NANYANG JUNIOR COLLEGE
DEPARTMENT OF MATHEMATICS

CHAPTER 13: COMPLEX NUMBERS

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At the end of this chapter, students should be able to

- Extend the number system from real numbers to complex numbers
- Solve for complex roots of quadratic equations
- Find the modulus, argument and conjugate of a complex number
- Conduct the four operations of complex numbers
- Understand the equality of complex numbers
- Find the conjugate roots of a polynomial equation with real coefficients
- Represent complex numbers in the Argand diagram
- Understand the geometrical effects of conjugate, negation, addition, subtraction and multiplication of i.

My Notes

The set of real numbers can be extended to complex numbers. This set of numbers arose, historically, from the question of whether a negative number can have a square root.

For example, what are the roots of $x^2 = -1$?

From this problem, a new number was discovered: the square root of negative one. This number is denoted by i, i.e. $\mathrm{i} = \sqrt{-1}$, a symbol assigned by Leonhard Euler (1707-83, Swiss mathematician).

Complex numbers have applications in a variety of sciences and related areas such as signal processing, control theory, electromagnetism, quantum mechanics, cartography, and many others.

13.1 Introduction

Let us start by solving the equation $x^2 - 8x + 20 = 0$. Applying the formula, we have

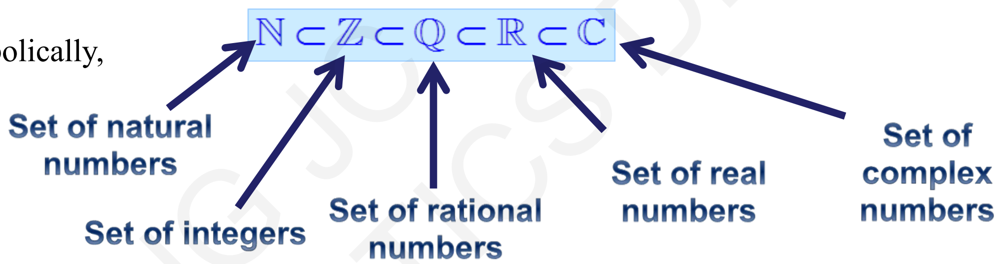
$$x = \frac{8 \pm \sqrt{64-80}}{2}$$
$$= 4 \pm 2\sqrt{-1}$$

If we let $\mathrm{i} = \sqrt{-1}$, it follows that $x = 4 + 2\mathrm{i}$ or $x = 4 - 2\mathrm{i}$.

The numbers $4 + 2\mathrm{i}$ and $4 - 2\mathrm{i}$ are called complex numbers.

The set of complex numbers is denoted by $\mathbb{C}$.

Symbolically,



A **complex number**, usually denoted by the letter z, is defined as any number of the form $z = x + y\mathrm{i}$ where $\mathrm{i} = \sqrt{-1}$ and $x, y \in \mathbb{R}$, and this is known as the **Cartesian form** of complex numbers.

x is known as the ***real*** part of the complex number z, and is denoted by **Re(z)**.
y is known as the ***imaginary*** part of the complex number z, and is denoted by **Im(z)**.

Example $\mathrm{Re}(-3+\frac{1}{5}\mathrm{i}) = -3$, $\mathrm{Im}(-3+\frac{1}{5}\mathrm{i}) = \frac{1}{5}$

If $x = 0$, then $z = y\mathrm{i}$, so the complex number is **purely imaginary**.
If $y = 0$, then $z = x$, so the complex number is **purely real**.

Note: $\mathrm{i} = \sqrt{-1}$ $\mathrm{i}^2 = -1$ $\mathrm{i}^3 = \mathrm{i} \times \mathrm{i}^2 = -\mathrm{i}$ $\mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = 1$	**THINK ZONE:** Simplify $\mathrm{i}^5,\ \mathrm{i}^6, \mathrm{i}^7,\ \mathrm{i}^8, \mathrm{i}^9, \mathrm{i}^{10}, \ldots$ Do you see any pattern?

13.2 Operation on Complex Numbers

We shall now see how we can add, subtract, multiply and divide two complex numbers.
Let $z_1 = x_1 + y_1\mathrm{i}$ and $z_2 = x_2 + y_2\mathrm{i}$, where $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

13.2.1 Equality of Complex Numbers

Two complex numbers z_1 and z_2 are equal **if and only if** their real and imaginary parts are equal.

$$\begin{aligned} z_1 = z_2 &\Leftrightarrow x_1 + y_1\mathrm{i} = x_2 + y_2\mathrm{i} \\ &\Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2 \end{aligned}$$

For example, $2 + a\mathrm{i} = b - 3\mathrm{i}$, where $a, b \in \mathbb{R}$, then $a = -3$ and $b = 2$.

Useful result: $a + b\mathrm{i} = 0$, where $a, b \in \mathbb{R} \Rightarrow a = 0$ and $b = 0$.

13.2.2 Addition and Subtraction of Complex Numbers

$$\begin{aligned} z_1 + z_2 &= x_1 + y_1\mathrm{i} + x_2 + y_2\mathrm{i} \\ &= (x_1 + x_2) + (y_1 + y_2)\mathrm{i} \end{aligned}$$

For example, $(2 + 3\mathrm{i}) + (4 - 2\mathrm{i}) = (2 + 4) + \mathrm{i}(3 - 2) = 6 + \mathrm{i}$.

$$\begin{aligned} z_1 - z_2 &= x_1 + y_1\mathrm{i} - (x_2 + y_2\mathrm{i}) \\ &= (x_1 - x_2) + (y_1 - y_2)\mathrm{i} \end{aligned}$$

For example, $(1 - 2\mathrm{i}) - (3 + 5\mathrm{i}) = (1 - 3) + \mathrm{i}(-2 - 5) = -2 - 7\mathrm{i}$

13.2.3 Multiplication of Complex Numbers

$$\begin{aligned} z_1 z_2 &= (x_1 + y_1\mathrm{i})(x_2 + y_2\mathrm{i}) \\ &= x_1x_2 + x_1y_2\mathrm{i} + x_2y_1\mathrm{i} + y_1y_2\mathrm{i}^2 \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)\mathrm{i} \quad \text{since } \mathrm{i}^2 = -1. \end{aligned}$$

For example,

$$\begin{aligned} (9 + \mathrm{i})(-2 + 3\mathrm{i}) &= -18 + 27\mathrm{i} - 2\mathrm{i} + 3\mathrm{i}^2 \\ &= -18 + 25\mathrm{i} - 3 \\ &= -21 + 25\mathrm{i} \end{aligned}$$

13.2.4 Multiplication of Complex Numbers by a Real Number

For any $k \in \mathbb{R}$, $\quad kz = k(x + \mathrm{i}y) = kx + ky\mathrm{i}.$

For example, $-2(4 - 9\mathrm{i}) = -8 + 18\mathrm{i}$.

Conjugate of a Complex Number

Recall when we have a surd in the form $a+\sqrt{b}$, we can write the conjugate of the surd as $a-\sqrt{b}$. Likewise, when we have a complex number, for e.g. $1+2\mathrm{i}$, we can write it as $1+2\sqrt{-1}$ and the conjugate of this surd is $1-2\sqrt{-1}$ which is equal to $1-2\mathrm{i}$.

Thus, if $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$, we can denote the **<u>conjugate of z</u>**, z^*, to be

$$z^* = x - y\mathrm{i}, \quad x, y \in \mathbb{R}.$$

For example, if $z = -3+4\mathrm{i}$, then $z^* = -3-4\mathrm{i}$.

Multiplication of z and z*

Let $z = x + y\mathrm{i}$, then $z^* = x - y\mathrm{i}$.

$$\boxed{zz^* = x^2 + y^2}$$

This is because

$$\begin{aligned} zz^* &= (x+y\mathrm{i})(x-y\mathrm{i}) \\ &= x^2 - (y\mathrm{i})^2 \\ &= x^2 + y^2 \text{ since } \mathrm{i}^2 = -1 \end{aligned}$$

This is a useful result, illustrating that zz^* gives you a **real** number, which is the sum of the squares of the real and imaginary part of z.

13.2.5 Division of Complex Numbers

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1+y_1\mathrm{i}}{x_2+y_2\mathrm{i}} \\ &= \frac{(x_1+y_1\mathrm{i})\ (x_2-y_2\mathrm{i})}{(x_2+y_2\mathrm{i})\ (x_2-y_2\mathrm{i})} \\ &= \frac{x_1x_2 - x_1y_2\mathrm{i} + x_2y_1\mathrm{i} - y_1y_2\mathrm{i}^2}{x_2^{\ 2} - y_2^{\ 2}\mathrm{i}^2} \\ &= \frac{(x_1x_2+y_1y_2)+(x_2y_1-x_1y_2)\mathrm{i}}{x_2^{\ 2}+y_2^{\ 2}} \\ &= \frac{x_1x_2+y_1y_2}{x_2^{\ 2}+y_2^{\ 2}} + \left(\frac{x_2y_1-x_1y_2}{x_2^{\ 2}+y_2^{\ 2}}\right)\mathrm{i} \end{aligned}$$

Important Idea:
The objective is to make the denominator into a real number in the fraction, so that we can split it into the real and imaginary parts in the final answer.

Example 1

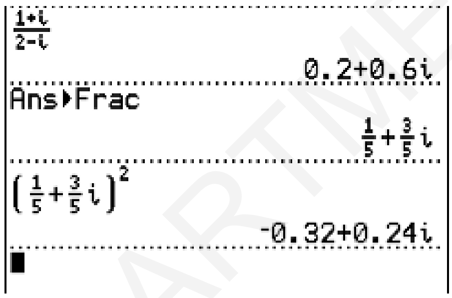
Express (i) $\dfrac{1}{2+3\mathrm{i}}$ and (ii) $\dfrac{2+\mathrm{i}}{1+\mathrm{i}}$ in the form of $a+b\mathrm{i}$ where *a* and *b* are exact real values to be found.

Solution:	
(i) $\dfrac{1}{2+3\mathrm{i}}$ $=\dfrac{1}{(2+3\mathrm{i})}\times\dfrac{2-3\mathrm{i}}{2-3\mathrm{i}}$ $=\dfrac{2-3\mathrm{i}}{2^2+3^2}$ $=\dfrac{2}{13}-\dfrac{3}{13}\mathrm{i}$	(ii) $\dfrac{2+\mathrm{i}}{1+\mathrm{i}}$ $=\dfrac{2+\mathrm{i}}{1+\mathrm{i}}\times\dfrac{1-\mathrm{i}}{1-\mathrm{i}}$ $=\dfrac{3-\mathrm{i}}{1+1}$ $=\dfrac{3}{2}-\dfrac{1}{2}\mathrm{i}$

Example 2

It is given that $z=\dfrac{1+\mathrm{i}}{2-\mathrm{i}}$. Without using a calculator, find the real and imaginary parts of

(a) z^2 (b) $z-\dfrac{1}{z}$.

Solution:	**Think Zone:**
(a) $z=\dfrac{1+\mathrm{i}}{2-\mathrm{i}}$ $=\dfrac{(1+\mathrm{i})}{(2-\mathrm{i})}\times\dfrac{(2+\mathrm{i})}{(2+\mathrm{i})}$ $=\dfrac{1}{2^2+1^2}(1+3\mathrm{i})$ $=\dfrac{1}{5}(1+3\mathrm{i})$ $z^2=\dfrac{1+3\mathrm{i}}{5}\times\dfrac{1+3\mathrm{i}}{5}$ $=\dfrac{1}{25}(-8+6\mathrm{i})=-\dfrac{8}{25}+\dfrac{6}{25}\mathrm{i}$ $\mathrm{Re}(z^2)=-\dfrac{8}{25}$, $\mathrm{Im}(z^2)=\dfrac{6}{25}$	Use GC to check answer: To enter the complex number i, press   Note: It is too tedious to evaluate $z^2=\left(\dfrac{1+\mathrm{i}}{2-\mathrm{i}}\right)\left(\dfrac{1+\mathrm{i}}{2-\mathrm{i}}\right)$ and simplify to get its real and imaginary parts.
(b) $\dfrac{1}{z}=\dfrac{2-\mathrm{i}}{1+\mathrm{i}}$ $=\dfrac{(2-\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}=\dfrac{1}{2}-\dfrac{3}{2}\mathrm{i}$ $z-\dfrac{1}{z}=\dfrac{1+3\mathrm{i}}{5}-\left(\dfrac{1}{2}-\dfrac{3}{2}\mathrm{i}\right)=-\dfrac{3}{10}+\dfrac{21}{10}\mathrm{i}$ $\mathrm{Re}\,(z-\dfrac{1}{z})=-\dfrac{3}{10}$, $\mathrm{Im}\,(z-\dfrac{1}{z})=\dfrac{21}{10}$	

Example 3

Do not use a calculator in answering this question.

(a) Find the roots of the equation $z^2=3-4\mathrm{i}$.

(b) Find z if $\dfrac{z}{1+z}=\dfrac{1}{1-3\mathrm{i}}$.

Solution:	**Think Zone:**
(a) Let $z=x+\mathrm{i}y$, where $x,y\in\mathbb{R}$ $z^2=3-4\mathrm{i}$ $(x+y\mathrm{i})^2=3-4\mathrm{i}$ $x^2-y^2+2xy\,\mathrm{i}=3-4\mathrm{i}$ Comparing the real and imaginary parts, we have	To solve $z^2=3-4\mathrm{i}$ means to find the square roots of 3 – 4i.

$x^2 - y^2 = 3$ ....(1) $2xy = -4 \Rightarrow y = -\dfrac{2}{x}$ ....(2) Substituting (2) into (1): $x^2 - \left(-\dfrac{2}{x}\right)^2 = 3$ $x^4 - 3x^2 - 4 = 0$ $(x^2 - 4)(x^2 + 1) = 0$ $x^2 = 4$ or $x^2 = -1$(reject $\because x \in \mathbb{R}$) $x = \pm 2$ Thus $y = \mp 1$ $\therefore$ the square roots of $3 - 4\text{i}$ are $2 - \text{i}$ and $-2 + \text{i}$.	How can you use your GC to check the answer?
(b) $\dfrac{z}{1+z} = \dfrac{1}{1-3\text{i}}$ $z(1 - 3\text{i}) = 1 + z$ $z - 3iz = 1 + z$ $z = \dfrac{1}{-3i}$ $z = \dfrac{1}{-3i}\left(\dfrac{i}{i}\right)$ $z = \dfrac{1}{3}i$	Multiplying $(1 + z)(1 - 3\text{i})$ on both sides and make z the subject of the equation. Note that there is no need to write $z = x + \text{i}y$ to solve for x and y. We also make the denominator real by multiplying both the numerator and denominator of the fraction by i.

Example 4 [N2012/I/Q6 (i) and (ii)]

The complex number z is given by $z = 1 + \text{i}c$, where c is a non-zero real number.

(i) Find z^3 in the form $x + y\text{i}$.

(ii) Given that z^3 is real, find the possible values of z.

Solution:	**Think Zone:**
(i) $(1+\text{i}c)^3 = 1^3 + \binom{3}{1}1^2(\text{i}c) + \binom{3}{2}(\text{i}c)^2 + (\text{i}c)^3$ $= 1 + 3\text{i}c + 3\text{i}^2c^2 + \text{i}^3c^3$ $= 1 - 3c^2 + \text{i}(3c - c^3)$	In general, if n is a positive integer, the expansion of $(a+b)^n$ is given by MF27 $(a+b)^n$ $= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots$ $+ \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$
(ii) Given that z^3 is real, the imaginary part is 0, i.e., $3c - c^3 = 0 \Rightarrow c(3 - c^2) = 0$ $c = 0$ or $c = \pm\sqrt{3}$ Since $c \neq 0$, $z = 1 + \text{i}\sqrt{3}$ or $1 - \text{i}\sqrt{3}$	

Example 5

Express $(2-\mathrm{i})^3$ in the form $x+y\mathrm{i}$. Hence, find a root of the equation $(z-\mathrm{i})^3=-11-2\mathrm{i}$.

Solution:	**Think Zone:**
$(2-\mathrm{i})^3=2^3+\binom{3}{1}2^2(-\mathrm{i})+\binom{3}{2}2(-\mathrm{i})^2+(-\mathrm{i})^3$ $=8+3\times4\times(-\mathrm{i})+3\times2\times\mathrm{i}^2-\mathrm{i}^3$ $=8-12\mathrm{i}-6+\mathrm{i}\;=\;2-11\mathrm{i}$ $(z-\mathrm{i})^3=-11-2\mathrm{i}$ $=-\mathrm{i}(2-11\mathrm{i})$ $=-\mathrm{i}(2-\mathrm{i})^3$ $=\mathrm{i}^3(2-\mathrm{i})^3$ $=\left[\mathrm{i}(2-\mathrm{i})\right]^3$ $z-\mathrm{i}=\mathrm{i}(2-\mathrm{i})=1+2\mathrm{i}$ $\therefore z=1+3\mathrm{i}$ is a root of the equation	We rewrite $(-11)-2\mathrm{i}$ as $-\mathrm{i}(2-11\mathrm{i})$ to link to the previous result. Why do we write $-\mathrm{i}=\mathrm{i}^3$? Alternatively, $-11-2\mathrm{i}=-\mathrm{i}(\frac{-11}{-\mathrm{i}}+2)$ $=-\mathrm{i}\left(\frac{11\mathrm{i}}{\mathrm{i}^2}+2\right)$ $=-\mathrm{i}(-11\mathrm{i}+2)$ $1+3\mathrm{i}$ is just one of the roots for $(z-\mathrm{i})^3=-11-2\mathrm{i}$. Since this is a cubic equation, there are 2 more roots. We will learn how to find them in Section 13.3.3.

13.2.6 Complex Numbers in Graphing Calculator

Press [mode] to display mode setting. Use arrow keys to select a+bi to display the complex number in Cartesian form.

Note:

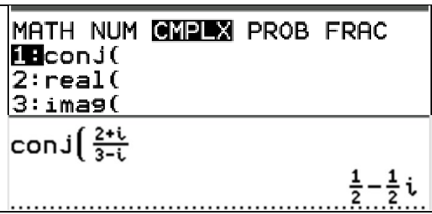
- The **radian mode** is to be used for calculations involving complex numbers
- To enter the complex number i, press [2nd][.]

You can find operations or functions for complex numbers in the **MATH CPLX menu** (press [math][▶][▶]:

1: conj(gives the conjugate of the complex number
2: real(gives the real part of a complex number
3: imag(gives the imaginary part of a complex number
4: angle(gives the principal argument of a complex number (to be discussed in 13.5)
5: abs(gives the modulus of a complex number (to be discussed in 13.5)
6: ▶Rect displays the result in Cartesian form

Example 6

Given $z=\frac{2+\mathrm{i}}{3-\mathrm{i}}$, use the graphing calculator to find z^*.

Solution:	**Think Zone:**
Use the command **"conj("** found under menu: **MATH**, submenu: **CMPLX** $\therefore z^* = \frac{1}{2} - \frac{1}{2}\mathrm{i}$	

13.2.7 Important Properties involving Conjugates

For the complex number $z = x + y\mathrm{i}, \quad x, y \in \mathbb{R}$, the conjugate pair z and z^* have the following properties:

Properties	Proof
(a) $z + z^* = 2\,\mathrm{Re}(z)$	$z + z^* = (x + y\mathrm{i}) + (x - y\mathrm{i})$ $= 2x = 2\,\mathrm{Re}(z)$
(b) $z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$	$z - z^* = (x + y\mathrm{i}) - (x - y\mathrm{i})$ $= 2\mathrm{i}y = 2\mathrm{i}\,\mathrm{Im}(z)$
(c) $zz^* = x^2 + y^2 = \lvert z\rvert^2$ where $\lvert z\rvert = \sqrt{x^2 + y^2}$ (to be discussed in 13.5.1)	$zz^* = (x + y\mathrm{i})(x + y\mathrm{i})^*$ $= (x + y\mathrm{i})(x - y\mathrm{i})$ $= x^2 - (y\mathrm{i})^2$ $= x^2 + y^2 = \lvert z\rvert^2$, where $\lvert z\rvert = \sqrt{x^2 + y^2}$
(d) $(z^*)^* = z$	$(z^*)^* = ((x + y\mathrm{i})^*)^*$ $= (x - y\mathrm{i})^*$ $= x + y\mathrm{i}$ $= z$
(e) $(z_1 + z_2)^* = z_1^* + z_2^*$ (f) $(z_1 - z_2)^* = z_1^* - z_2^*$	
(g) $(z_1 z_2)^* = z_1^* z_2^*$ When $z_1 = z_2 = z$, then $(z^2)^* = (z^*)^2$ In general, $(z^n)^* = (z^*)^n$ where n is a positive integer.	
(h) $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$.	

Exercise: Derive the proof of the results (f)-(h).

Example 7

If $z=\frac{1-\mathrm{i}}{2+\mathrm{i}}$, find $z-\frac{1}{z}$. Hence, or otherwise, find the complex number w in cartesian form $a+b\mathrm{i}$ such that $w-\frac{1}{w}=-\frac{3}{10}+\frac{21}{10}\mathrm{i}$ where $wz^*\neq -1$.

Solution:	Think Zone:
$z-\frac{1}{z}=\frac{1-\mathrm{i}}{2+\mathrm{i}}-\frac{2+\mathrm{i}}{1-\mathrm{i}}$ $=-\frac{3}{10}-\frac{21}{10}\mathrm{i}$ (using GC)	
$w-\frac{1}{w}=-\frac{3}{10}+\frac{21}{10}\mathrm{i}$ $=\left(-\frac{3}{10}-\frac{21}{10}\mathrm{i}\right)^*$ $=\left(z-\frac{1}{z}\right)^*$ $=z^*-\frac{1}{z^*}$	Use the property of conjugate: $(z_1\pm z_2)^*=z_1^*\pm z_2^*$
$w-z^*+\frac{1}{z^*}-\frac{1}{w}=0$ $(w-z^*)+\frac{(w-z^*)}{wz^*}=0$ $(w-z^*)\left(1+\frac{1}{wz^*}\right)=0$ $w=z^*$ or $\frac{1}{wz^*}=-1\Rightarrow wz^*=-1$ But since $wz^*\neq -1$, $w=z^*$. Thus, $w=\left(\frac{1-\mathrm{i}}{2+\mathrm{i}}\right)^*$ $=\frac{1}{5}+\frac{3}{5}\mathrm{i}$ (using GC).	Method if GC is not allowed: $w=\left(\frac{1-\mathrm{i}}{2+\mathrm{i}}\right)^*$ $=\frac{1+\mathrm{i}}{2-\mathrm{i}}$ $=\frac{(1+\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}$ $=\frac{1}{5}+\frac{3}{5}\mathrm{i}$ Upon getting the answer for w, use GC to check your answer!

Self-Review 1: If $z = 4 - 3\mathrm{i}$, express $z + \dfrac{1}{z}$ in Cartesian form. Hence, find the complex number w in Cartesian form such that $-w - \dfrac{1}{w} = \dfrac{104}{25} + \dfrac{72}{25}\mathrm{i}$ where $wz^* \neq -1$. [$w = -4 - 3\mathrm{i}$]

Solution:

13.3 Solving Equations Involving Complex Numbers

13.3.1 Solving Equations in 1 Unknown

Example 8

Find the two roots of the equation $ww^* = 4 + 2\mathrm{i} + 2\mathrm{i}\,w^*$, giving your answers in the form $a + \mathrm{i}b$, where $a,\ b \in \mathbb{R}$.

Solution:	**Think Zone:**
Let $w = a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.	Since w^* occurs in the equation, we let $w = a + b\mathrm{i}$.
$ww^* = 4 + 2\mathrm{i} + 2\mathrm{i}\,w^*$ $(a + b\mathrm{i})(a - b\mathrm{i}) = 4 + 2\mathrm{i} + 2\mathrm{i}(a - b\mathrm{i})$ $a^2 + b^2 = 4 + 2b + \mathrm{i}(2 + 2a)$	$ww^* = \lvert w \rvert^2 = a^2 + b^2$
Equating real and imaginary parts, $a^2 + b^2 = 4 + 2b$ ----(1) and $0 = 2 + 2a$ ----(2) From (2), $a = -1$ Sub. $a = -1$ in (1):	When equating the real and imaginary parts, the problem develops into a system of 2 unknowns and subsequently 2 equations.

$(-1)^2+b^2-4-2b=0$ $b^2-2b-3=0$ $(b-3)(b+1)=0$ $b=-1$ or $b=3$ Hence the two roots are $-1-\mathrm{i}$ and $-1+3\mathrm{i}$.	

13.3.2 Solving Simultaneous Equations with 2 Unknowns

In the O Level syllabus, we solve simultaneous equations in two variables (involving real numbers) using the substitution and elimination methods. By doing so, we can reduce the simultaneous equations into one equation with just a single variable.

When solving simultaneous equations involving complex numbers, we do the following.
Step 1: Using substitution or elimination, express the equations into one single variable.
Step 2: If z is the remaining variable, and

(a) if z can be made the subject easily, solve for z directly;
(b) if z cannot be made the subject easily, substitute $z=a+b\mathrm{i}$ into the equation. Equate the real and imaginary parts on the LHS and RHS.

Example 9

Two complex numbers w and z are such that $z-\mathrm{i}w=2$ and $2w+(1+2\mathrm{i})z=\mathrm{i}$. Find w and z, giving each answer in the form $x+y\mathrm{i}$.

Solution:	**Think Zone:**
$z-\mathrm{i}w=2 \quad \Rightarrow \quad z=2+\mathrm{i}w$ ---(1) $2w+(1+2\mathrm{i})z=\mathrm{i}$ --- (2) Substitute (1) into (2): $2w+(1+2\mathrm{i})(2+\mathrm{i}w)=\mathrm{i}$ $\Rightarrow \quad 2w+2+\mathrm{i}w+4\mathrm{i}-2w=\mathrm{i}$ $\Rightarrow \quad \mathrm{i}w=-2-3\mathrm{i}$ $\Rightarrow \quad w=\dfrac{-2-3\mathrm{i}}{\mathrm{i}}=-3+2\mathrm{i}$ using GC Substitute $w=-3+2\mathrm{i}$ into (1): $z=2+\mathrm{i}(-3+2\mathrm{i})=-3\mathrm{i}$ using GC	We DO NOT let $z=a+ib$ and $w=c+id$, as we would eventually end up with solving for 4 unknowns. If there are 2 or more unknown complex numbers involved, this method would be too tedious as an approach and inadvisable in general.

Example 10 [RVHS/Prelim 2020/I/Q9]

Solve the simultaneous equations $z-2w^*=\mathrm{i}$, $\mathrm{i}z-w=\mathrm{i}$, giving your answers in the form $x+\mathrm{i}y$, where $x,\ y\in\mathbb{R}$.

Solution:	**Think Zone:**
$z-2w^*=\mathrm{i} \ \cdots(1) \qquad \mathrm{i}z-w=\mathrm{i} \ \cdots(2)$ From (2), $\mathrm{i}z=w+\mathrm{i}$ $z=1-w\mathrm{i} \quad (3)$ Sub (3) into (1): $1-w\mathrm{i}-2w^*=\mathrm{i}$ Let $w=a+b\mathrm{i}$,	Since z is the common variable in both equations, we can eliminate it by making it the subject. Note that it is not easy to make w, the remaining variable, the

$1-(a+bi)i-2(a-bi)=i$ $1-ai+b-2a+2bi=i$ $1-2a+b+(2b-a)i=i$ Comparing real part, $1-2a+b=0$ $2a-b=1 \quad \cdots(4)$ Comparing imaginary part, $2b-a=1 \quad \cdots(5)$ Solving (4) and (5) using GC, $a=1, b=1$ Therefore, $w=1+i$, $z=1-(1+i)i$ $=2-i$	subject in the equation (due to presence of w and w^*).

Example 11

The complex numbers p and q satisfy $p=qi+2$ and $p^2-q+6+2i=0$. By eliminating q or otherwise, solve the simultaneous equations.

Solution:	**Think Zone:**
$p=qi+2$ ---(1) $q=\dfrac{p-2}{i}=\dfrac{p-2}{i}\times\dfrac{-i}{-i}=2i-pi$ Sub $q=2i-pi$ into $p^2-q+6+2i=0$. We have $p^2-(2i-pi)+6+2i=0$ $p^2+ip+6=0$ $p=\dfrac{-(i)\pm\sqrt{(i)^2-4(1)(6)}}{2(1)}$ $=\dfrac{-i\pm\sqrt{-25}}{2}$ $p=\dfrac{-i+5i}{2}$ or $p=\dfrac{-i-5i}{2}$ $=2i$ $\quad$ $=-3i$ Thus, if $p=2i$, $q=2i-(2i)i$ $=2+2i$ if $p=-3i$, $q=2i-(-3i)i$ $=-3+2i$	To solve $az^2+bz+c=0$, we use the quadratic formula $z=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. This is still true if $a,b,c\in\mathbb{C}$.

Self-Review 2 [HCI/Prelim 2018/I/Q9(a)]

Showing your working clearly, find the complex numbers z and w which satisfy the simultaneous equations

$$4iz-w=9i-13,$$
$$(4+2i)w^*=z+3i.$$

[$w=1-i$, $z=2+3i$]

Solution:	Think Zone:
	Make z the subject and eliminate it. To solve for w, let $w = x + \mathrm{i}y$

13.3.3 Solving Polynomial Equations

The following results are useful in solving polynomial equations.

Useful result 1: <u>Fundamental Theorem of Algebra</u>
The Fundamental Theorem of Algebra states that **any polynomial of degree *n* has *n* roots**.

Note: These n roots can be real or complex, and some of which may be repeated.

Useful result 2: <u>Factor and Remainder Theorem (Refer to Chapter 0 Notes)</u>
Consider the equation $P(x) = 0$, where $P(x)$ is a polynomial of degree n.
Remainder Theorem states that:
If we divide $P(x)$ by $(x - a)$, then the remainder is $P(a)$.
Thus, if the remainder is 0, then $(x - a)$ is a factor of $P(x)$.

This leads us to Factor Theorem:
If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Useful result 3: <u>Conjugate Root Theorem</u>
Let f(z) be a polynomial in z with **real coefficients**. If α is a complex root of $\mathrm{f}(z) = 0$, then α^* is also a complex root.

Example, Suppose the equation $az^4 + bz^3 + cz^2 + dz + e = 0$, where a, b, c, d and e are real numbers, has a complex root α.

Then $a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0$

$\Rightarrow (a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e)^* = 0^* = 0$

$\Rightarrow (a\alpha^4)^* + (b\alpha^3)^* + (c\alpha^2)^* + (d\alpha)^* + e^* = 0$

$\Rightarrow \quad a^*(\alpha^*)^4 + b^*(\alpha^*)^3 + c^*(\alpha^*)^2 + d^*(\alpha^*) + e = 0$

$\Rightarrow \quad a(\alpha^*)^4 + b(\alpha^*)^3 + c(\alpha^*)^2 + d(\alpha^*) + e = 0$

Then α^* is also a root.

In general, whenever a polynomial equation with **real coefficients** has complex roots, by Conjugate Root Theorem, the complex roots will occur in **conjugate pairs**.

13.3.3.1 Solving Quadratic Equations (Polynomial Equations of degree 2)

Useful result 4: <u>Formula for solving quadratic equations</u>

Consider a degree 2 polynomial, i.e. the general quadratic equation $az^2 + bz + c = 0$ where a, b, c can be real or complex.

Using the formula, $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we find that if the discriminant $b^2 - 4ac < 0$, the solutions are not real. Thus, we say that the roots of the given equations are complex (non-real).

In fact, the solutions of a quadratic equation with real coefficients will form a conjugate pair, i.e. $z = \dfrac{-b}{2a} + \dfrac{\sqrt{b^2 - 4ac}}{2a}$ or $\dfrac{-b}{2a} - \dfrac{\sqrt{b^2 - 4ac}}{2a}$ $(b^2 - 4ac < 0)$

<u>EXTENSION</u>

- This formula has been expanded to find not only real roots but also complex roots.
- While in ‘O’ level, a quadratic equation with $b^2 - 4ac < 0$ would be said to have no real roots, in ‘A’ level, it would be stated that the equation has complex roots instead.

Example 12 (degree 2 polynomial with real and complex numbers as coefficients)

Solve the equation $z^2 + \mathrm{i}z + 1 = 0$.

Solution:	**Think Zone:**
$z^2 + \mathrm{i}z + 1 = 0$ $z = \dfrac{-\mathrm{i} \pm \sqrt{\mathrm{i}^2 - 4}}{2} = \dfrac{-\mathrm{i} \pm \sqrt{-5}}{2}$ $= \dfrac{-\mathrm{i} \pm \sqrt{5\mathrm{i}^2}}{2} = \dfrac{-\mathrm{i} \pm \mathrm{i}\sqrt{5}}{2}$ Hence $z = \left(\dfrac{-1+\sqrt{5}}{2}\right)i$ or $z = \left(\dfrac{-1-\sqrt{5}}{2}\right)i$	We can use the quadratic formula. Note: 1) We cannot use the GC to solve the equation. 2) The roots are not conjugate pairs of each other.

Useful result 5: <u>Relationship between Roots and Coefficients of a degree 2 polynomial</u>

Suppose α and β are two roots of the quadratic equation $ax^2+bx+c=0$ ($a\neq 0$)

Dividing by $a(\neq 0)$ gives $x^2+\frac{b}{a}x+\frac{c}{a}=0$ and still have roots α and β.

We can therefore write

$$x^2+\frac{b}{a}x+\frac{c}{a}=(x-\alpha)(x-\beta)$$
$$=x^2-(\alpha+\beta)x+\alpha\beta$$

Equating coefficients of x and the constants, we have the useful results:

$$\text{Sum of roots, } \alpha+\beta=-\frac{b}{a}; \quad \text{Product of roots, } \alpha\beta=\frac{c}{a}.$$

Example 13 [AJC/Prelim 2018/I/7] (degree 2 polynomial with real & complex coefficients)

Given that $z=-2+3\mathrm{i}$ is a root of the equation $2z^2+(-1+4\mathrm{i})z+c=0$, find the complex number c and the other root.

Solution:	**Think Zone:**
Let $\alpha=-2+3\mathrm{i}$ and the other root be β $\alpha+\beta=-\frac{-1+4\mathrm{i}}{2}$ $\Rightarrow \beta=-\frac{-1+4\mathrm{i}}{2}-(-2+3\mathrm{i})$ $=\frac{5}{2}-5\mathrm{i}$ (using GC) $\alpha\beta=\frac{c}{2}$ $\Rightarrow c=2\alpha\beta$ $=2(-2+3\mathrm{i})(\frac{5}{2}-5\mathrm{i})$ $=20+35\mathrm{i}$ (using GC)	Recall that, if α, β are the roots of the quadratic equation $ax^2+bx+c=0$, then, $\alpha+\beta=-\frac{b}{a}, \quad \alpha\beta=\frac{c}{a}$ Alternative Method: Since $z=-2+3\mathrm{i}$ is a root of the equation $2z^2+(-1+4\mathrm{i})z+c=0$, $2(-2+3\mathrm{i})^2+(-1+4\mathrm{i})(-2+3\mathrm{i})+c=0$ $c=20+35\mathrm{i}$ (using GC) Let $2z^2+(-1+4\mathrm{i})z+20+35\mathrm{i}$ $=[z-(-2+3\mathrm{i})][2z-\alpha]$ Comparing coefficients of z, $-\alpha-2(-2+3\mathrm{i})=-1+4\mathrm{i}$ $\alpha=5-10\mathrm{i}$ Alternatively, comparing constants, $\alpha(-2+3\mathrm{i})=20+35\mathrm{i}$ $\alpha=\frac{20+35\mathrm{i}}{-2+3\mathrm{i}}=5-10\mathrm{i}$ (using GC) Hence the other root is $\frac{5}{2}-5\mathrm{i}$ Why can't we use Conjugate Root Theorem?

13.3.3.2 Solving Polynomial Equations of degree 3 and higher

Example 14 (degree 4 polynomial with all real coefficients)

Without the use of GC, solve the equation $z^4+z^3-8z^2+14z-8=0$ given that $1+\mathrm{i}$ is one of the roots. Hence solve the equation $z^4-\mathrm{i}z^3+8z^2+14\mathrm{i}z-8=0$.

Solution:

Since $z^4+z^3-8z^2+14z-8=0$ has **real coefficients**, by Conjugate Root Theorem, $1-\mathrm{i}$ is another root.

Then $[z-(1+\mathrm{i})]$ and $[z-(1-\mathrm{i})]$ are factors of the expression on the LHS of the given equation.

Hence, $[z-(1+\mathrm{i})][z-(1-\mathrm{i})]$ is also a factor of it.

$[z-(1+\mathrm{i})][z-(1-\mathrm{i})]$
$=[(z-1)-\mathrm{i}][(z-1)+\mathrm{i}]$
$=(z-1)^2-(\mathrm{i})^2$
$=z^2-2z+1-(-1)$
$=z^2-2z+2$

Therefore
$z^4+z^3-8z^2+14z-8$
$=(z^2-2z+2)(z^2+Bz-4)$ by inspection
Comparing coefficient of
z^3: $1=B-2 \Rightarrow B=3$

Thus
$z^4+z^3-8z^2+14z-8$
$=(z^2-2z+2)(z^2+3z-4)$
$=(z^2-2z+2)(z+4)(z-1)$
The roots of the equation are $1+\mathrm{i},\ 1-\mathrm{i},\ -4,\ 1$.

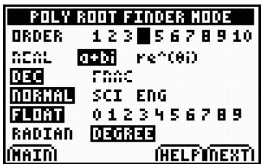
$z^4-\mathrm{i}z^3+8z^2+14\mathrm{i}z-8=0$
$\Rightarrow (\mathrm{i}z)^4+(\mathrm{i}z)^3-8(\mathrm{i}z)^2+14(\mathrm{i}z)-8=0$
$\Rightarrow w^4+w^3-8w^2+14w-8=0$ where $w=\mathrm{i}z$
By previous result,
$w=1+\mathrm{i},1-\mathrm{i},-4,1$
$\mathrm{i}z=1+\mathrm{i},1-\mathrm{i},-4,1$
$z=\frac{1+\mathrm{i}}{\mathrm{i}},\frac{1-\mathrm{i}}{\mathrm{i}},\frac{-4}{\mathrm{i}},\frac{1}{\mathrm{i}}$
$=1-\mathrm{i},-1-\mathrm{i},4\mathrm{i},-\mathrm{i}$

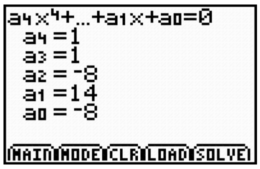
Think Zone:

Alternatively, we can obtain the expanded result for the quadratic factor using result 5, since $1+\mathrm{i}$ and $1-\mathrm{i}$ are roots of equation, $[z-(1+\mathrm{i})]$ and $[z-(1-\mathrm{i})]$ are factors of polynomial.
Sum of roots
$=(1+\mathrm{i})+(1-\mathrm{i})=2$
Product of roots
$=(1+\mathrm{i})(1-\mathrm{i})=2$
So, the quadratic factor $[z-(1+\mathrm{i})][z-(1-\mathrm{i})]$ is z^2-2z+2.

$z^4+z^3-8z^2+14z-8$
$=(z^2-2z+2)(Az^2+Bz+C)$
By inspection (coefficient of z^4 and constant), $A=1$ and $C=-4$

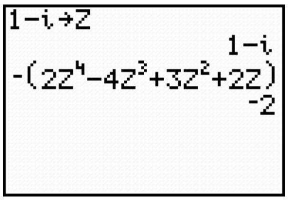
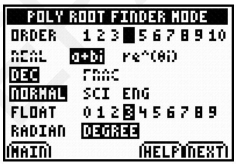
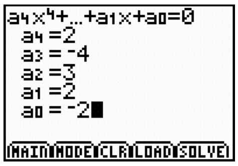
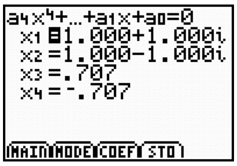
Use of GC to check answer:
press **APPS > 5:PlySmlt2 >1: POLY ROOT FINDER**
select **ORDER 4** and **a+bi**
and press **graph**





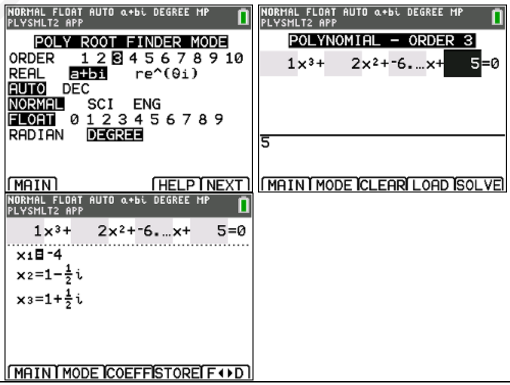
Example 15 (degree 4 polynomial with all real coefficients)

Find the real value of k such that $2x^4-4x^3+3x^2+2x+k=0$ has a complex root $1-\mathrm{i}$. Hence factorise $2x^4-4x^3+3x^2+2x+k$ into a product of one quadratic and two linear factors with real coefficients.

Solution:	**Think Zone:**
$k=-\left(2(1-\mathrm{i})^4-4(1-\mathrm{i})^3+3(1-\mathrm{i})^2+2(1-\mathrm{i})\right)=-2$ (using GC)	Use of GC to get answer. 
Since $2x^4-4x^3+3x^2+2x-2=0$ has real coefficients, by Conjugate Root Theorem, $1+\mathrm{i}$ is also a complex root.	
Then $[x-(1+\mathrm{i})]$ and $[x-(1-\mathrm{i})]$ are factors of the expression on the LHS of the given equation. Hence, the quadratic factor is x^2-2x+2.	Sum of roots = $(1+\mathrm{i})+(1-\mathrm{i})=2$ Product of roots = $(1+\mathrm{i})(1-\mathrm{i})=2$
Thus, we can let, $2x^4-4x^3+3x^2+2x-2=\left(x^2-2x+2\right)\left(2x^2+ax+b\right)$ Comparing constants: $-2=2b \Rightarrow b=-1$ Comparing coeff of x^3: $-4=a-4 \Rightarrow a=0$	We can use the GC to check the accuracy of our answers too: 
Hence, $2x^4-4x^3+3x^2+2x-2$ $=\left(x^2-2x+2\right)\left(2x^2-1\right)$ $=\left(x^2-2x+2\right)\left[\left(\sqrt{2}x\right)^2-1^2\right]$ $=\left(x^2-2x+2\right)\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)$	

Self-Review 3 [N2021/II/Q1]

One of the roots of the equation $x^3+2x^2+ax+b=0$, where a and b are real, is $1+\frac{1}{2}\mathrm{i}$. Find the other roots of the equation and the values of a and b. [$a=-6.75$; $b=5$; $1-\frac{1}{2}\mathrm{i}, -4$]

Solution:	Think Zone:
	Use GC to obtain $\left(1+\frac{1}{2}\text{i}\right)^3+2\left(1+\frac{1}{2}\text{i}\right)^2=\frac{7}{4}+\frac{27}{8}\text{i}$ $\left(1+\frac{1}{2}\text{i}\right)+\left(1-\frac{1}{2}\text{i}\right)=2$ $\left(1+\frac{1}{2}\text{i}\right)\left(1-\frac{1}{2}\text{i}\right)=1+\frac{1}{4}=\frac{5}{4}$ Alternatively, after getting the *a* and *b* values, you can use GC to solve for the roots. 

Example 16 [ACJC/Prelim 2017/II/Q1 Modified]

Explain why a cubic polynomial equation with real coefficients must have at least 1 real root. Given that $1+\text{i}$ is a root of the equation $z^3-4(1+\text{i})z^2+(-2+9\text{i})z+5-\text{i}=0$, find the other roots of the equation.

Solution:	Think Zone:
By Fundamental Theorem of Algebra, a cubic polynomial equation has 3 roots. Since polynomial equation has all real coefficients, if there are complex roots, they must exist in conjugate pairs, i.e. even number of complex roots. Hence equation must have at least 1 real root (or at most 2 complex roots).	

$z^3-4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z+5-\mathrm{i}=0$ $\left(z-(1+\mathrm{i})\right)\left(Az^2+Bz+C\right)=0$ By comparing coefficients, $z^3: A=1$ $z^0: -(1+\mathrm{i})C=5-\mathrm{i}\Rightarrow C=\dfrac{5-\mathrm{i}}{-(1+\mathrm{i})}=-2+3\mathrm{i}$ $z^2: B-(1+\mathrm{i})=-4(1+\mathrm{i})\Rightarrow B=-3(1+\mathrm{i})$ $\left(z-(1+\mathrm{i})\right)\left(z^2-3(1+\mathrm{i})z-2+3\mathrm{i}\right)=0$ Solving $\left(z^2-3(1+\mathrm{i})z-2+3\mathrm{i}\right)=0$, $z=\dfrac{-\left(-3(1+\mathrm{i})\right)\pm\sqrt{\left(-3(1+\mathrm{i})\right)^2-4(1)\left(-2+3\mathrm{i}\right)}}{2(1)}$ $=\dfrac{3+3\mathrm{i}\pm\sqrt{8+6\mathrm{i}}}{2}$ $=\dfrac{3+3\mathrm{i}\pm(3+\mathrm{i})}{2}=3+2\mathrm{i}$ or i $\therefore$ The other 2 roots are $z=3+2\mathrm{i}$ or $z=\mathrm{i}$	Notice that not all the coefficients are real. Hence, we cannot use Conjugate Root Theorem here. When $1+\mathrm{i}$ is a root, $\left[z-(1+\mathrm{i})\right]$ is a factor. You can use the GC to solve for $\sqrt{8+6\mathrm{i}}$. Alternatively, let $w=\sqrt{8+6\mathrm{i}}\Rightarrow w^2=8+6\mathrm{i}$ and solve for w.

Self-Review 4 [TJC/Prelim 2020/I/4(a)]

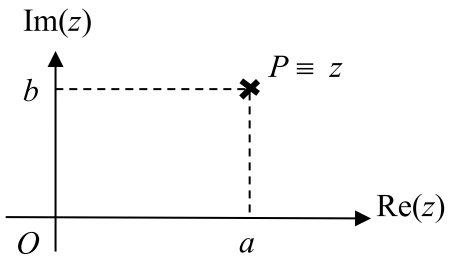
The complex number w is such that $w^3=-8\mathrm{i}$. Given that one possible value of w is 2i, use a **non-calculator method** to find the other values of w. Give your answers in the form $a+b\mathrm{i}$, where a and b are exact values. [$\sqrt{3}-\mathrm{i}$ or $-\sqrt{3}-\mathrm{i}$]

Solution:	**Alternative Solution**

13.4 The Argand Diagram

The x-axis of the Cartesian plane can be used to represent the real part of a complex number, while the y-axis represents the imaginary part. Hence, the x-axis is called real axis while the y-axis is the imaginary axis.

We represent the complex number $z = a + b\text{i}$ using the point P with coordinates (a, b) as shown below.



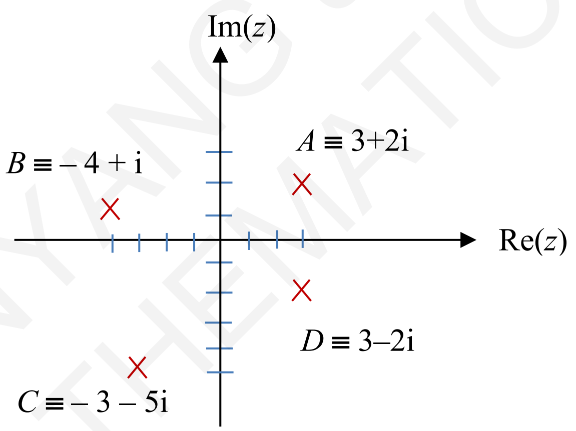
This way of representing complex numbers using a diagram was an idea introduced by the French Mathematician Argand, hence this diagram is known as the **Argand diagram**. We say that the complex number z is represented by the point P. We label the complex number as $P \equiv z$.

DIAGRAMS

The Argand diagram provides a way to represent and visualise complex numbers geometrically, like the Cartesian coordinate system and the number line for real numbers.

Example 17

Illustrate on an Argand diagram the following complex numbers $A \equiv 3 + 2\text{i}$, $B \equiv -4 + \text{i}$, $C \equiv -3 - 5\text{i}$ and $D \equiv 3 - 2\text{i}$.

Solution:	**Think Zone:**
	It is always good to draw to scale for all Argand diagrams. A distorted diagram will distort your view. Notice, $3 + 2\text{i}$ and $3 - 2\text{i}$ are reflections of each other in the real axis. Recall that they are also conjugates of each other.

13.4.1 Geometrical Representation of z^* and $-z$ where z is a complex number

For simplicity, let $z = a + b\mathrm{i}$ where $a > 0$ and $b > 0$. Let $P \equiv z$.

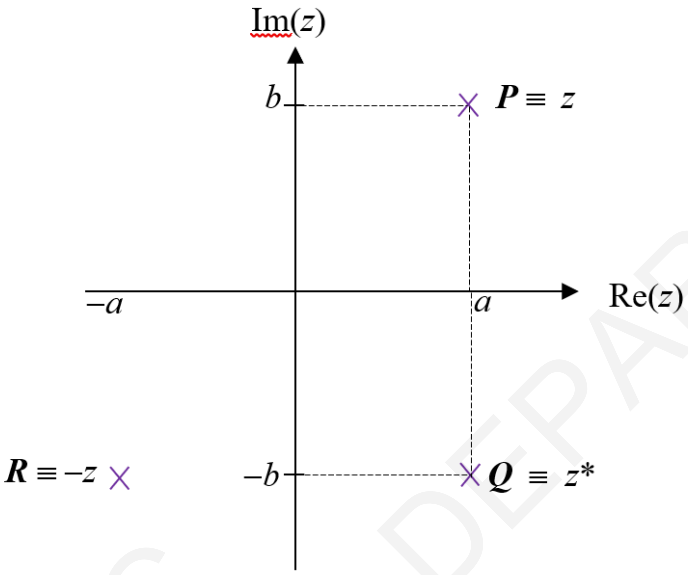
(a) The conjugate of z, $z^* = a - b\mathrm{i}$.

Let $Q \equiv z^*$.

z^* will be represented by $Q\ (a, -b)$. Geometrically, point Q is the mirror image of point P reflected in the real axis.

(b) Let $R \equiv -z$.

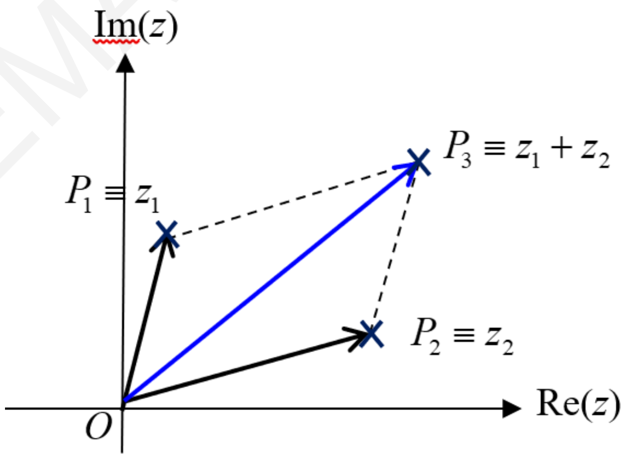
$-z$ will be represented by $R\ (-a, -b)$. Geometrically, point R is the mirror image of point P (a, b) representing z in the origin.



13.4.2 Geometrical Interpretation of complex number addition and subtraction

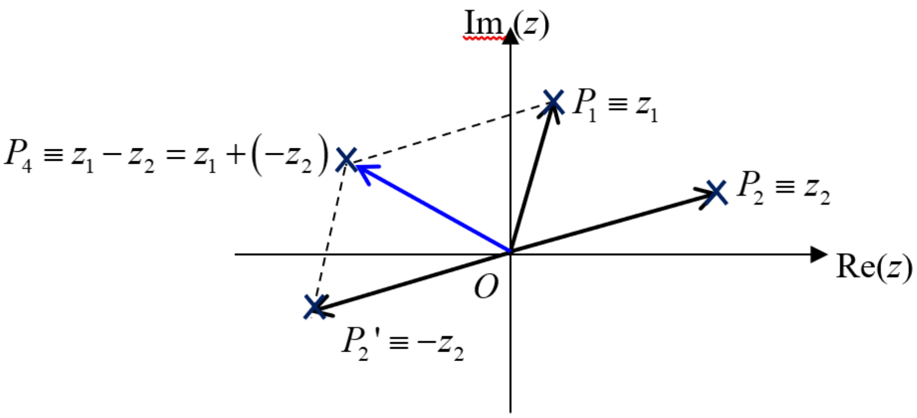
Addition of Two Complex Numbers in an Argand Diagram

If P_1 and P_2 represent the complex numbers z_1 and z_2 respectively, then $P_3 \equiv z_1 + z_2$ is the vertex of the parallelogram $OP_1P_3P_2$ as shown in the diagram.



Subtraction of Two Complex Numbers in an Argand Diagram

If P_1 and P_2' represent the complex numbers z_1 and $-z_2$ respectively, then $P_4 \equiv z_1 - z_2$, also written as $z_1 + (-z_2)$, is the vertex of the parallelogram $OP_1P_4P_2'$ as shown in the diagram.

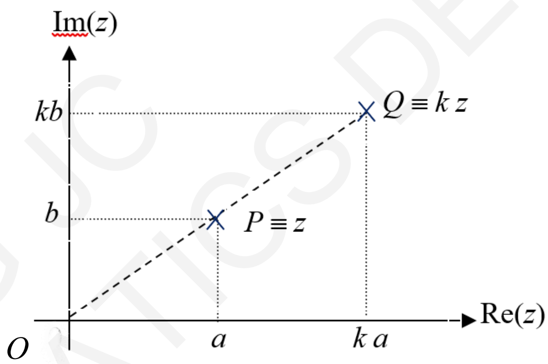


VECTORS:
Complex numbers can be thought as 2-dimensional position vectors. Thus, addition and subtraction of complex numbers are analogous to addition and subtraction of position vectors.

13.4.3 Geometrical Interpretation of *kz* where $k \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C}$

If $k > 0$, then the complex number $kz = k(a + bi)$ is represented by the point Q such that the points O, P ($\equiv z$) and Q ($\equiv k\,z$) are collinear and $OQ = k\,OP$.

Note that P and Q are on the same side as O.



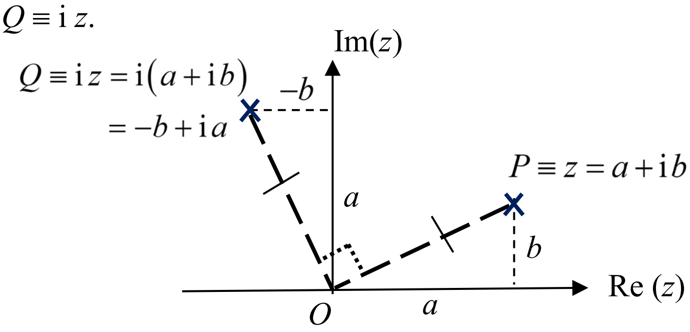
In general, for any point P representing $z \in \mathbb{C}$, the point Q representing kz where k is a non-zero real number lies on the straight line passing through the origin O and the point P. If $k < 0$, then Q is obtained by rotating P 180° about O and scaled by a factor of $|k|$ so that P and Q are on opposite sides of O such that $OQ = |k|OP$.

13.4.4 Geometrical Interpretation of Multiplication of a complex number by i

For the sake of simplicity, let $z = a + bi$ where $a, b \in \mathbb{R}$ and $a > 0$, $b > 0$.

$\mathrm{i}\,z = \mathrm{i}\,(a + bi) = a\,\mathrm{i} + b\,\mathrm{i}^2 = -b + a\mathrm{i}$.

Let $P \equiv z$ and $Q \equiv \mathrm{i}\,z$.

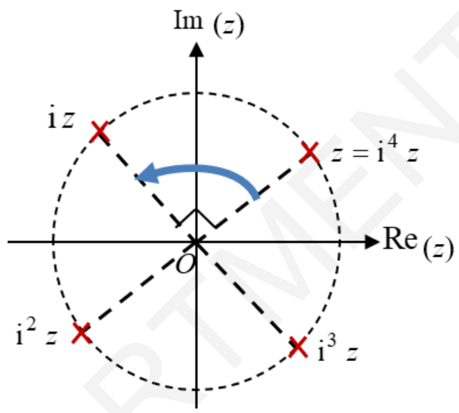


From above, we see that $\angle POQ = 90^\circ$ and
length $OP = |\mathrm{i}z| = |\mathrm{i}|\,|z| = |z| = \sqrt{a^2+b^2}$ = length OQ

Therefore, the vector representing the complex number iz (i.e. $\overrightarrow{OQ}$) is obtained by **rotating the vector representing z (i.e. $\overrightarrow{OP}$) about O through 90° in an anticlockwise sense**.

In fact, we have the following:

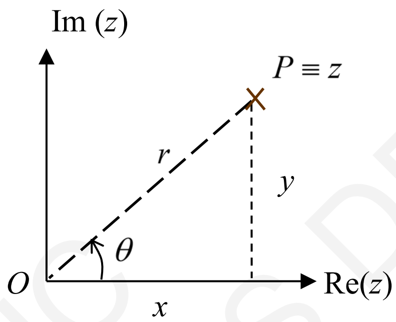
iz:	Rotation anticlockwise by 90° about the origin.
i^2z (or $-z$):	Rotation by 180^0 about the origin.
$-\mathrm{i}z$ (or i^3z):	Rotation clockwise by 90° about the origin.



13.5 Modulus and Argument

- Let point P represent the complex number $z = x + \mathrm{i}y$.

Let the length of the line segment OP be r and the angle made b
θ.



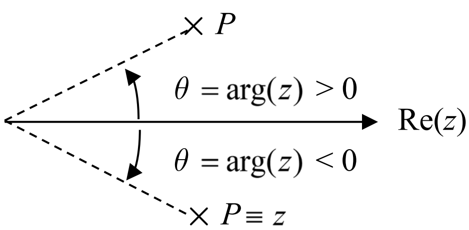
Then $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{x^2+y^2}$.

r is called the **modulus of** z and is denoted by $|z|$, i.e. $r = |z| = |x+\mathrm{i}y| = \sqrt{x^2+y^2}$.

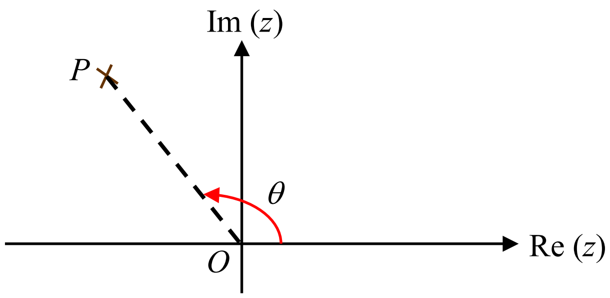
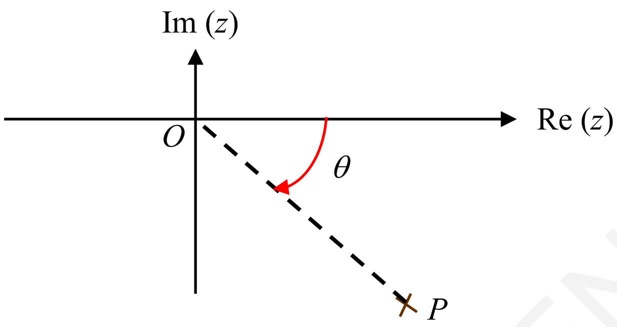
θ is called the **argument of** z and is denoted by $\arg z$, i.e. $\theta = \arg z = \arg(x+\mathrm{i}y)$.

- **Finding argument of z**

Convention:



Conventionally, we restrict θ to the range $-\pi < \theta \le \pi$, called the **principal range**.
The **unique** value of θ lying in the principal range is called the **principal argument**.
Thus, in order that the argument is in the principal range, this is how we will measure the argument.

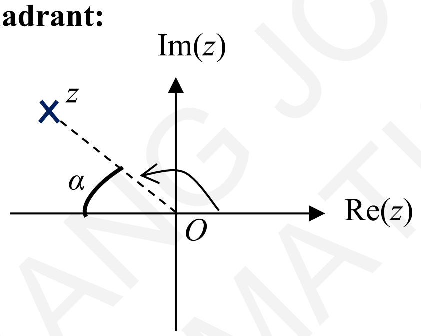
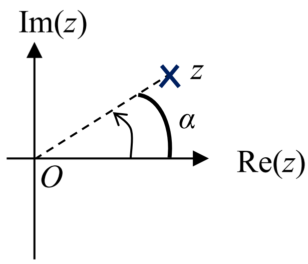
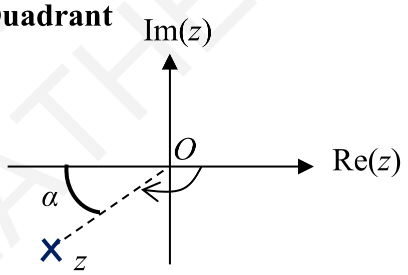
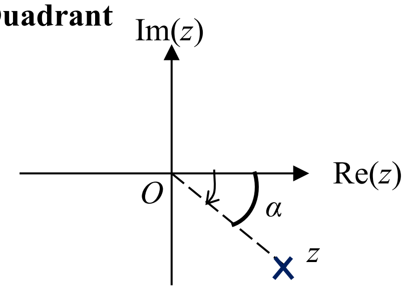
If P lies **above** the real axis, θ is measured in an **anticlockwise direction** and θ is **positive**. 	If P lies **below** the real axis, θ is measured in a **clockwise direction** and θ is **negative**. 

Following are the steps (**ABS**) to get the principal argument correctly.

Step 1. Argand diagram: To find the argument of a complex number, always **draw an Argand diagram** and indicate on the diagram the quadrant where the complex number lies.

Step 2. Basic angle: Find basic angle $= \alpha = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

Step 3. Sign: Find the sign and magnitude of the argument of a complex number. This depends on the quadrant that the complex number is in.

2nd Quadrant:  $\arg(z) = \pi - \alpha$ Argument is positive and obtuse.	**1st Quadrant:**  $\arg(z) = \alpha$ Argument is positive and acute.
3rd Quadrant  $\arg(z) = -(\pi - \alpha)$ Argument is negative and obtuse.	**4th Quadrant**  $\arg(z) = -\alpha$ Argument is negative and acute.

Argument of:

Real number	$k\pi$
Positive real number	$2k\pi$
Negative real number	$(2k+1)\pi$
Purely imaginary number	$\frac{1}{2}\pi + k\pi$
Purely positive imaginary number	$\frac{1}{2}\pi + 2k\pi$
Purely negative imaginary number	$-\frac{1}{2}\pi + 2k\pi$
z^*	$-\arg(z)$

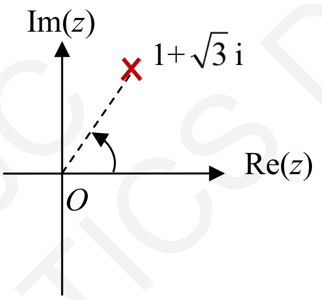
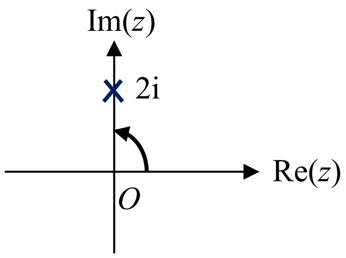
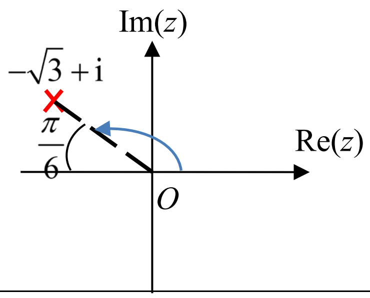
where $k \in \mathbb{Z}$.

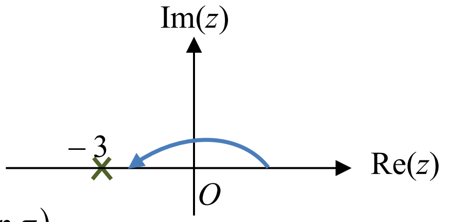
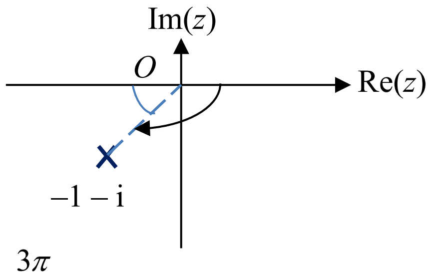
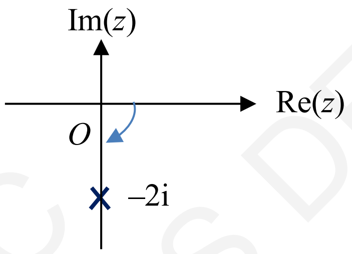
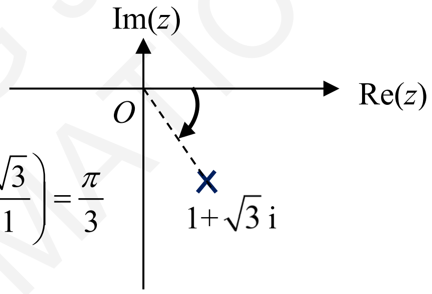
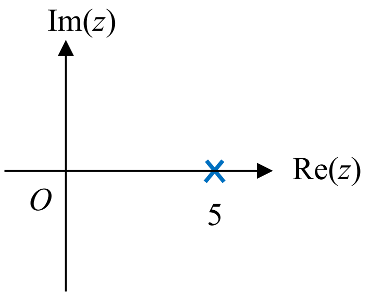
Example 18

Find the modulus and argument of each of the following complex numbers:

(a) $z = 1+\sqrt{3}\,\mathrm{i}$ (b) $z = 2\mathrm{i}$ (c) $z = -\sqrt{3}+\mathrm{i}$ (d) $z = -3$

(e) $z = -1-\mathrm{i}$ (f) $z = -2\mathrm{i}$ (g) $z = 1-\sqrt{3}\,\mathrm{i}$ (h) $z = 5$

Solution:	**Think Zone:**
(a) $z = 1+\sqrt{3}\,\mathrm{i}$ $r = \lvert z\rvert = \sqrt{1+3} = 2$ basic $\measuredangle = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ $= \frac{\pi}{3}$ $\arg z = \frac{\pi}{3}$ rad 	Always represent the complex number on an Argand diagram to visualize. Followed the **ABS** approach. Use of GC to check/find the modulus and argument of a complex number θ is in the 1^{st} quadrant. It is positive and acute.
(b) $z = 2\mathrm{i}$ $\lvert z\rvert = 2$ $\arg z = \frac{\pi}{2}$ 	$z = 2\mathrm{i}$ is purely imaginary.
(c) $z = -\sqrt{3}+\mathrm{i}$ $\lvert z\rvert = \sqrt{3+1} = 2$ basic $\measuredangle = \tan^{-1}\frac{1}{\sqrt{3}}$ $= \frac{\pi}{6}$ 	

$\arg z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$	θ is in 2nd quadrant. It is positive and obtuse.
(d) $z = -3$ $\lvert z \rvert = 3$ $\arg z = \pi$ $z = 3(\cos\pi + \mathrm{i}\sin\pi)$  	$z = -3$ is purely real (and negative).
(e) $z = -1 - \mathrm{i}$ $\lvert z \rvert = \sqrt{1+1} = \sqrt{2}$ basic $\measuredangle = \tan^{-1}\frac{1}{1}$ $= \frac{\pi}{4}$ $\arg z = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$.  	θ is in the 3rd quadrant. It is negative and obtuse. When asked to write in trigonometric form, we **do not** simplify $\cos(-\theta)$ to $\cos\theta$ and $\sin(-\theta)$ to $-\sin(\theta)$.
(f) $z = -2\mathrm{i}$ $\lvert z \rvert = 2$ $\arg z = -\frac{\pi}{2}$  	$z = -2\mathrm{i}$ is purely imaginary.
(g) $z = 1 - \sqrt{3}\,\mathrm{i}$ $\lvert z \rvert = \sqrt{1+3} = 2$ basic $\measuredangle = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$ $\arg z = -\frac{\pi}{3}$  	θ is in the 4th quadrant. It is negative and acute.
(h) $z = 5$ $\lvert z \rvert = 5$ $\arg z = 0$  	$z = 5$ is purely real (and positive).

Remark:

It is important to remember the trigonometric ratios of special angles: i.e. 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$.

Example 18 illustrates the relationship between "special" complex numbers with these special angles.

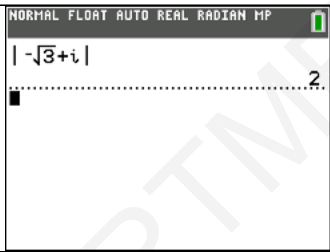
GC Basics to Calculate $|z|$ **and** $\arg(z)$

You may use the GC to calculate the modulus and the argument of a complex number.

For instance. in Example 18(c), $z = -\sqrt{3} + \mathrm{i}$.

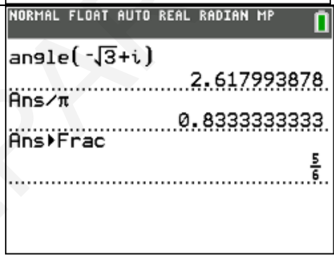
To find the modulus, we can key [math] [▸] [▸] [5] to access the modulus function, and type in our complex number accordingly.

(note: if it is a irrational number, you can square the number, then square root it back to get the exact surd form)

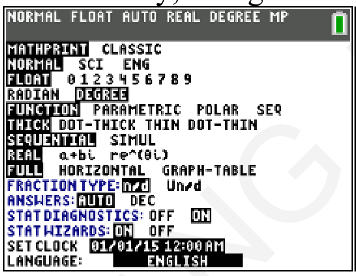


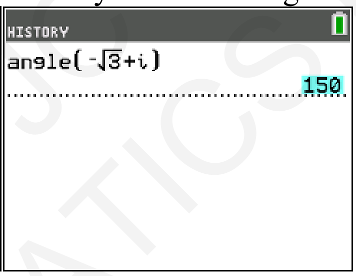
To find the angle, we can key [math] [▸] [▸] [4] to get the argument of the complex number

(Note: the number given is usually irrational, we can divide our answer by π to see if arg(z) is an exact number in terms of π.)



Alternatively, change the mode in your GC to Degree.





Then convert the angle to radian to give $\frac{5\pi}{6}$.

Properties of modulus and argument of complex numbers:

The table below is a summary of the useful properties of modulus and argument of complex numbers.

Modulus	Argument
$\lvert z_1z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$
$\lvert z^n\rvert = \lvert z\rvert^n$	$\arg(z^n) = n\arg(z)$
$\left\lvert\dfrac{z_1}{z_2}\right\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$	$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
$\left\lvert\dfrac{1}{z}\right\rvert = \dfrac{1}{\lvert z\rvert}$	$\arg\left(\dfrac{1}{z}\right) = \arg(1) - \arg(z) = 0 - \arg(z) = -\arg(z)$
$\lvert z^*\rvert = \lvert z\rvert$	$\arg z^* = -\arg z$
$zz^* = \lvert z\rvert^2$	$\arg(zz^*) = 0$

TRANSORMATION:

$|z^*| = |z|$ means that the modulus of a complex number is invariant when we take conjugate of a complex number.

Geometrically $|z^*| = |z|$ holds because z and z^* are mutual reflections in the real axis and therefore they have the same modulus. Can you also explain geometrically why $\arg z^* = -\arg z$?

Proof of $|z_1z_2| = |z_1||z_2|$:

Let $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$. Then

$$|z_1z_2| = |(x_1+\mathrm{i}y_1)(x_2+\mathrm{i}y_2)| = |x_1x_2 - y_1y_2 + \mathrm{i}(x_1y_2 + x_2y_1)| = \sqrt{(x_1x_2-y_1y_2)^2 + (x_1y_2+x_2y_1)^2}$$
$$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2y_1y_2}$$
$$= \sqrt{(x_1^2+y_1^2)x_2^2 + (x_1^2+y_1^2)y_2^2}$$
$$= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$
$$= \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$$
$$= |z_1||z_2|$$

You can treat is as an exercise to prove the rest of the properties.

Note:

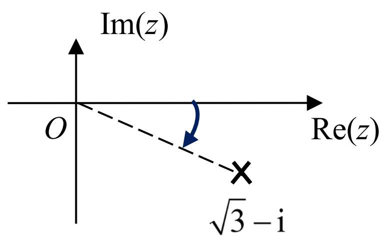
i) It is worth noting that the complex function '**arg**' behaves just like the **logarithmic function** 'ln' in view of the properties above.

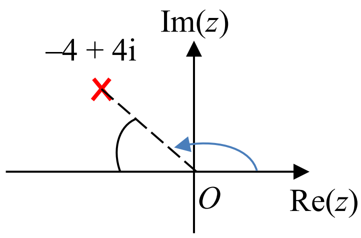
ii) The values of $\arg(z_1z_2)$, $\arg\left(\dfrac{z_1}{z_2}\right)$ and $\arg(z^n)$ obtained may not be the principal argument. If this occurs, we have to **add or subtract multiples of 2π** to obtain the desired principal argument (i.e. $-\pi < \theta \le \pi$).

Example 19

The complex numbers p and q are given by $p=\sqrt{3}-\mathrm{i}$ and $q=-4+4\mathrm{i}$. Without the use of GC, by finding the modulus and argument of p and q. Using the geometrical significance of multiplication by i and a real number, find the modulus and principal argument of

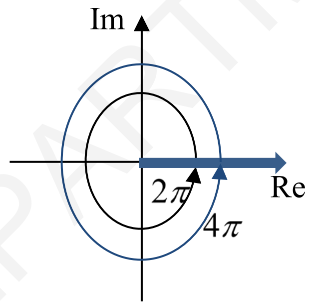
(a) $\mathrm{i}p$ (b) $-\mathrm{i}p$ (c) $2\mathrm{i}q$.

<table>
<tr><th>Solution:</th><th>Think Zone:</th></tr>
<tr><td>$|p|=\sqrt{3+1}=2$
basic $\measuredangle=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$
$\arg p=-\frac{\pi}{6}$


$|q|=\sqrt{16+16}=4\sqrt{2}$
basic $\measuredangle=\tan^{-1}\left(\frac{4}{4}\right)=\frac{\pi}{4}$
$\arg q=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

</td><td>Use your GC to check and verify your answers!</td></tr>
<tr><td>(a) Since $\mathrm{i}p$ is an anticlockwise rotation of p about O through 90°, therefore
$|\mathrm{i}p|=|p|=2$ and
$\arg(\mathrm{i}p)=\frac{\pi}{2}+\left(-\frac{\pi}{6}\right)=\frac{\pi}{3}$.</td><td>Note that we can use the properties in the above table to obtain the modulus and argument iz:
$|\mathrm{i}pq|=|\mathrm{i}||p|=2$
$\arg(\mathrm{i}p)=\arg\mathrm{i}+\arg p$
$=\frac{\pi}{2}+\left(-\frac{\pi}{6}\right)=\frac{\pi}{3}$</td></tr>
<tr><td>(b) Since $-\mathrm{i}p$ is a clockwise rotation of p about O through 90°, therefore
$|-\mathrm{i}p|=|p|=2$ and $\arg(-\mathrm{i}p)=-\frac{\pi}{2}+\left(-\frac{\pi}{6}\right)=-\frac{2\pi}{3}$</td><td>$|-\mathrm{i}p|=|-\mathrm{i}||p|=2$
$\arg(-\mathrm{i}p)=\arg(-i)+\arg(p)$
$=-\frac{\pi}{2}+\left(-\frac{\pi}{6}\right)=-\frac{2\pi}{3}$</td></tr>
<tr><td>(c) Since $2\mathrm{i}q$ is an anticlockwise rotation of q about O through 90° followed by a scaling of factor 2,
$|2\mathrm{i}q|=2|q|=8\sqrt{2}$ and $\arg(2\mathrm{i}q)=\frac{\pi}{2}+\frac{3\pi}{4}=\frac{5\pi}{4}$
Since $-\pi<\arg(z)\le\pi$, the principal argument
$\arg(2\mathrm{i}q)=\frac{5\pi}{4}-2\pi=-\frac{3\pi}{4}$</td><td>$|2\mathrm{i}q|=|2\mathrm{i}||q|$
$=2(4\sqrt{2})$
$=8\sqrt{2}$
$\arg(2\mathrm{i}q)=\arg(2\mathrm{i})+\arg(q)$
$=\frac{\pi}{2}+\frac{3\pi}{4}=\frac{5\pi}{4}$
Since $-\pi<\arg(z)\le\pi$, we have
$\arg(2\mathrm{i}q)=\frac{5\pi}{4}-2\pi=-\frac{3\pi}{4}$</td></tr>
</table>

Example 20 [N08/II/3(a)]

The complex number w has modulus r and argument θ, where $0<\theta<\frac{1}{2}\pi$, and w^* denotes the conjugate of w. Given $\left\lvert\frac{z_1}{z_2}\right\rvert=\frac{\lvert z_1\rvert}{\lvert z_2\rvert}$ and $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$, state the modulus and argument of p, where $p=\frac{w}{w^*}$. Given that p^5 is real and positive, find the possible values of θ.

Solution:	**Think Zone:**
$\lvert p\rvert=\left\lvert\frac{w}{w^*}\right\rvert=\frac{\lvert w\rvert}{\lvert w^*\rvert}=\frac{\lvert w\rvert}{\lvert w\rvert}=1$ $\arg(p)=\arg\left(\frac{w}{w^*}\right)=\arg(w)-\arg(w^*)=\arg(w)+\arg(w)$ $=2\arg(w)=2\theta$ For p^5 to be real and positive, p^5 would lie totally on the Real axis on the 1st quadrant. $\arg(p^5)=5\arg(p)=10\theta$, which is out of the principal argument range. Since p^5 is real and positive, $\arg(p^5)=0$. Since $0<\theta<\frac{1}{2}\pi$, $0<10\theta<5\pi$ Thus $10\theta=2\pi,\ 4\pi$ $\theta=\frac{\pi}{5},\frac{2\pi}{5}$.	Note that: $0<\theta<\frac{1}{2}\pi\Rightarrow\ 0<2\theta<\pi$ So 2θ is in the principal argument range. 

Self-Review 5:

Without the use of calculator, find the modulus and argument of $\frac{1+\mathrm{i}}{1-\mathrm{i}}$ and illustrate the complex numbers $1+\mathrm{i}$, $1-\mathrm{i}$ and $\frac{1+\mathrm{i}}{1-\mathrm{i}}$ on an Argand diagram. [1, $\frac{\pi}{2}$]

Solution: