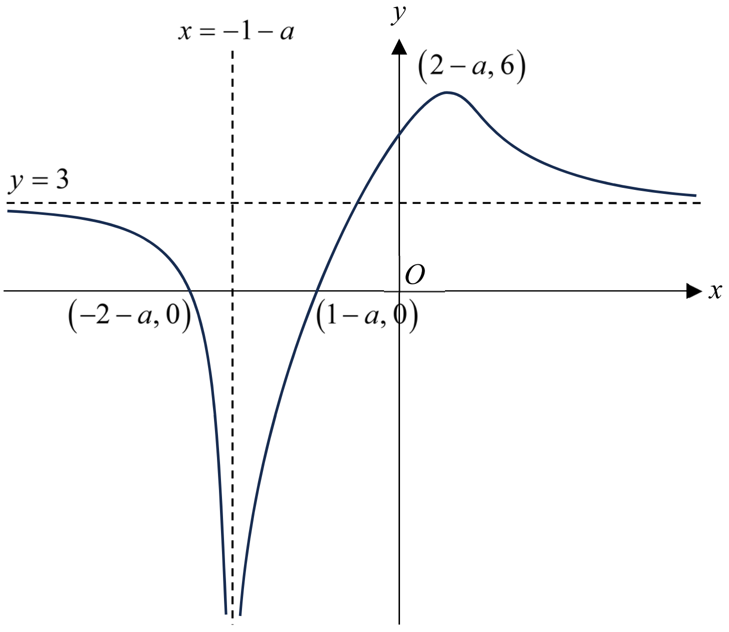
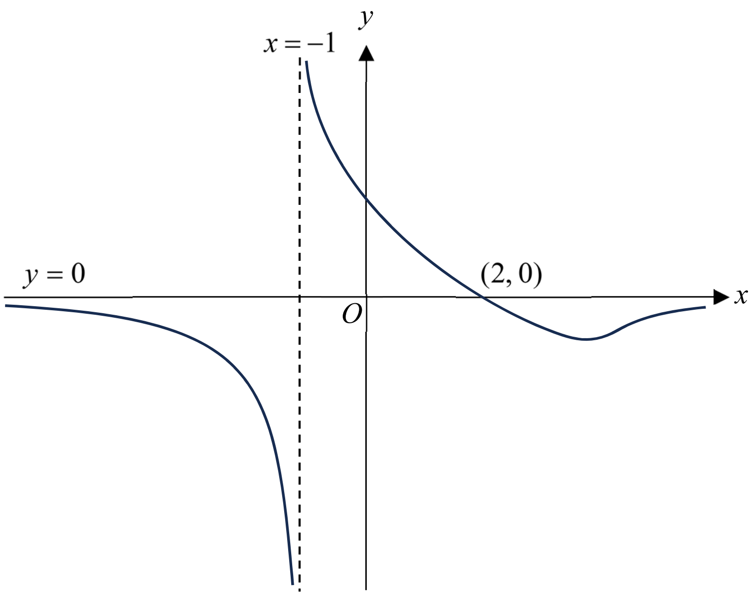
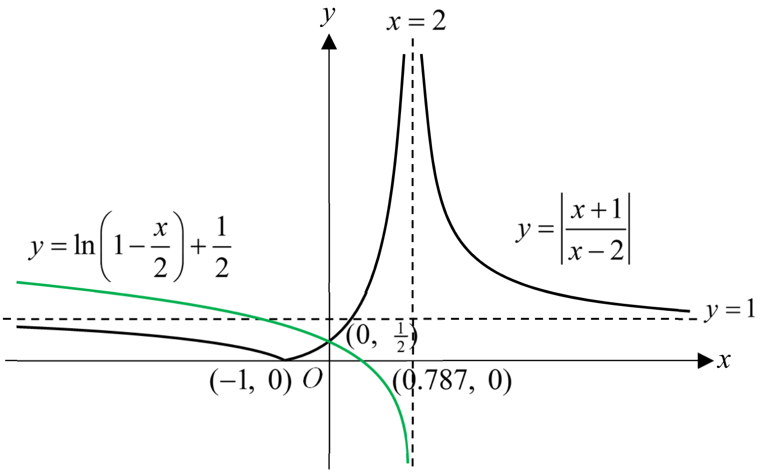
2023 EJC H2 Math Promo Solutions

1	**Solution**
	At $\left(-2,1\right)$, $1=\left(-2\right)^3+a\left(-2\right)^2+b\left(-2\right)+c\Rightarrow 4a-2b+c=9$ --- (1) At $\left(2,-3\right)$, $-3=\left(2\right)^3+a\left(2\right)^2+b\left(2\right)+c\Rightarrow 4a+2b+c=-11$ --- (2) Since $\left(2,1\right)$ lies on $y=\mathrm{f}\left(x+1\right)$, <u>Method 1:</u> consider that $\left(3,1\right)$ lies on $y=\mathrm{f}\left(x\right)$: $1=\left(3\right)^3+a\left(3\right)^2+b\left(3\right)+c\Rightarrow 9a+3b+c=-26$ --- (3) <u>Method 2</u>: use $y=\mathrm{f}\left(x+1\right)=\left(x+1\right)^3+a\left(x+1\right)^2+b\left(x+1\right)+c$ $1=\left(2+1\right)^3+a\left(2+1\right)^2+b\left(2+1\right)+c\Rightarrow 9a+3b+c=-26$ --- (3) Solving (1), (2) and (3), $a=-2, b=-5, c=7$ [or $\mathrm{f}(x)=x^3-2x^2-5x+7$]
2	**Solution**
(a)	

(b)	
3	**Solution**
(a)	$y=\left\|\frac{x+1}{x-2}\right\|=\left\|1+\frac{3}{x-2}\right\|$. Horizontal Asymptote: $y=1$. Vertical Asymptote: $x=2$ For $\ln\left(1-\frac{x}{2}\right)$: $1-\frac{x}{2}\neq 0\Rightarrow x\neq 2$, i.e. $x=2$ is a Vertical Asymptote. Graph exist when $1-\frac{x}{2}>0\Rightarrow x<2$ 
(b)	From the graph, $0\leq x<2$

4 **Solution**

(a)

$$\begin{aligned}
\text{LHS} &= (\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q}) \\
&= (\mathbf{r}-\mathbf{p})\times\mathbf{r}-(\mathbf{r}-\mathbf{p})\times\mathbf{q} \\
&= \mathbf{r}\times\mathbf{r}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{q}+\mathbf{p}\times\mathbf{q} \\
&= \mathbf{r}\times\mathbf{p}+\mathbf{q}\times\mathbf{r}+\mathbf{p}\times\mathbf{q} \qquad (\because \mathbf{r}\times\mathbf{r}=0 \text{ and } \mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}) \\
&= \mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p} \\
&= \text{RHS}
\end{aligned}$$

$\therefore\ \mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}=(\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})$ (shown)

(b)

$$\begin{aligned}
\frac{1}{2}\left|\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\right| &= \frac{1}{2}\left|(\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})\right| \\
&= \frac{1}{2}\left|\left(\overrightarrow{OR}-\overrightarrow{OP}\right)\times\left(\overrightarrow{OR}-\overrightarrow{OQ}\right)\right| \\
&= \frac{1}{2}\left|\overrightarrow{PR}\times\overrightarrow{QR}\right|
\end{aligned}$$

$\therefore\ \frac{1}{2}\left|\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\right|$ represents the area of ΔPQR.

<u>Alternative</u>

$\frac{1}{2}\left|\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\right|$ represents half the area of a parallelogram with *PR* and *QR* as adjacent sides.

(c)

Given $\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}=\mathbf{0}$,

$\Rightarrow\ (\mathbf{r}-\mathbf{p})\times(\mathbf{r}-\mathbf{q})=\mathbf{0}$ (from result in **(a)**)

$\Rightarrow\ \overrightarrow{PR}\times\overrightarrow{QR}=\mathbf{0}$

$\because$ P, Q, R are distinct points,

$\Rightarrow\ \overrightarrow{PR}\neq\mathbf{0}$, and $\overrightarrow{QR}\neq\mathbf{0}$,

$\Rightarrow\ \overrightarrow{PR}\parallel\overrightarrow{QR}$, i.e. P, Q, R are <u>collinear</u> points.

Given also that $PR=3QR$, and $PQ>PR$,

$\therefore$ Point R divides PQ internally in the ratio 3:1,

i.e. $PR:RQ=3:1$.

<u>Alternative (to show collinear)</u>

Given $\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}=\mathbf{0}$,

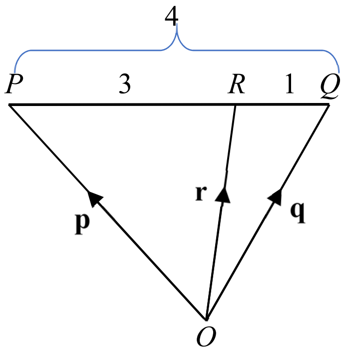
Area of ΔPQR

$=\frac{1}{2}\left|\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{r}+\mathbf{r}\times\mathbf{p}\right|$ (by part **(b)**)

$=\frac{1}{2}\left|\mathbf{0}\right|$

$=0$ i.e. ΔPQR is a degenerate triangle

i.e. P, Q, R are <u>collinear</u> points.



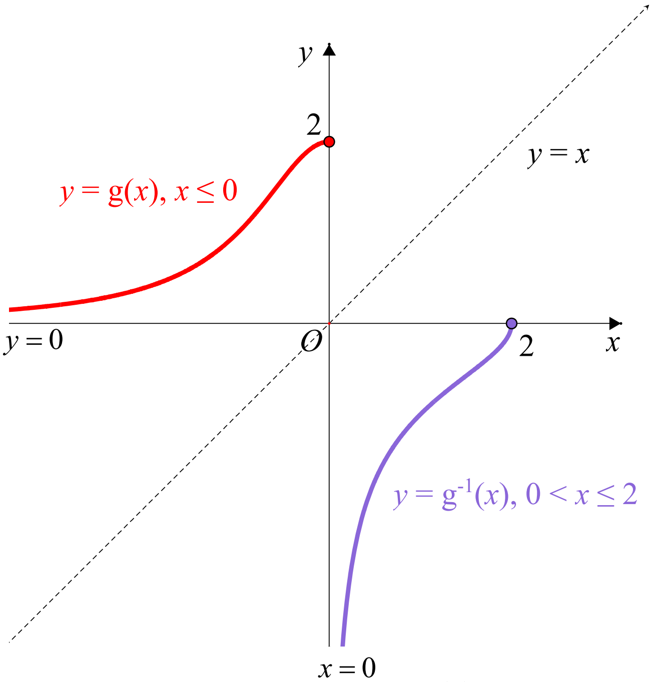
By the ratio theorem, position vector $\mathbf{r}=\dfrac{\mathbf{p}+3\mathbf{q}}{4}$.

5	**Solution**	
(a)	$\begin{aligned} \frac{3}{(r+1)!}-\frac{2}{r!}-\frac{1}{(r-1)!} &= \frac{3-2(r+1)-1(r)(r+1)}{(r+1)!} \\ &= \frac{-r^2-3r+1}{(r+1)!} \text{ (verified)} \end{aligned}$	
(b)	$\begin{aligned} &\sum_{r=1}^{n}\frac{-r^2-3r+1}{(r+1)!} \\ &=\sum_{r=1}^{n}\left(\frac{3}{(r+1)!}-\frac{2}{r!}-\frac{1}{(r-1)!}\right) \\ &=\cancel{\frac{3}{2!}}-\frac{2}{1!}-\frac{1}{0!} \\ &+\cancel{\frac{3}{3!}}-\cancel{\frac{2}{2!}}-\frac{1}{1!} \\ &+\cancel{\frac{3}{4!}}-\cancel{\frac{2}{3!}}-\cancel{\frac{1}{2!}} \\ &+\cancel{\frac{3}{5!}}-\cancel{\frac{2}{4!}}-\cancel{\frac{1}{3!}} \\ &\vdots \\ &+\cancel{\frac{3}{(n-1)!}}-\cancel{\frac{2}{(n-2)!}}-\cancel{\frac{1}{(n-3)!}} \\ &+\frac{3}{n!}-\cancel{\frac{2}{(n-1)!}}-\cancel{\frac{1}{(n-2)!}} \\ &+\frac{3}{(n+1)!}-\frac{2}{n!}-\cancel{\frac{1}{(n-1)!}} \\ &=\frac{3}{(n+1)!}+\frac{1}{n!}-4 \end{aligned}$	
(c)	Method 1: change of variable $\begin{aligned} \sum_{r=3}^{n}\frac{-r^2-r+3}{r!} &= \sum_{r+1=3}^{r+1=n}\frac{-(r+1)^2-(r+1)+3}{(r+1)!} \\ &=\sum_{r=2}^{n-1}\frac{-r^2-3r+1}{(r+1)!} \\ &=\sum_{r=1}^{n-1}\frac{-r^2-3r+1}{(r+1)!}-\left(-\frac{3}{2}\right) \\ &=\left[\left(\frac{3}{n!}+\frac{1}{(n-1)!}-4\right)\right]+\frac{3}{2} \\ &=\left(\frac{3}{n!}+\frac{1}{(n-1)!}\right)-\frac{5}{2} \end{aligned}$	Method 2: Listing $\begin{aligned} \sum_{r=3}^{n}\frac{-r^2-r+3}{r!} &= \frac{-3^2-3+3}{3!}+\ldots+\frac{-n^2-n+3}{n!} \\ &=\sum_{r=2}^{n-1}\frac{-(r+1)^2-(r+1)+3}{(r+1)!} \\ &=\sum_{r=1}^{n-1}\frac{-r^2-3r+1}{(r+1)!}-\left(-\frac{3}{2}\right) \\ &=\left(\frac{3}{n!}+\frac{1}{(n-1)!}-4\right)-\left(-\frac{3}{2}\right) \\ &=\left(\frac{3}{n!}+\frac{1}{(n-1)!}\right)-\frac{5}{2} \end{aligned}$

6	**Solution**
(a)(i)	$d = u_n - u_{n-1}$ $= \log_a 3^{2n-1} - \log_a 3^{2(n-1)-1}$ $= \log_a \dfrac{3^{2n-1}}{3^{2n-3}}$ $= \log_a 9$ which is a constant independent of n Therefore, the series is an arithmetic series.
(a)(ii)	<u>Method 1: use $S_n = \frac{n}{2}(a+l)$</u> $S_{30} = \dfrac{30}{2}\left[\log_a 3 + \log_a 3^{2(30)-1}\right] = 300$ $15\left(\log_a 3^{60}\right) = 300$ $900 \log_a 3 = 300$ $\log_a 3 = \dfrac{1}{3}$ $a^{\frac{1}{3}} = 3$ $a = 27$ <u>Method 2: use $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$</u> $S_{30} = \dfrac{30}{2}\left[2(\log_a 3) + 29\left(\log_a 9\right)\right] = 300$ $15\left[2\left(\log_a 3\right) + 29\left(\log_a 3^2\right)\right] = 300$ $900\left(\log_a 3\right) = 300$ $\log_a 3 = \dfrac{1}{3}$ $a^{\frac{1}{3}} = 3$ $a = 27$
(b)	$b + 4d = cr \quad (1)$ $b + 7d = cr^2 \quad (2)$ $b + 9d = cr^3 \quad (3)$ (Eliminate b): $(2)-(1): \quad cr^2 - cr = 3d$ $(3)-(2): \quad cr^3 - cr^2 = 2d$ (Eliminate d):

	$\frac{cr^2-cr}{3}=\frac{cr^3-cr^2}{2}$ Since $c, r \neq 0$, we divide both sides by *c* and *r,* and rearrange to get $3r^2-5r+2=0$ (shown) Solving, $r=1$ (rejected $\because d \neq 0$) or $r\ =\frac{2}{3}$ $S_\infty=\frac{c}{1-\frac{2}{3}}=3c$
7	**Solution**
(a)	<u>Explanation 1 (“Horizontal Line Test”)</u> g is not a one-to-one function as the horizontal line $y=0$ meets the graph of $y=\mathrm{g}(x)$ more than once. <u>Explanation 2 (State two inputs with same output)</u> g is not a one-to-one function as there are distinct inputs producing the same output under function g, e.g. $\mathrm{g}(1)=\mathrm{g}(2)=0$.
(b)	Greatest *k* = 0.
(c)	Let $y=\mathrm{g}(x),\ x \leq 0$. Then $x=\mathrm{g}^{-1}(y)$. $y=\mathrm{g}(x)=\frac{2}{1+x^2}$, since $x \leq 0$. $1+x^2=\frac{2}{y}$ $x^2=\frac{2}{y}-1$, $\quad x=\pm\sqrt{\frac{2}{y}-1}$ Since $x \leq 0$, $x=-\sqrt{\frac{2}{y}-1}=\mathrm{g}^{-1}(y)$ $\therefore\ \mathrm{g}^{-1}(x)=-\sqrt{\frac{2}{x}-1}$ $\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{R}_{\mathrm{g}}=(0,2]$.

(d)



The line in which the graph of $y = \mathrm{g}(x)$ is reflected to obtain the graph of $y = \mathrm{g}^{-1}(x)$ is $y = x$.

8 **Solution**

(a)

Method 1: Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ then simplify

Differentiating implicitly w.r.t. *x*,

Method A: Consider Chain Rule	Method B: Consider product rule
$1+\frac{\mathrm{d}y}{\mathrm{d}x}=2(x-y)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$	$x+y=(x-y)^2=x^2-2xy+y^2$ $1+\frac{\mathrm{d}y}{\mathrm{d}x}=2x-2x\frac{\mathrm{d}y}{\mathrm{d}x}-2y+2y\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\Rightarrow 1+\frac{\mathrm{d}y}{\mathrm{d}x}=2x-2y-(2x-2y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow 1-2x+2y=-(1+2x-2y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow -\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-2x+2y}{1+2x-2y}$$

Add 1 to both sides,

$$1-\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-2x+2y+(1+2x-2y)}{1+2x-2y}$$

$$=\frac{2}{1+2x-2y} \text{ (shown)}$$

	Method 2: Consider adding $1-\frac{\mathrm{d}y}{\mathrm{d}x}$ to both sides Differentiating implicitly w.r.t. *x*, $1+\frac{\mathrm{d}y}{\mathrm{d}x}=2(x-y)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ Add $1-\frac{\mathrm{d}y}{\mathrm{d}x}$ to both sides, $1+\frac{\mathrm{d}y}{\mathrm{d}x}+1-\frac{\mathrm{d}y}{\mathrm{d}x}=2(x-y)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $2=(2x-2y+1)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $1-\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2x-2y+1}$ (shown)
(b)	Diff implicitly w.r.t. x, $-\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2(1+2x-2y)^{-1-1}\left(2-2\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=-\frac{4}{(1+2x-2y)^2}\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=-\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$ (shown)
(c)	$\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1>0 \quad \therefore$ minimum point
9	**Solution**
(a)	$y=\ln\left(2-\mathrm{e}^{-2x}\right)\Rightarrow\underbrace{\mathrm{e}^{y}=2-\mathrm{e}^{-2x}}_{\text{Eqn 1}}\Rightarrow\underbrace{\mathrm{e}^{-2x}=2-\mathrm{e}^{y}}_{\text{Eqn 2}}$ Method 1: implicit differentiation Differentiating Eqn 1 implicitly w.r.t. *x*, $\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^{-2x}$ Then $\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^{-2x}\mathrm{e}^{-y}=2\underbrace{\left(2-\mathrm{e}^{-y}\right)}_{\text{from Eqn 2}}\mathrm{e}^{-y}=4\mathrm{e}^{-y}-2$ (shown) Method 2: direct differentiation

	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\mathrm{e}^{-2x}}{2-\mathrm{e}^{-2x}} = \frac{2\overbrace{\left(2-\mathrm{e}^{y}\right)}^{\text{from Eqn 2}}}{\underbrace{\mathrm{e}^{y}}_{\text{from Eqn 1}}} = 4\mathrm{e}^{-y}-2$ (shown) Method 3: make x the subject, implicit differentiation From Eqn 2, $\mathrm{e}^{-2x} = 2-\mathrm{e}^{y} \Rightarrow -2x = \ln\left(2-\mathrm{e}^{y}\right)$ Differentiating implicitly w.r.t. x, $-2 = \frac{1}{2-\mathrm{e}^{y}}\left(-\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ Then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\left(2-\mathrm{e}^{y}\right)}{-\mathrm{e}^{y}} = 4\mathrm{e}^{-y}-2$ (shown)
(b)	Method 1: further differentiation of result in **(a)** Differentiating $\frac{\mathrm{d}y}{\mathrm{d}x} = 4\mathrm{e}^{-y}-2$ implicitly w.r.t. x, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\mathrm{e}^{-y}\left(-\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -4\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{-y}$ When $x=0$, $y=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=2$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4(2)\mathrm{e}^{-0} = -8$ $y = (0)+(2)x+\left(\frac{-8}{2!}\right)x^2+\ldots = 2x-4x^2+\ldots$ Method 2: direct differentiation of 1st derivative in x $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-4\mathrm{e}^{-2x}\left(2-\mathrm{e}^{-2x}\right)-2\mathrm{e}^{-2x}\left(2\mathrm{e}^{-2x}\right)}{\left(2-\mathrm{e}^{-2x}\right)^2}$ $= \frac{-8\mathrm{e}^{-2x}}{\left(2-\mathrm{e}^{-2x}\right)^2}$ When $x=0$, $y=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=2$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-8\mathrm{e}^{-0}}{\left(2-\mathrm{e}^{-0}\right)^2} = -8$ $y = (0)+(2)x+\left(\frac{-8}{2!}\right)x^2+\ldots = 2x-4x^2+\ldots$
(c)	From MF26, $\mathrm{e}^{-2x} = 1+(-2x)+\frac{(-2x)^2}{2}+\ldots = 1-2x+2x^2+\ldots$ Then

$$
\begin{aligned}
y &= \ln\left(2-\mathrm{e}^{-2x}\right)\\
&= \ln\left[2-\left(1-2x+2x^2+...\right)\right]\\
&= \ln\left[1+\left(2x-2x^2+...\right)\right]\\
&= \underbrace{\left(2x-2x^2+...\right)-\frac{\left(2x-2x^2+...\right)^2}{2}+...}_{\text{Using the expansion for } \ln\left[1+\mathrm{f}(x)\right]}\\
&= 2x-2x^2-\frac{4x^2}{2}+...\\
&= 2x-4x^2+...
\end{aligned}
$$

This is the same expression as found part **(b)** and hence we can conclude that the expansion is correct.

10 **Solution**

(a)

$$
\begin{aligned}
\int \sin 3x\cos x\,\mathrm{d}x &= \frac{1}{2}\int 2\sin 3x\cos x\,\mathrm{d}x\\
&= \frac{1}{2}\int \sin(3x+x)+\sin(3x-x)\,\mathrm{d}x\\
&= \frac{1}{2}\int \sin 4x+\sin 2x\,\mathrm{d}x\\
&= \frac{1}{2}\left[\int \sin 4x\,\mathrm{d}x+\int \sin 2x\,\mathrm{d}x\right]\\
&= \frac{1}{2}\left[\frac{1}{4}\int 4\sin 4x\,\mathrm{d}x+\frac{1}{2}\int 2\sin 2x\,\mathrm{d}x\right]\\
&= \frac{1}{2}\left[\frac{1}{4}(-\cos 4x)+\frac{1}{2}(-\cos 2x)\right]+c\\
&= -\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x+c,
\end{aligned}
$$

where c is an arbitrary constant.

(b)

[Since $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+4x+13\right)=2x+4$, we re-write x as $x=A(2x+4)+B$ in order to split the numerator into 2 parts. Compare coefficients to get A and B.]

$$
\begin{aligned}
\int \frac{x}{x^2+4x+13}\,\mathrm{d}x &= \int \frac{\frac{1}{2}(2x+4)-2}{x^2+4x+13}\,\mathrm{d}x\\
&= \frac{1}{2}\int \frac{2x+4}{x^2+4x+13}\,\mathrm{d}x-2\int \frac{1}{x^2+4x+13}\,\mathrm{d}x\\
&= \frac{1}{2}\ln\left|x^2+4x+13\right| -2\int \frac{1}{(x+2)^2+3^2}\,\mathrm{d}x\\
&= \frac{1}{2}\ln\left(x^2+4x+13\right)-\frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c,
\end{aligned}
$$

where c is an arbitrary constant

(c)	Let $x = 3\sin\theta$. Then $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\cos\theta$. Substituting, $\int\sqrt{9-x^2}\ \mathrm{d}x = \int\sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,\mathrm{d}\theta$ $= \int\sqrt{9\left(1-\sin^2\theta\right)}\cdot 3\cos\theta\ \mathrm{d}\theta$ $= \int\sqrt{9\cos^2\theta}\cdot 3\cos\theta\ \mathrm{d}\theta$ $= \int 3\cos\theta\cdot 3\cos\theta\ \mathrm{d}\theta$ $= \int 9\cos^2\theta\ \mathrm{d}\theta$ $= \frac{9}{2}\int\cos 2\theta + 1\ \mathrm{d}\theta$ (double angle formula) $= \frac{9}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c$ $= \frac{9}{2}\sin\theta\cos\theta + \frac{9}{2}\theta + c$ $\left[x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3} \Rightarrow \cos\theta = \sqrt{1-\left(\frac{x}{3}\right)^2} = \frac{\sqrt{9-x^2}}{3}\right]$ So $\int\sqrt{9-x^2}\ \mathrm{d}x = \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + c$ $= \frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + c,$ where c is an arbitrary constant.
11	**Solution**
(a)	$V = \pi r^2 h + \frac{2}{3}\pi r^3 = k$ $\Rightarrow h = \frac{k - \frac{2}{3}\pi r^3}{\pi r^2} = \frac{k}{\pi r^2} - \frac{2}{3}r$ $C = 3\left(2\pi r^2\right) + 2.5\left(2\pi r h\right)$ $= 6\pi r^2 + 5\pi r\left(\frac{k}{\pi r^2} - \frac{2}{3}r\right)$ $= \frac{8}{3}\pi r^2 + \frac{5k}{r}$ (shown)
(b)	$\frac{\mathrm{d}C}{\mathrm{d}r} = \frac{16\pi r}{3} - \frac{5k}{r^2}$ $\frac{\mathrm{d}C}{\mathrm{d}r} = 0 \Rightarrow \frac{16\pi r}{3} - \frac{5k}{r^2} = 0$

$$\Rightarrow r^3 = \frac{15k}{16\pi}$$

$$\Rightarrow r = \sqrt[3]{\frac{15k}{16\pi}}$$

Check min using second derivative method

$$\frac{\mathrm{d}^2C}{\mathrm{d}r^2} = \frac{16\pi}{3} + \frac{10k}{r^3} > 0 \quad (\because k, r > 0)$$

So C is minimum when $r = \sqrt[3]{\frac{15k}{16\pi}}$.

Check min using first derivative method

$$\frac{\mathrm{d}C}{\mathrm{d}r} = \frac{16\pi r}{3} - \frac{5k}{r^2} = \frac{16\pi r^3 - 15k}{3r^2}$$

r	$\sqrt[3]{\frac{15k}{16\pi}}^{-}$	$\sqrt[3]{\frac{15k}{16\pi}}$	$\sqrt[3]{\frac{15k}{16\pi}}^{+}$
Sign of $\frac{\mathrm{d}C}{\mathrm{d}r}$	$16\pi r^3 - 15k < 0$ $3r^2 > 0$ $\therefore \frac{16\pi r^3 - 15k}{3r^2} < 0$	0	$16\pi r^3 - 15k > 0$ $3r^2 > 0$ $\therefore \frac{16\pi r^3 - 15k}{3r^2} > 0$
Slope	\	-	/

So C is minimum when $r = \sqrt[3]{\frac{15k}{16\pi}}$.

(c) When $k = 50$,

$r = \sqrt[3]{\frac{15(50)}{16\pi}} = 2.4619 = 2.46$ (3s.f.) and

$$h = \frac{50}{\pi(2.4619)^2} - \frac{2}{3}(2.4619) = 0.985 \text{ (3 s.f.)}$$

(d) [Since the leak is at the joint between cylinder and hemisphere, we need to consider only the volume in the cylindrical part.]

Let V_c be volume of water in the cylindrical part and l be level of water in the cylindrical part.

Note that $r = 2.4619$ is a constant, so $V_c = (2.4619)^2 \pi l$

Method 1 – differentiate w.r.t. t

$$\frac{\mathrm{d}V_c}{\mathrm{d}t} = (2.4619)^2 \pi \frac{\mathrm{d}l}{\mathrm{d}t}.$$

$$\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{1}{\pi(2.4619)^2}\frac{\mathrm{d}V_c}{\mathrm{d}t}$$

$$=\frac{1}{\pi(2.4619)^2}(-0.002) \quad .$$

$$=-1.05\times10^{-4} \text{ m per minute}$$

So the water level decreases at 1.05×10^{-4} m/min.

<u>Method 2 – connected rate of change</u>

$$\frac{\mathrm{d}V_c}{\mathrm{d}l}=\pi(2.4619)^2$$

$$\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{\mathrm{d}l}{\mathrm{d}V_c}\frac{\mathrm{d}V_c}{\mathrm{d}t}$$

$$=\frac{1}{\pi(2.4619)^2}(-0.002)$$

$$=-1.05\times10^{-4} \text{ m per minute}$$

So the water level decreases at 1.05×10^{-4} m/min.

<u>Method 3 – consider proportionality</u>

Since surface area is a constant,

$$\frac{\mathrm{d}V_c}{\mathrm{d}t}=A\frac{\mathrm{d}l}{\mathrm{d}t}$$

$$\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{1}{\pi(2.4619)^2}(-0.002)$$

$$=-1.05\times10^{-4} \text{ m per minute}$$

12 **Solution**

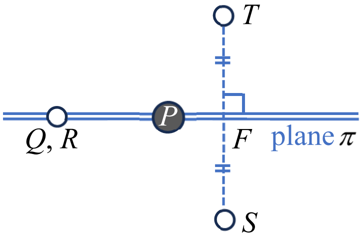
(a)

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP} \qquad \overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}$$

$$=\begin{pmatrix}-2\\-1\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \qquad =\begin{pmatrix}-2\\1\\3\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

$$\text{Angle } QPR=\cos^{-1}\left(\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|}\right)$$

	Angle $QPR = \cos^{-1}\left(\dfrac{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}}{\left\|\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\|\left\|\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\|}\right)$ $= \cos^{-1}\left(\dfrac{1+(-1)+(-1)}{\sqrt{3}\sqrt{3}}\right)$ $= \cos^{-1}\left(-\dfrac{1}{3}\right)$ $= 109.47^\circ$ (2 d.p.)
(b)	$QS^2 = RS^2 \quad (\because\ QS = RS)$ $\left(0-(-2)\right)^2 + \left(-1-(-1)\right)^2 + \left(a-1\right)^2 = \left(0-(-2)\right)^2 + \left(-1-1\right)^2 + \left(a-3\right)^2$ $\cancel{2^2} + 0^2 + (a-1)^2 = \cancel{2^2} + (-2)^2 + (a-3)^2$ $(\cancel{a^2} - 2a + 1) = 4 + (\cancel{a^2} - 6a + 9)$ $4a = 12$ $a = 3$ (shown)
(c)	$\overrightarrow{PQ} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$ from **(a)** A vector normal to plane π is $\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\times\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}(-1)(1)-(-1)(1)\\(-1)(-1)-(-1)(1)\\(-1)(1)-(-1)(-1)\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix} = 2\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\therefore \mathbf{n} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\therefore\ \pi : \mathbf{r}\cdot\mathbf{n} = \mathbf{r}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix} = -2$ $\pi : y - z = -2$
(d)(i)	F is the foot of perpendicular from S to plane π, and SF is parallel to a normal vector used for plane π. $\therefore$ Line $SF : \mathbf{r} = \begin{pmatrix}0\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\-1\end{pmatrix},\ \lambda\in\mathbb{R}.$

(d)(ii)	$\because$ F lies on line SF, $\overrightarrow{OF}=\begin{pmatrix}0\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\-1+\lambda\\3-\lambda\end{pmatrix}$, for some $\lambda\in\mathbb{R}$. $\because$ F lies on π, $\overrightarrow{OF}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=-2$. $\Rightarrow\ \begin{pmatrix}0\\-1+\lambda\\3-\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=-2,$ $\Rightarrow\ (-1+\lambda)-(3-\lambda)=-2,$ $-4+2\lambda=-2$ $\lambda=1$ $\therefore\ \overrightarrow{OF}=\left.\begin{pmatrix}0\\-1+\lambda\\3-\lambda\end{pmatrix}\right\rvert_{\lambda=1}=\begin{pmatrix}0\\0\\2\end{pmatrix}$ $F(0,\ 0,\ 2)$ (shown) Alternative; $\begin{pmatrix}-1\\0\\2\end{pmatrix}+\mu\begin{pmatrix}-1\\-1\\-1\end{pmatrix}+\alpha\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $-1-\mu-\alpha=0$ $-\mu+\alpha=-1+\lambda$ $2-\mu+\alpha=3-\lambda$ $\mu=-\frac{1}{2},\alpha=-\frac{1}{2},\lambda=1$
(d)(iii)	$\because$ T is the mirror image of S in π, the foot of perpendicular from S to π, i.e. point F, is the midpoint of S and T. By the midpoint theorem (special case of ratio theorem), $\overrightarrow{OF}=\frac{\overrightarrow{OS}+\overrightarrow{OT}}{2}$ $\overrightarrow{OT}=2\overrightarrow{OF}-\overrightarrow{OS}$ $=2\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}0\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$  Alternatively, $\because$ T is the mirror image of S in plane π,

$$\overrightarrow{SF} = \overrightarrow{FT}$$

$$\begin{aligned}
\overrightarrow{OT} &= \overrightarrow{OF} + \overrightarrow{FT} \\
&= \overrightarrow{OF} + \overrightarrow{SF} \\
&= \overrightarrow{OF} + \overrightarrow{OF} - \overrightarrow{OS} \\
&= 2\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}$$