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2021 H2 Math

JC1 Promo Exam Paper

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3	CATHOLIC JUNIOR COLLEGE	EOY
4	DUNMAN HIGH SCHOOL	EOY
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11	TEMASEK JUNIOR COLLEGE	EOY
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13	VICTORIA JUNIOR COLLEGE	EOY

2021 ACJC H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**
Attempt all questions.

1 State a sequence of transformations that will transform the curve with equation $y=2\sin(2x+\alpha)\cos(x)$ on to the curve with equation $y=-2\sin(4x+3\alpha)\cos(2x+\alpha)$, where α is a positive constant. [3]

2 Solve algebraically the inequality $\dfrac{x+3}{x^2+x-2}>-1$. [3]

Hence solve the inequality $\dfrac{x+3x^2}{1+x-2x^2}>-1$. [2]

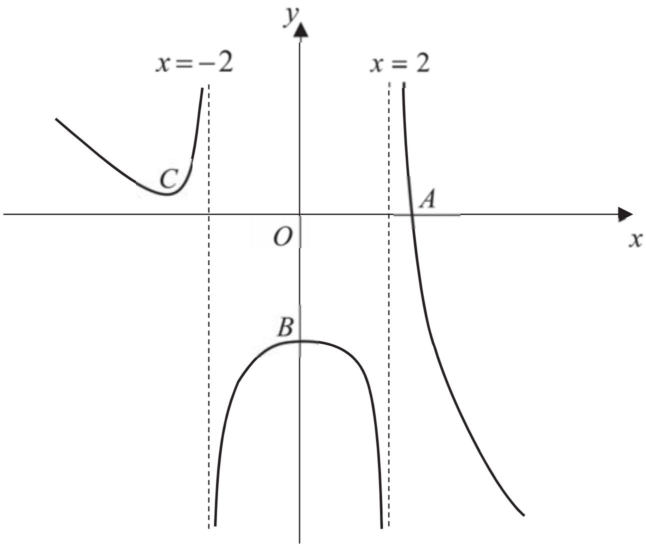
3 A curve C has equation

$$\frac{x^2-4y^2}{x^2+xy^2+100}=\frac{1}{2}\ ,\ x\in\mathbb{R}\ ,\ x\neq-8.$$

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x-y^2}{2xy+16y}$. [2]

Hence, prove that curve C does not have any stationary point. [3]

4



The diagram shows the curve $y=\mathrm{f}(x)$. There are two vertical asymptotes with equations $x=-2$ and $x=2$ respectively. The curve crosses the x-axis at the point A and has a maximum turning point at B where it crosses the y-axis.
The curve also has a minimum turning point at C. The coordinates of A, B and C are $(a,\ 0)$, $(0,\ -10)$ and $(p,\ q)$ respectively, where a, p and q are constants.
Sketch the following curves and state the equations of the asymptotes, the coordinates of the turning points and of points where the curve crosses the axes, if any. Leave your answers in terms of a, p or q where necessary.

(i) $y=\dfrac{1}{\mathrm{f}(x)}$, and [3]

(ii) $y=\mathrm{f}(2-|x|)$. [3]

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The modulus of $\mathbf{a}$ is 2 and $\mathbf{b}$ is a unit vector. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$. Point C lies on AB, between A and B, such that $AC = kCB$, where $0 < k < 1$.

(i) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) Show that the length of projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is given by $\dfrac{k+4}{2(k+1)}$. [3]

(iii) Find, in terms of k, the area of triangle OAC. [3]

6 The Cartesian equation of line L_1 is $\dfrac{x-2}{a} = \dfrac{y+2}{b} = \dfrac{z-3}{c}$, where a, b, c are constants. The line L_2 is parallel to the vector $4\mathbf{i}+3\mathbf{j}$. The line L_3 passes through the origin and the point with position vector $\mathbf{j}+\mathbf{k}$.

(i) Given that L_1 is perpendicular to L_2, form an equation relating a and b. [1]

(ii) Given that L_1 intersects L_3, show that $5a + 2b - 2c = 0$. [3]

(iii) Hence express a and b in terms of c. [1]

(iv) Find the acute angle between L_1 and L_3. [2]

7 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1}{\left|1-x^2\right|},\ x \in \mathbb{R},\ \ -2 \le x < -1,$$

$$\mathrm{g}: x \mapsto -(x-2)^2 + k,\ x \in \mathbb{R},\ \ x \ge 0 \text{ where } k \text{ is a constant.}$$

(i) Sketch on the same diagram the graphs of

(a) $y = \mathrm{f}(x)$

(b) $y = \mathrm{f}^{-1}(x)$

(c) $y = \mathrm{f}^{-1}\mathrm{f}(x)$

stating the equations of any asymptotes and the coordinates of any endpoints. [3]

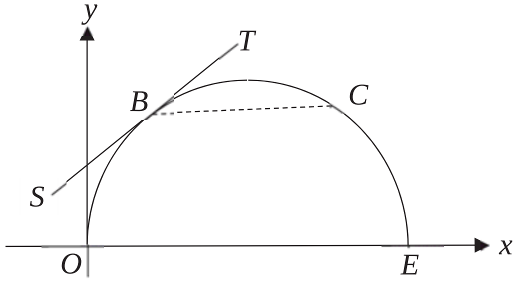
(ii) Find f^{-1} and state the domain of f^{-1}. [3]

(iii) Show that the composite function gf exists and find its range. [2]

8 The figure below shows a cross-section *OBCE* of a car headlight whose reflective surface is modelled in suitable units by the curve with parametric equations

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

for $0 \le \theta \le 2\pi$, where a is a positive constant.



(i) Find in terms of a

(a) the length of *OE*, [2]

(b) the maximum height of the curve *OBCE*. [1]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cot\dfrac{\theta}{2}$. [3]

Point *B* lies on the curve and has parameter β. *TS* is tangential to the curve at *B* and *BC* is parallel to the *x*-axis. Given that $\angle TBC = \dfrac{\pi}{6}$,

(iii) show that $\beta = \dfrac{2\pi}{3}$. [2]

(iv) Show that the equation of normal to the curve at the point *B* is

$$ky = -k^2 x + 2\pi a,$$

where k is an exact constant to be determined. [3]

9 **(a)** Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, find $\sum_{r=7}^{n+1}\left(2^r + r^2 - r\right)$ in terms of n. [4]

(b) **(i)** Use the method of differences to show that $\sum_{r=2}^{n} \dfrac{1}{r^2-1} = \dfrac{3}{4} + \dfrac{A}{n} + \dfrac{A}{n+1}$, where A is a constant to be determined. [3]

(ii) Explain why the series $\sum_{r=2}^{\infty} \dfrac{1}{r^2-1}$ converges, and write down its value. [2]

(iii) Hence deduce that $\dfrac{2}{2^2} + \dfrac{2}{3^2} + \dfrac{2}{4^2} + \ldots$ is less than $\dfrac{3}{2}$. [2]

10 Referred to the origin *O*, the points *A*, *B* and *C* have position vectors $4\mathbf{i} - 2\mathbf{j}$, $\alpha\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} - 7\mathbf{j} + \beta\mathbf{k}$ respectively, where α and β are constants.

(i) Given that *A*, *B* and *C* are collinear, show that $\alpha = 5$, and find the value of β. [3]

The plane π contains the line *L*, which has equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$. The plane π is also parallel to the line that passes through the points *A* and *B*.

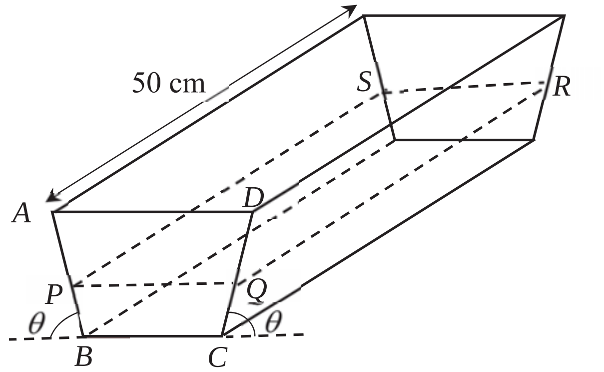
(ii) Find the shortest distance from point *A* to the line *L*. [2]

(iii) Show that the cartesian equation of the plane π is $x + y - z = 5$. [2]

(iv) Find the position vector of the foot of the perpendicular from point *A* to the plane π. [3]

(v) Hence find the reflection of the line that passes through points *A* and *B* about the plane π. [2]

11 The figure below shows a container with an open top. The uniform cross section *ABCD* of the container is a trapezium with *AB* = *BC* = *CD* = 10 cm. *AB* and *CD* are each inclined to the line *BC* at an acute angle of θ radians.
The length of the container is 50 cm and the container is placed on a horizontal table.



(i) Show that the volume *V* of the container is given by

$$V = 5000\left(\sin\theta\right)(1+\cos\theta)\ \text{cm}^3. \qquad [2]$$

Hence using differentiation, find the exact maximum value of *V*, proving that it is a maximum. [5]

(ii) For the remaining part of the question, θ is fixed at $\dfrac{\pi}{4}$.

Water fills the container at a rate of $100\,\text{cm}^3\text{s}^{-1}$. At time *t* seconds, the depth of the water is *h* cm. The surface of the water is a rectangle *PQRS*. When $h = 3$ cm, find the rate of change of

(a) the depth of the water, *h*, [3]

(b) the surface area of the water *PQRS*. [2]

12 Mrs Tan plans to start a business which requires a start-up capital of \$700,000. She decided to first save \$200,000 by depositing money every month into a savings plan. For the remaining \$500,000, she intends to take a loan from a finance company.
She deposited \$3000 into the savings plan in the first month and on the first day of each subsequent month, she deposited \$100 more than the previous month. Mrs Tan will continue depositing money into the savings plan until the total amount in her savings plan reaches \$200,000. It is given that this savings plan pays no interest.

(i) Find the month in which Mrs Tan's monthly deposit will exceed \$6,550. [2]

(ii) Find the number of months that it will take for Mrs Tan to save \$200,000 and hence find the amount that she would have deposited in the last month. [4]

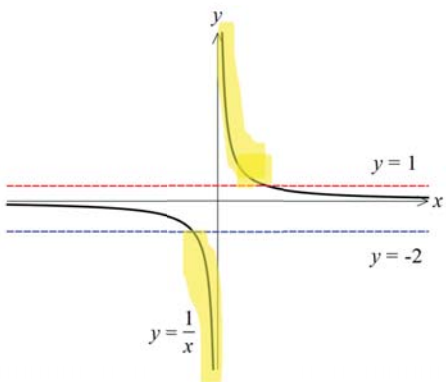
After Mrs Tan has saved \$200,000, she took a loan of \$500,000 from a finance company. To repay the loan from the finance company, Mrs Tan would have to pay a monthly payment of \$*x* at the beginning of each month, starting from the first month. An interest of 0.3% per month will be charged on the outstanding loan amount at the end of the month.

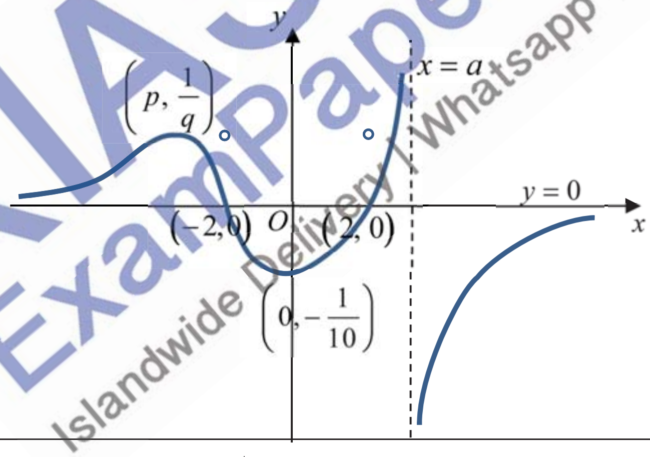
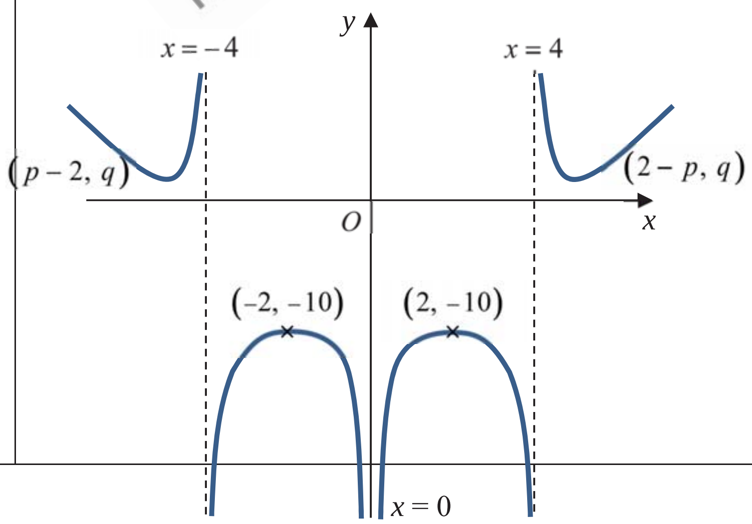
(iii) Show that the outstanding amount at the end of n^{th} month, after the interest has been charged, is $A\left(1.003^n\right) - Bx\left(1.003^n - 1\right)$, where *A* and *B* are exact constants to be determined. [3]

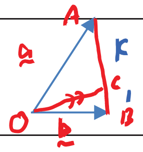
(iv) Find the amount of \$*x*, to 2 decimal places, if Mrs Tan wants to fully repay her loan in 8 years. [2]

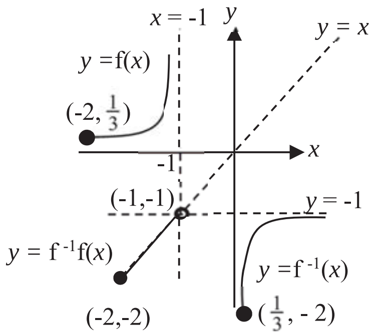
(v) Using the value of *x* found in part **(iv)**, calculate the total interest that the finance company will earn from Mrs Tan at the end of 8 years. [2]

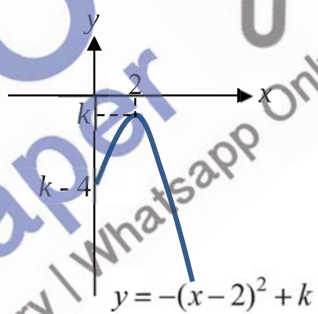
2021 ACJC H2 Math Promo Paper Solution

Qn	Solution
1	$y=2\sin(2x+\alpha)\cos(x) \xrightarrow{\text{replace } x \text{ by } x+\alpha} y=2\sin(2x+3\alpha)\cos(x+\alpha)$ $\xrightarrow{\text{replace } x \text{ by } 2x} y=2\sin(4x+3\alpha)\cos(2x+\alpha)$ $\xrightarrow{\text{replace } y \text{ by } -y} y=-2\sin(4x+3\alpha)\cos(2x+\alpha)$ 1) Translation of the graph by α units in the negative x-direction, followed by 2) Scaling parallel to $x-$axis by a factor of ½ . 3) Reflection in the $x-$axis.
OR	$y=2\sin(2x+\alpha)\cos(x) \xrightarrow{\text{replace } x \text{ by } 2x} y=2\sin(4x+\alpha)\cos(2x)$ $\xrightarrow{\text{replace } x \text{ by } x+\frac{\alpha}{2}} y=2\sin\left(4\left(x+\frac{\alpha}{2}\right)+\alpha\right)\cos\left(2\left(x+\frac{\alpha}{2}\right)\right)$ $=2\sin(4x+3\alpha)\cos(2x+\alpha)$ $\xrightarrow{\text{replace } y \text{ by } -y} y=-2\sin(4x+3\alpha)\cos(2x+\alpha)$ 1) Scaling parallel to $x-$axis by a factor of ½ , followed by 2) Translation of the graph by $\frac{\alpha}{2}$ units in the negative $x-$axis direction. 3) Reflection in the $x-$axis.
2	$\frac{x+3}{x^2+x-2}>-1$ $\frac{x+3}{x^2+x-2}+1>0$ $\frac{x^2+2x+1}{x^2+x-2}>0$ $\frac{(x+1)^2}{(x+2)(x-1)}>0$ Since $(x+1)^2\ge 0$ for $x\in\mathbb{R}$, $(x+2)(x-1)>0$. $\therefore x<-2$ or $x>1$. $\frac{x+3x^2}{1+x-2x^2}>-1$ $\frac{\frac{x+3x^2}{x^2}}{\frac{1+x-2x^2}{x^2}}>-1$ $\frac{\frac{1}{x}+3}{\left(\frac{1}{x}\right)^2+\left(\frac{1}{x}\right)-2}>-1$ Replace x by $\frac{1}{x}$, $\frac{1}{x}<-2$ or $\frac{1}{x}>1$ $\therefore -\frac{1}{2}<x<0$ or $0<x<1$ 

3	$2x^2-8y^2=x^2+xy^2+100$ $x^2-8y^2=xy^2+100$ $2x-16y\frac{dy}{dx}=2xy\frac{dy}{dx}+y^2$ $\frac{dy}{dx}=\frac{2x-y^2}{2xy+16y}$ (shown)
	Suppose $\frac{dy}{dx}=\frac{2x-y^2}{2xy+16y}=0$ $2x=y^2$ $\therefore \frac{x^2-8x}{x^2+2x^2+100}=\frac{1}{2}$ $2x^2-16x=3x^2+100$ $x^2+16x+100=0$ $(x+8)^2+36=0$ No solution since $(x+8)^2+36>0$ for $x\in\mathbb{R}$. [Or using discriminant: Since $16^2-4(1)(100)=-144<0$, there's no real roots.] $\therefore$ There is no stationary point since $\frac{dy}{dx}\neq 0$ for $x\in\mathbb{R}$.
4 (i)	
(ii)	 $y=f(x)$ ↓ replace x by $x+2$ $y=f(x+2)$ ↓ replace x by $-x$ $y=f(2-x)$ ↓ replace x by $\lvert x\rvert$ $y=f(2-\lvert x\rvert)$

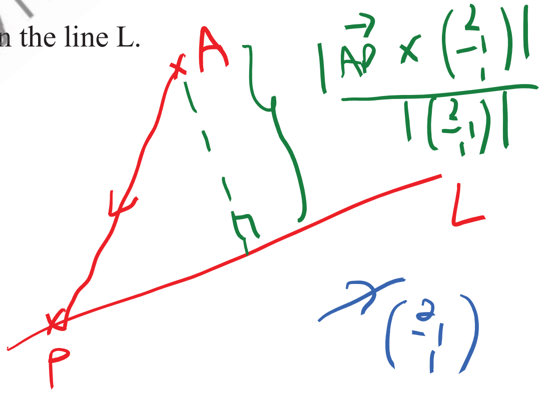
5(i)	Using Ratio Theorem $\overrightarrow{OC}=\frac{1}{k+1}(\mathbf{a}+k\mathbf{b})$ 
(ii)	Length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OA}$ $=\frac{\lvert\overrightarrow{OC}\bullet\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert}=\frac{\left\lvert\frac{1}{k+1}(\mathbf{a}+k\mathbf{b})\bullet\mathbf{a}\right\rvert}{2}$ $=\frac{1}{2(k+1)}\lvert(\mathbf{a}+k\mathbf{b})\bullet\mathbf{a}\rvert$ since $0<k<1$ $=\frac{1}{2(k+1)}\lvert(\mathbf{a}\bullet\mathbf{a}+k\mathbf{b}\bullet\mathbf{a})\rvert$ $=\frac{1}{2(k+1)}\left\lvert\lvert\mathbf{a}\rvert^2+k\mathbf{b}\bullet\mathbf{a}\right\rvert$ $=\frac{1}{2(k+1)}\left\lvert 4+k\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos(60^o)\right\rvert$ $=\frac{1}{2(k+1)}\left\lvert 4+k(2)(1)\left(\frac{1}{2}\right)\right\rvert=\frac{4+k}{2(k+1)}$ (*proved*)
(iii)	Area of triangle *OAC* $=\frac{1}{2}\lvert\mathbf{a}\times\mathbf{c}\rvert=\frac{1}{2}\left\lvert\mathbf{a}\times\frac{1}{k+1}(\mathbf{a}+k\mathbf{b})\right\rvert$ $=\frac{1}{2(k+1)}\lvert(\mathbf{a}\times\mathbf{a})+k(\mathbf{a}\times\mathbf{b})\rvert$ since $0<k<1$ $=\frac{1}{2(k+1)}\lvert(k(\mathbf{a}\times\mathbf{b})\rvert$ since $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ $=\frac{k}{2(k+1)}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin(60^o)\rvert=\frac{k}{2(k+1)}\left\lvert 2(1)\frac{\sqrt{3}}{2}\right\rvert=\frac{\sqrt{3}k}{2(k+1)}$
6 **(i)**	$L_1:\frac{x-2}{a}=\frac{y+2}{b}=\frac{z-3}{c}$ $\mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}a\\b\\c\end{pmatrix}$ --------(1) L_1 is perpendicular to L_2, $\begin{pmatrix}4\\3\\0\end{pmatrix}\bullet\begin{pmatrix}a\\b\\c\end{pmatrix}=0$ $4a+3b=0$
(ii)	Equation of line L_3 : $\mathbf{r}=\mu\begin{pmatrix}0\\1\\1\end{pmatrix}$ ---------(2) Since L_1 intersects L_3, sub (1) into (2):

	$2+\lambda a=0 \quad \Rightarrow \lambda=-\dfrac{2}{a}$ ------(3) $-2+\lambda b=\mu$ ------(4) $3+\lambda c=\mu$ ------(5) Sub (4) into (5): $-2+\lambda b=3+\lambda c$ --------(6) Sub (3) into (6): $-2+\left(\dfrac{-2}{a}\right)b=3+\left(\dfrac{-2}{a}\right)c$ $-2a-2b=3a-2c$ $5a+2b-2c=0$ (Shown)
(iii)	Using results in (i) & (ii), use GC to solve: $4a+3b+0c=0$ $5a+2b-2c=0$ $a=\dfrac{6}{7}c$ $b=-\dfrac{8}{7}c$
(iv)	Using result in (iii), $L_1: \quad \mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}\frac{6}{7}c\\-\frac{8}{7}c\\c\end{pmatrix}$ $L_1: \quad \mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\left(\dfrac{1}{7}\right)\begin{pmatrix}6c\\-8c\\7c\end{pmatrix}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\mu\begin{pmatrix}6\\-8\\7\end{pmatrix}$ Angle between L_1 and L_3: $\cos\theta=\left\|\dfrac{\begin{pmatrix}0\\1\\1\end{pmatrix}\bullet\begin{pmatrix}6\\-8\\7\end{pmatrix}}{\sqrt{2}\sqrt{6^2+8^2+7^2}}\right\|=\left\|\dfrac{-1}{\sqrt{2}\sqrt{149}}\right\|$ $\theta=86.7^0$ Acute angle between the two planes is 86.7^0
7 (i)	 Note that $D_{f^{-1}f}=D_f=[-2,-1)$

(ii)	Considering the interval $-2 \le x < -1$, $\dfrac{1}{\lvert 1-x^2 \rvert} = -\dfrac{1}{1-x^2}$ $y = -\dfrac{1}{1-x^2}$ $y = \dfrac{1}{x^2-1}$ $yx^2 - y = 1$ $x^2 = \dfrac{1+y}{y}$ $x = \pm\sqrt{\dfrac{1+y}{y}}$ $x = -\sqrt{\dfrac{1+y}{y}}$ (since $-2 \le x < -1$) $f^{-1}(x) = -\sqrt{\dfrac{1+x}{x}} = -\sqrt{\dfrac{1}{x}+1}$ $D_{f^{-1}} = R_f = [\dfrac{1}{3}, \infty)$
(iii)	Since $R_f = [\frac{1}{3}, \infty) \subseteq [0, \infty) = D_g$ Hence gf exists.  $[-2,-1) \xrightarrow{f} [\frac{1}{3}, \infty) \xrightarrow{g} (-\infty, k]$ $R_{gf} = (-\infty, k]$
8 **(i) (a)**	At E, $y = a(1-\cos\theta) = 0$. Hence $\cos\theta = 1$ $\therefore \theta = 2\pi$ $\therefore x = 2a\pi$ Hence $OE = 2a\pi$
(b)	When y is a maximum, $\cos\theta = -1$ OR $\dfrac{dy}{d\theta} = a(\sin\theta) = 0$ $\therefore \theta = \pi$ and $y = 2a$
(ii)	$\dfrac{dy}{d\theta} = a(\sin\theta) = 2a\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ and $\dfrac{dx}{d\theta} = a(1-\cos\theta) = a(1-1+2\sin^2\theta) = 2a\sin^2\dfrac{\theta}{2}$

	$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2}$
(iii)	At B, $\frac{dy}{dx}=\cot\frac{\beta}{2}=\frac{1}{\tan\frac{\beta}{2}}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$ Hence $\tan\frac{\beta}{2}=\sqrt{3}$. $\frac{\beta}{2}=\frac{\pi}{3}$ $\beta=\frac{2\pi}{3}(shown)$
(iv)	Since $\frac{dy}{dx}=\cot\frac{\theta}{2}$ Gradient of normal at point B is $-\tan\frac{\pi}{3}=-\sqrt{3}$. Equation of normal : $y-\frac{3}{2}a=-\sqrt{3}\left(x-\left[a\left(\frac{2\pi}{3}-\sin\frac{2\pi}{3}\right)\right]\right)$ $y-\frac{3}{2}a=-\sqrt{3}\left(x-\left[a\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\right]\right)$ $y=-\sqrt{3}\left(x-\frac{2\pi a}{3}+\frac{a\sqrt{3}}{2}\right)+\frac{3}{2}a$ $y=-\sqrt{3}x+\frac{2\pi a}{\sqrt{3}}$ $\sqrt{3}y=-3x+2\pi a$ $\sqrt{3}y=-\left(\sqrt{3}\right)^2x+2\pi a$
9(a)	$\sum_{r=7}^{n+1}\left(2^r+r^2-r\right)$ $=\sum_{r=7}^{n+1}\left(2^r\right)+\sum_{r=7}^{n+1}\left(r^2\right)-\sum_{r=7}^{n+1}\left(r\right)$ $=\frac{2^7\left(2^{n-5}-1\right)}{2-1}+\sum_{r=1}^{n+1}\left(r^2\right)-\sum_{r=1}^{6}\left(r^2\right)-\left(\frac{n-5}{2}\right)\left(7+n+1\right)$ $=2^7\left(2^{n-5}-1\right)+\frac{(n+1)}{6}(n+2)(2n+3)-\left(\frac{6}{6}\right)(7)(13)-\left(\frac{n-5}{2}\right)(8+n)$ $=2^7\left(2^{n-5}-1\right)+\frac{(n+1)}{6}(n+2)(2n+3)-91-\frac{(n-5)(8+n)}{2}$ Alternative Method:

	$\sum_{r=7}^{n+1}\left(2^r+r^2-r\right)$ $=\sum_{r=1}^{n+1}\left(2^r+r^2-r\right)-\sum_{r=1}^{6}\left(2^r+r^2-r\right)$ $=\frac{2\left(2^{n+1}-1\right)}{2-1}+\frac{1}{6}(n+1)(n+2)(2n+3)-\frac{n+1}{2}(1+n+1)$ $-\frac{2\left(2^6-1\right)}{2-1}-91+\frac{6}{2}(1+6)$ $=2^{n+2}+\frac{1}{6}(n+1)(n+2)(2n+3)-\frac{(n+1)(n+2)}{2}-198$ $=2^{n+2}+\left[\frac{(n+1)(n+2)}{6}\right](2n+3-3)-198$ $=2^{n+2}+\frac{1}{3}n(n+1)(n+2)-198$
9(b)(i)	Using partial fractions, $\frac{1}{r^2-1}=\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$ $\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$ $=\frac{1}{2}\begin{bmatrix}\frac{1}{1} & -\frac{1}{3}\\ +\frac{1}{2} & -\frac{1}{4}\\ +\frac{1}{3} & -\frac{1}{5}\\ +\frac{1}{4} & -\frac{1}{6}\\ \vdots & \\ +\frac{1}{n-3} & -\frac{1}{n-1}\\ +\frac{1}{n-2} & -\frac{1}{n}\\ +\frac{1}{n-1} & -\frac{1}{n+1}\end{bmatrix}$ $=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$ $=\frac{3}{4}+\frac{-\frac{1}{2}}{n}+\frac{-\frac{1}{2}}{n+1}$ $\therefore A=-\frac{1}{2}.$

(ii)	As $n \to \infty, \frac{1}{n} \to 0, \frac{1}{n+1} \to 0$, therefore $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$ converges. $\sum_{r=2}^{\infty} \frac{1}{r^2-1} = \frac{3}{4}$.
(iii)	$\frac{2}{2^2}+\frac{2}{3^2}+\frac{2}{4^2}+\ldots=\sum_{r=2}^{\infty}\frac{2}{r^2}$ Since $r^2-1<r^2$, $\therefore \frac{2}{r^2-1} > \frac{2}{r^2}$. $\sum_{r=2}^{\infty}\frac{2}{r^2-1} > \sum_{r=2}^{\infty}\frac{2}{r^2}$ $\sum_{r=2}^{\infty}\frac{2}{r^2} < 2\left(\frac{3}{4}\right)$ $\therefore \frac{2}{2^2}+\frac{2}{3^2}+\frac{2}{4^2}+\ldots<\frac{3}{2}$ (shown)
10(i)	$\overrightarrow{OA}=\begin{pmatrix}4\\-2\\0\end{pmatrix} \quad \overrightarrow{OB}=\begin{pmatrix}\alpha\\-1\\2\end{pmatrix} \quad \overrightarrow{OC}=\begin{pmatrix}-1\\-7\\\beta\end{pmatrix}$ Since A, B and C are collinear, $\overrightarrow{AB}=k\overrightarrow{AC}$ $\begin{pmatrix}\alpha-4\\1\\2\end{pmatrix}=k\begin{pmatrix}-5\\-5\\\beta\end{pmatrix} \Rightarrow \begin{cases}\alpha-4=-5k\\1=-5k\\2=k\beta\end{cases} \Rightarrow \begin{cases}k=-\frac{1}{5}\\\alpha=5\\\beta=-10\end{cases}$
(ii)	$\overrightarrow{OP}=\begin{pmatrix}2\\3\\0\end{pmatrix}$ is position vector of a point on the line L. $\overrightarrow{AP}=\begin{pmatrix}2\\3\\0\end{pmatrix}-\begin{pmatrix}4\\-2\\0\end{pmatrix}=\begin{pmatrix}-2\\5\\0\end{pmatrix}$ Distance from A to L $=\left\|\overrightarrow{AP}\times\frac{1}{\sqrt{4+1+1}}\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\|$ $=\frac{1}{\sqrt{6}}\left\|\begin{pmatrix}-2\\5\\0\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\|$ $=\frac{1}{\sqrt{6}}\left\|\begin{pmatrix}5\\2\\-8\end{pmatrix}\right\|=\frac{1}{\sqrt{6}}\sqrt{25+4+64}=\sqrt{\frac{93}{6}}=\sqrt{\frac{31}{2}}$ 

(iii)	Normal of plane $\pi = \overrightarrow{AB} \times \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\-3\end{pmatrix} = 3\begin{pmatrix}1\\1\\-1\end{pmatrix}$ Equation of plane π: $\mathbf{r} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\3\\0\end{pmatrix} \bullet \begin{pmatrix}1\\1\\-1\end{pmatrix} = 2+3=5$ Cartesian equation is $x+y-z = 5$
(iv)	Let F be the foot of perpendicular from $A(4,-2,0)$ to the plane π Equation of line AF : $\mathbf{r} = \begin{pmatrix}4\\-2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}4+\lambda\\-2+\lambda\\-\lambda\end{pmatrix}$ To find the point of intersection of line AF and plane π, substitute equation of line into equation of plane $x+y-z = 5$, $4+\lambda-2+\lambda+\lambda=5 \quad \Rightarrow \quad 3\lambda+2=5 \Rightarrow \lambda=1$ $\therefore \overrightarrow{OF} = \begin{pmatrix}4\\-2\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}5\\-1\\-1\end{pmatrix}$
	Let the reflection of A about plane π be A'(x,y,z) $\overrightarrow{AF} = \overrightarrow{FA'}$ $\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$ $\overrightarrow{OA'} = 2\begin{pmatrix}5\\-1\\-1\end{pmatrix} - \begin{pmatrix}4\\-2\\0\end{pmatrix} = \begin{pmatrix}6\\0\\-2\end{pmatrix}$ Alternatively: $\frac{4+x}{2} = 5 \Rightarrow x = 6$ $\frac{-2+y}{2} = -1 \Rightarrow y = 0$ $\frac{0+z}{2} = -1 \Rightarrow z = -2$ Since the line AB is parallel to π, then the reflected line about π will also be parallel to π, i.e. also parallel to the line AB. Equation of reflected line is: $\mathbf{r} = \begin{pmatrix}6\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\2\end{pmatrix}, \lambda \in \mathbb{R}$
11(i) (a)	$A = \frac{1}{2}(10\sin\theta)(10+10+2(10\cos\theta)) = \frac{1}{2}(10\sin\theta)(20+20\cos\theta) = (100\sin\theta)(1+\cos\theta)$ $V = (50)(100\sin\theta)(1+\cos\theta) = (5000\sin\theta)(1+\cos\theta)$

(b)

$\frac{dV}{d\theta} = (5000\cos\theta)(1+\cos\theta) + (5000\sin\theta)(-\sin\theta)$

$= 5000(\cos\theta + \cos^2\theta - \sin^2\theta)$

$= 5000(2\cos^2\theta + \cos\theta - 1)$

$\frac{dV}{d\theta} = 0$

$(2\cos^2\theta + \cos\theta - 1) = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

Since θ is acute $\cos\theta \neq -1$

$\therefore \theta = \frac{\pi}{3}$

$\frac{d^2V}{d\theta^2} = 5000(-4\cos\theta\sin\theta - \sin\theta) = 5000(-\frac{3\sqrt{3}}{2}) \approx -12990 < 0$ when $\theta = \frac{\pi}{3}$

V is a maximum when $\theta = \frac{\pi}{3}$

Max $V = (5000\frac{\sqrt{3}}{2})(1+\frac{1}{2}) = \frac{15000\sqrt{3}}{4}$

Maximum volume is $\frac{15000\sqrt{3}}{4}$ cm^3 = $3750\sqrt{3}$ cm^3.

(ii) (a)

Volume of water = $V = [\frac{1}{2}h(20 + 2h\tan\frac{\pi}{4})]50$

$V = [h(10+h)]50 = 500h + 50h^2$

$\frac{dV}{dh} = 500 + 100h$.

When $h = 3$ cm, $\frac{dV}{dh} = 800$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{100}{800}$ cm s^{-1} = $\frac{1}{8}$ cm s^{-1} = 0.125 cm s^{-1}

(b)

When the depth of the water is h cm,

area of water surface = $y = (10 + 2h\tan\frac{\pi}{4})(50) = 500 + 100h$

$\frac{dy}{dt} = 100\frac{dh}{dt} = \frac{100}{8}$ cm^2 s^{-1} = 12.5 cm^{2}s^{-1}

12(i)

$U_n > 6550$

$3000 + (n-1)100 > 6550$

$n > 36.5$

$\therefore 37^{\text{th}}$ month

12(ii)

$S_n = \frac{n}{2}\left[6000 + (n-1)100\right]$

$\frac{n}{2}\left[6000 + (n-1)100\right] \geq 200,000$

Method 1

By GC,

When $n = 40, y = 198,000 < 200,000$
When $n = 41, y = 205,000 > 200,000$

<u>Method 2</u>

$\frac{n}{2}\left[6000+(n-1)100\right] \geq 200,000$

$100n^2 + 5900n - 400,000 \geq 0$

$(n-40.287)(n+99.287) \geq 0$

$n \leq -99.3$ (rej) or $n \geq 40.3$

$\therefore$ 41 months

$S_{40} = \frac{40}{2}\left[6000+(40-1)100\right] = \$198,000$

$\$200,000 - \$198,000 = \$2000$

(iii)

n	**End of the month**
1	$1.003(500000-x)$
2	$1.003^2(500000-x)-(1.003)x$
3	$1.003^3(500000-x)-(1.003)^2x-(1.003)x$
⋮	⋮
n	

At the end of the *n*th month, the outstanding amount would be

$1.003^n(500000-x)-(1.003)^{n-1}x-...-(1.003)x$

$=1.003^n(500000)-(1.003^n)x-...-(1.003)x$

$=1.003^n(500000)-x\left[1.003+1.003^2+...+1.003^n\right]$

$=1.003^n(500000)-x\left[\frac{1.003(1.003^n-1)}{1.003-1}\right]$

$=1.003^n(500000)-\frac{1003}{3}x(1.003^n-1)$

$\therefore A = 500000, B = \frac{1003}{3}$

(iv)

$1.003^{96}(500000)-\frac{1003}{3}x(1.003^{96}-1)=0$

Using GC, $x = \$5984.09$.

(v)

Total paid: $\$5984.09 \times 12 \times 8 = \$574,472.64$

Interest: $\$574,472.64 - \$500,000 = \$74,472.64$

ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Promotional Exam Paper (100 marks) **1 Oct 2021**

3 hours

Additional Material(s): List of Formulae (MF26)

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1 It is given that $y^2 = \sin x + \cos x$. Show that $y\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} + A\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, where A is a real constant to be determined. [4]

2 A graph with the equation $y=\dfrac{3}{x^2+2}$ undergoes, in succession, the following transformations:

A: A translation of 1 unit in the direction of the negative x-axis.

B: A scaling parallel to the x-axis by a scale factor of $\dfrac{1}{3}$.

C: A reflection in the y-axis.

Determine the equation of the resulting curve. [4]

3 **(a)** Find $\int x^2 \tan^{-1}(2x)\ \mathrm{d}x$. [3]

(b) Find $\int \frac{x}{\sqrt{8-2x-x^2}}\ \mathrm{d}x$. [3]

4 A curve C has equation

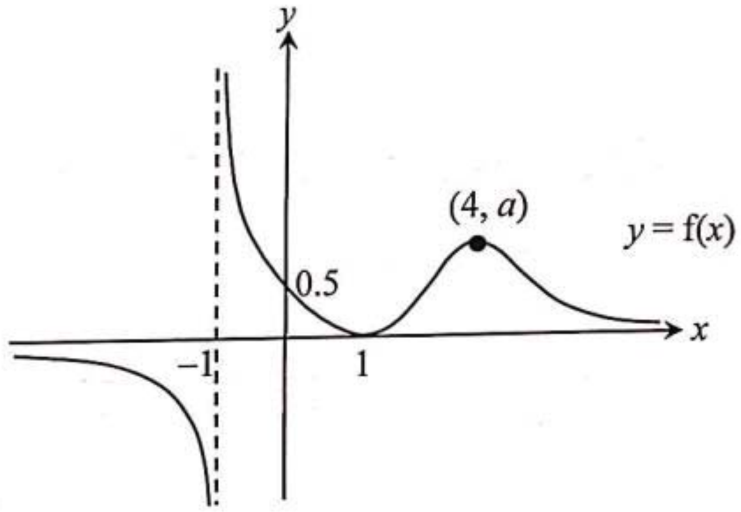
$$y^3 - 2xy^2 + 3x^2 - 3 = 0.$$

(i) Find $\frac{dy}{dx}$ in terms of x and y. [2]

(ii) Find the equation of the normal to the curve at the point $P(2, 3)$. [2]

(iii) Given that C meets the y-axis at the point R and the normal in **(ii)** meets the y-axis at the point A, find the area of triangle APR in the form $a-\sqrt[3]{b}$, where a and b are integers to be determined. [3]

5



The diagram above shows the graph of $y = \mathrm{f}(x)$. It has a maximum point at $(4, a)$, where $a > 0$, and meets the axes at $(1, 0)$ and $(0, 0.5)$. The curve has asymptotes with equations $y = 0$ and $x = -1$.

On separate diagrams, sketch the graphs of

(a) $y = \dfrac{1}{\mathrm{f}(x)}$; [3]

(b) $y = \mathrm{f}'(x)$, [3]

stating the equation(s) of any asymptotes and where possible, the coordinates of any turning point(s) and axial intercept(s).

(a)

(b)

6 The functions f and g are defined by

$$f : x \mapsto 6 + \lambda x - x^2, \quad x \in \mathbb{R},$$

$$g : x \mapsto 1 + 7e^{-x}, \quad x \geq 0.$$

(i) For this part of the question, it is given that $\lambda = 5$. Show that the composite function fg exists. Hence find the range of fg. [3]

(ii) It is now given that λ is a real constant that is not necessary equals to 5. If the domain of f is restricted to $x \leq \frac{\lambda}{2}$, find f^{-1} in a similar form. [4]

7 (a) Find $\frac{\mathrm{d}}{\mathrm{dx}}(\tan^3 x)$. [1]

Hence find $\int \sec^4 x \; \mathrm{d}x$. [2]

(b) Find $\int \sin x \; (\sin x + \sin 3x) \, \mathrm{d}x$. [3]

(c) For $p > 2$, find the value of $\int_0^p x\,|2 - x|\,\mathrm{dx}$ in terms of p. [3]

8 Relative to origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. The point C is on BA produced such that $BA : BC = 3 : 5$ and OC is perpendicular to OB.

(i) Find $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) Show that $\mathbf{a} \cdot \mathbf{b} = \frac{2}{5}|\mathbf{b}|^2$. [2]

The point P is on the line OB such that it is the image of B in the line OC.

(iii) Find the area of triangle PCB. Leave your answer in the form of $k|\mathbf{a} \times \mathbf{b}|$, where k is an exact real constant. [3]

The point F is the foot of perpendicular of P to the line AB.

(iv) Given that $|\mathbf{a}|^2 = \frac{7}{25}|\mathbf{b}|^2$, find the position vector of F in terms of **a** and **b**. [4]

9 The curve C has equation

$$y = \frac{4x^2 + px - q}{x^2 - s}, \quad x \in \mathbb{R}, \qquad x^2 \neq s$$

where $p,\ q\ and$ s are non-zero constants.

(a) It is given that C passes through the point $\left(0, \frac{1}{2}\right)$ and has a vertical asymptote $x = 2$.

(i) State the value of s and show that the value of q is 2. [2]

(ii) It is given further that the line $y = 1$ is a tangent to C and it does not meet the curve again. Find the exact value of p if p is a negative real value. [3]

(b) It is now given instead that $p = 4$, $q = -1$ and $s = 1$.

(i) Sketch the curve C, showing clearly the coordinates of any turning point(s), equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(ii) Find the equation of the additional curve that needs to be added to the curve sketched in (b)(i) to determine the number of distinct real roots for the equation

$$10(x+2)^2 = 3\left(10 - \left(\frac{(4x^2+4x+1)}{(x^2-1)}\right)^2\right).$$

[2]

10 (a)

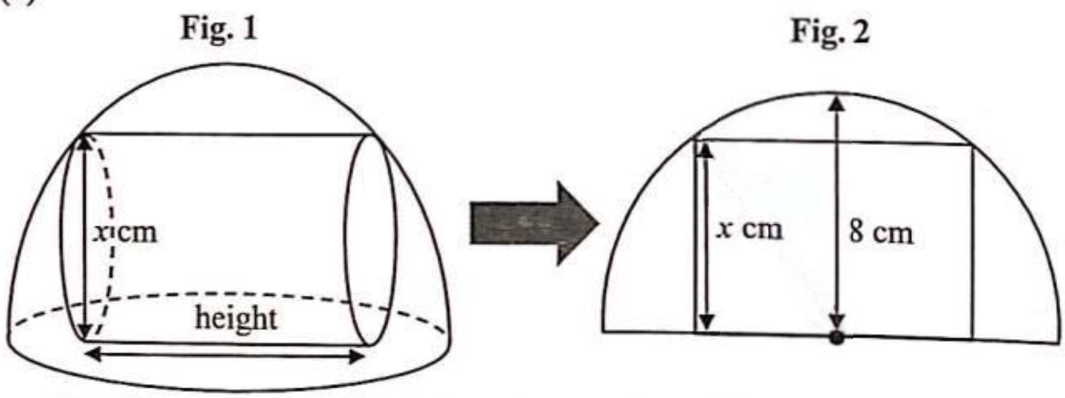


Fig. 1 shows the cylindrical-shaped water pipe, with negligible thickness and open on both ends, inscribed in a hemisphere with fixed radius 8 cm. The cross sectional view of the pipe and the hemisphere is shown in Fig. 2.

(i) If the diameter of the pipe is x cm, show that the curved surface area, S, of the pipe is $2\pi x\sqrt{64-x^2}$ cm^2. [2]

(ii) It is given that as x varies, the maximum value of S occurs when the ratio of the diameter of the pipe to its height is $1 : k$. Find the exact value of k and the exact maximum value of S. [6]

(b) A particle is moving on the curve with equation

$$y = \sin^{-1}(3x) \ , \quad -\frac{1}{3} \leq x \leq \frac{1}{3} \ ,$$

where $(x \, , y)$ is the coordinate of the particle at time t relative to a fixed point O. The x and y values represent the horizontal and vertical displacements.
When the y −coordinate of the particle is $\frac{\pi}{3}$, the rate at which the y −coordinate is decreasing with time t is 2 units per second. At this instant, find the exact rate at which the x −coordinate of the particle changes with time. [4]

11 (a) Ivy took a \$40 000 tuition fee loan for her 4-year university course that commences on 1st January 2021. The loan is interest-free during the period of study. Immediately after graduation, interest is charged at 4% per annum of the outstanding amount owe at the end of each year. The maximum loan repayment period is at most 15 years upon graduation. Ivy is planning to pay \$550 every month upon graduation.

(i) Show that the amount she owes at the end of the n years after graduation is $\$171600 - 131600(1.04)^n$. [3]

(ii) Will she be able to finish repaying the loan by the end of 2030? Justify your answer clearly. [2]

(iii) Find the minimum monthly repayment Ivy should make if she intends to utilize fully the loan repayment period. [2]

(b) To save for her tuition fee loan repayment, Ivy wishes to start a new savings plan on the first day of November 2021. In this plan, she needs to invest \$200 into the account on the first day of each month. Every \$200 invested earns a fixed interest of $d\%$ of \$200 at the end of each month until a withdrawal is made from the account. The interest is added to the account and does not accumulate further interest.

(i) How much interest, in terms of d, will the first \$200 deposited earn at the end of 2022? [2]

(ii) Find the least value of d such that the total amount in the account exceed \$10 000 at the end of 36 months. [3]

12 Relative to the origin O, a point A has position vector $-\mathbf{j}+2\mathbf{k}$. The plane p_1 has equation $\mathbf{r}.\begin{pmatrix}1\\-1\\0\end{pmatrix}=3$.

(i) Find the position vector of the foot of perpendicular from point A to the plane p_1. [4]

The line l has equation $\mathbf{r}=\begin{pmatrix}0\\-1\\2\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}, \mu\in\mathbb{R}$.

(ii) Find the acute angle between the plane p_1 and the line l. [2]

(iii) The point $B(-\alpha, 2, \alpha)$ is equidistant from the plane p_1 and the line l. Find the possible values of α. [4]

The plane p_2 has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 5 - \beta$, $\beta \neq 8$.

(iv) Show that the point $C(4, 1, \beta)$ lies on both p_1 and p_2. Hence find the vector equation of the line of intersection between the planes p_1 and p_2. [3]

1	It is given that $y^2 = \sin x + \cos x$. Show that $y\frac{d^3y}{dx^3} + A\frac{dy}{dx}\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$, where A is a real constant to be determined.	[4]
	Solution	
	$y^2 = \sin x + \cos x$ $2y\frac{dy}{dx} = \cos x - \sin x$	
	$2\left[\frac{dy}{dx}\cdot\frac{dy}{dx} + y\cdot\frac{d^2y}{dx^2}\right] = -\sin x - \cos x$ (product rule on LHS)	
	$\therefore\ 2\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] = -y^2$	
	$2\left[2\left(\frac{dy}{dx}\right)^1\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3}\right] = -2y\frac{dy}{dx}$	
	$y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$ Therefore $A = 3$.	
2	A graph with the equation $y = \frac{3}{x^2+2}$ undergoes, in succession, the following transformations: A: A translation of 1 unit in the direction of the negative x-axis. B: A scaling parallel to the x-axis by a scale factor of $\frac{1}{3}$. C: A reflection in the y-axis. Determine the equation of the resulting curve.	[4]
	Solution	
	$y = f(x) = \frac{3}{x^2+2}$ $\downarrow$ after A (replace x by $x+1$) $y = f(x+1)$	
	$\downarrow$ after B (replace x by $3x$) $y = f(3x+1)$	
	$\downarrow$ after C (replace x by $-x$) $y = f(-3x+1)$	
	$y = f(-3x+1) = \frac{3}{(-3x+1)^2+2}$	
	$= \frac{3}{9x^2-6x+3} \Rightarrow y = \frac{1}{3x^2-2x+1}$	

Commented [LMH1]: Things to note
When the question requires you to obtain the result in such a manner, it tells you that your approach must be to apply **implicit differentiation.**

However, several students ended with **highly complicated expressions** because they insisted in making dy/dx, d^2y/dx^2, ... the subject, which was totally unnecessary.

Commented [LMH2]: Misconception
When differentiating $y\frac{dy}{dx}$, a segment of students appear oblivious to the need for **product rule**, and they wrote $\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$ as the result.
Also, some students didn't realise $\left(\frac{dy}{dx}\right)^2 \neq \frac{d^2y}{dx^2}$.

Commented [SH3]: Carelessness / Conceptual Understanding
1. Reflection in the y-axis implies that the line of reflection is y-axis which implies that every x will be replaced with $-x$.

Commented [SH4]: Presentation
Answers must be simplified. Majority were penalized for not simplifying the expression

3	**(a)** Find $\int x^2 \tan^{-1}(2x)\,dx$.	[3]
	(b) Find $\int \frac{x}{\sqrt{8-2x-x^2}}\,dx$.	[3]
	Solution	
	(a) $\int x^2 \tan^{-1}(2x)\,dx = \frac{x^3}{3}\tan^{-1}(2x) - \int \left(\frac{x^3}{3}\right)\left(\frac{2}{1+4x^2}\right) dx$	
	$= \frac{x^3}{3}\tan^{-1}(2x) - \frac{2}{3}\int \left(\frac{1}{4}x - \frac{x}{4(1+4x^2)}\right) dx$	
	$= \frac{x^3}{3}\tan^{-1}(2x) - \frac{1}{12}x^2 + \frac{1}{6}\int \frac{x}{1+4x^2}\,dx$	
	$= \frac{x^3}{3}\tan^{-1}(2x) - \frac{1}{12}x^2 + \frac{1}{48}\int \frac{8x}{1+4x^2}\,dx$	
	$= \frac{x^3}{3}\tan^{-1}(2x) - \frac{1}{12}x^2 + \frac{1}{48}\ln\left(1+4x^2\right) + c$	
	(b) $\int \frac{x}{\sqrt{8-2x-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x-2}{\sqrt{8-2x-x^2}}\,dx - \int \frac{1}{\sqrt{8-2x-x^2}}\,dx$	
	$= -\sqrt{8-2x-x^2} - \int \frac{1}{\sqrt{9-(x+1)^2}}\,dx$	
	$= -\sqrt{8-2x-x^2} - \sin^{-1}\left(\frac{x+1}{3}\right) + c$	
4	A curve C has equation $$y^3 - 2xy^2 + 3x^2 - 3 = 0.$$	
	(i) Find $\frac{dy}{dx}$ in terms of x and y.	[2]
	(ii) Find the equation of the normal to the curve at the point $P(2, 3)$.	[2]
	(iii) Given that C meets the y-axis at the point R and the normal in **(ii)** meets the y-axis at the point A, find the area of triangle APR in the form $a - \sqrt[3]{b}$, where a and b are integers to be determined.	[3]
	Solution	
	(i) $y^3 - 2xy^2 + 3x^2 - 3 = 0$	
	Differentiate with respect to x: $3y^2\frac{dy}{dx} - (2y^2 + 4xy\frac{dy}{dx}) + 6x = 0$	

Commented [CKJ5]: Common Mistakes
1. Some students did not know how to apply integration by parts

2. Some students did not know how to identify the part to be differentiated / integrated. Students could use **LIATE** as a guide in deciding the part to be differentiated.

Commented [CKJ6]: Common Mistakes
Many students were not able to do long division correctly.

Commented [CKJ7]: Common Mistakes
Some students did not know how to introduce (-2x-2) into the integral.

Commented [CKJ8]: Common Mistakes
Some students complete the square incorrect.

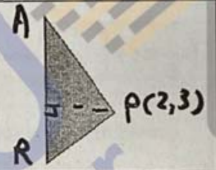
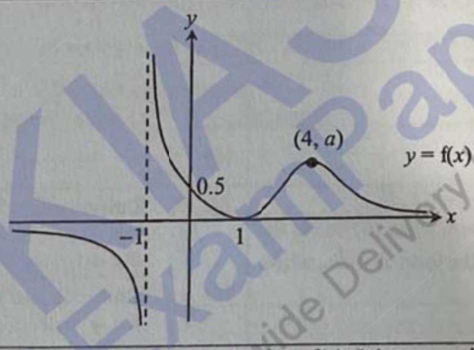
Commented [CKJ9]: Question Reading
Some students gave the equation of the tangent instead of the equation of the normal.

Commented [CKJ10]: Things to note
Many students did not find the gradient of the normal using the point $P(2, 3)$.

Commented [CKJ11]: Question Reading
Many students did not give the values of a and b as required in the question.

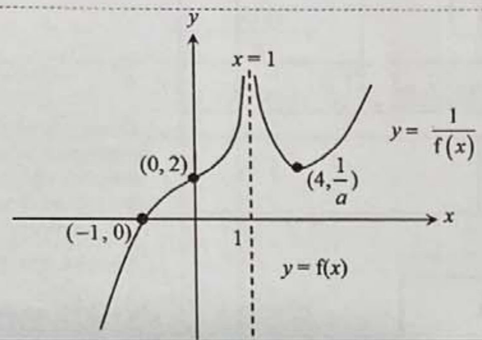
Commented [CKJ12]: Common Mistake
Many students did not put the **brackets** while performing the product rule, resulting the whole expression to be wrong.

[Turn Over

	$(3y^2-4xy)\frac{dy}{dx}=2y^2-6x$ $\frac{dy}{dx}=\frac{2y^2-6x}{3y^2-4xy}$	
	(ii) Gradient of tangent to curve at $P(2, 3)$ $=\frac{2(3)^2-6(2)}{3(3)^2-4(2)(3)}=2$ Gradient of normal to curve at $P=-\frac{1}{2}$	
	Equation of normal at P: $y-3=-\frac{1}{2}(x-2)$ $\Rightarrow y=-\frac{1}{2}x+4$	
	(iii) When $x=0$, normal cuts y-axis at $A(0, 4)$; C: $y^3-2xy^2+3x^2-3=0$ When $x=0$, $y^3=3\Rightarrow y=\sqrt[3]{3}$ C meets y-axis at $R(0, \sqrt[3]{3})$.	
	Area of triangle APR $=\frac{1}{2}\times 2\times(4-\sqrt[3]{3})$ $=4-\sqrt[3]{3}$ where $a=4$, $b=3$ 	
5		
	The diagram above shows the graph of $y=f(x)$. It has a maximum point at $(4, a)$, where $a>0$, and meets the axes at $(1, 0)$ and $(0, 0.5)$. The curve has asymptotes with equations $y=0$ and $x=-1$.	
	On separate diagrams, sketch the graphs of	
	(a) $y=\frac{1}{f(x)}$;	[3]
	(b) $y=f'(x)$,	[3]
	stating the equation(s) of any asymptotes and where possible, the coordinates of any turning point(s) and axial intercept(s).	

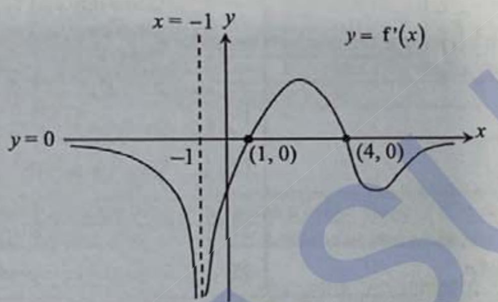
Solution

(a)



> **Commented [TCK13]: Misconceptions**
> Labelling horizontal asymptotes as x = c instead of y = c and vertical asymptotes as y = k instead of x = k.
> **Presentation**
> Curve should be continuous and smooth unless broken up by a vertical asymptote. Behavior of curve near a asymptote should be drawn approaching the asymptote clearly.
> **Things to note**
> Turning points and intercepts should be given in coordinates form as required.

(b)



6 The functions f and g are defined by

$$\mathrm{f}: x \mapsto 6 + \lambda x - x^2, \quad x \in \mathbb{R},$$
$$\mathrm{g}: x \mapsto 1 + 7\mathrm{e}^{-x}, \quad x \geq 0.$$

(i) For this part of the question, it is given that $\lambda = 5$. Show that the composite function fg exists. Hence find the range of fg. [3]

(ii) It is now given that λ is a real constant that is not necessary equals to 5. If the domain of f is restricted to $x \leq \frac{\lambda}{2}$, find f^{-1} in a similar form. [4]

Solution

(i)

$R_g = (1, 8]$ $\qquad$ $D_f = \mathbb{R}$

Since $R_g \subseteq D_f$, fg exists.

$$\mathrm{f}(x) = -\left(x - \frac{5}{2}\right)^2 + \frac{49}{4}$$

$$\mathrm{f}(8) = -\left(8 - \frac{5}{2}\right)^2 + \frac{49}{4} = -18$$

> **Commented [TCK14]: Careless Mistakes**
> Range of g is given as $(0, 8]$ instead. This mistake arises due to lack of familiarity with the characteristics of exponential function.
> **Misconceptions**
> Wrong criterion for existence of composite function fg.
> **Interpretation of Question**
> Not sure how to make use of the range of g to find the range of fg.

[Turn Over

	$(1,8] \xrightarrow{f} [-18, \frac{49}{4}]$ $\therefore R_{fg} = [-18, \frac{49}{4}]$	
	(ii) $y = -(x-\frac{\lambda}{2})^2 + (6+\frac{\lambda}{4})^2$	
	$x = \frac{\lambda}{2} \pm \sqrt{6+\frac{\lambda^2}{4} - y}$	
	$\therefore x = \frac{\lambda}{2} - \sqrt{6+\frac{\lambda^2}{4} - y}$ $(\because x \le \frac{\lambda}{2})$	
	$f^{-1}(x) \to \frac{\lambda}{2} - \sqrt{6+\frac{\lambda^2}{4} - x}$, $x \in \mathbb{R}$, $x \le 6 + \frac{\lambda^2}{4}$	
7	**(a)** Find $\frac{d}{dx}(\tan^3 x)$.	[1]
	Hence find $\int \sec^4 x \, dx$.	[2]
	(b) Find $\int \sin x(\sin x + \sin 3x) \, dx$.	[3]
	(c) For $p > 2$, find the value of $\int_0^p x\|2-x\| \, dx$ in terms of p.	[3]
	Solution	
	(a) $\frac{d}{dx}(\tan^3 x) = 3\tan^2 x \sec^2 x$	
	$\int \sec^4 x \, dx$ $\int \sec^2 x(1+\tan^2 x) \, dx$	
	$= \int (\sec^2 x + \sec^2 x \tan^2 x) \, dx$	
	$= \tan x + \frac{1}{3}\tan^3 x + C$	
	(b) $\int \sin x(\sin x + \sin 3x) \, dx$ $= \int \sin^2 x + \sin x \cdot \sin 3x \, dx$	
	$\therefore p = 4, q = 2$ $= \int (\sin^2 x + \sin 3x \cdot \sin x) \, dx$	
	$= \int \left(\frac{1-\cos 2x}{2} - \frac{\cos 4x - \cos 2x}{2}\right) dx$	

Commented [TCK15]: Things to Note
Many candidates do not know how to do complete the square of a quadratic function.
Recommendation
Lots of practice on completing the square.

Commented [TCK16]: Presentation
It is necessary to explain why the negative square root is the answer.

Commented [TCK17]: Interpretation of Question
Many candidates do not seem to understand what 'find f inverse in a similar form' requires.

Commented [LMH18]: Recommendation
When the previous part of the question involves differentiation, often you will need to use that result for the subsequent integration working. In this case, it is to get the expression "$3\tan^2 x \sec^2 x$".

Commented [LMH19]: Misconception
A significant number of students erroneously think that $\int \sin^2 x \, dx = \frac{\sin^3 x}{3} + c$.
Also, take note that $\sin x \sin 3x \ne \sin^2 3x$.

Commented [LMH20]: Recommendation
Remember to switch the trigo term with 'bigger angle' to the front before applying **factor formula.**

$\int \sin x(\sin x + \sin 3x) \, dx$

$= \int \sin^2 x + \sin x(\sin 3x) \, dx$

$= \int \sin^2 x \, dx + \int \sin x(\sin 3x) \, dx$

$= \int \frac{1-\cos 2x}{2} \, dx - \frac{1}{2}\int \cos 4x - \cos 2x \, dx$

$= \frac{1}{2}\int 1 - \cos 2x \, dx - \frac{1}{2}\int \cos 4x - \cos 2x \, dx$

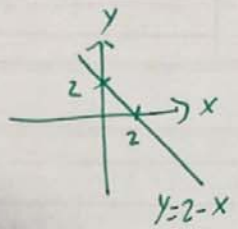
$$=\frac{1}{2}\int(1-\cos 4x)\ dx$$

$$=\frac{1}{2}x-\frac{\sin 4x}{8}+c$$

(c) $\int_0^p x|2-x|\ dx$ $=\int_0^2 x(2-x)\,dx+\int_2^p x(x-2)\,dx$

$=\left[x^2-\frac{x^3}{3}\right]_0^2+\left[\frac{x^3}{3}-x^2\right]_2^p$

$=4-\frac{8}{3}+\frac{p^3}{3}-p^2-\left(\frac{8}{3}-4\right)$



$=\frac{8}{3}+\frac{p^3}{3}-p^2$

8	Relative to origin O, the points A and B have position vectors **a** and **b** respectively, where **a** and **b** are non-zero vectors. The point C is on BA produced such that $BA:BC=3:5$ and OC is perpendicular to OB.	
	(i) Find $\overrightarrow{OC}$ in terms of **a** and **b**.	[1]
	(ii) Show that $\mathbf{a}\cdot\mathbf{b}=\frac{2}{5}\lvert\mathbf{b}\rvert^2$.	[2]
	The point P is on the line OB such that it is the image of B in the line OC.	
	(iii) Find the area of triangle PCB. Leave your answer in the form of $k\lvert\mathbf{a}\times\mathbf{b}\rvert$, where k is an exact real constant.	[3]
	The point F is the foot of perpendicular of P to the line AB.	
	(iv) Given that $\lvert\mathbf{a}\rvert^2=\frac{7}{25}\lvert\mathbf{b}\rvert^2$, find the position vector of F in terms of **a** and **b**.	[4]

Solution

(i) By Ratio Theorem,

$\mathbf{a}=\frac{3\mathbf{c}+2\mathbf{b}}{5}$

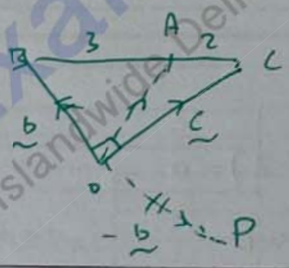
$\Rightarrow \mathbf{c}=\frac{5\mathbf{a}-2\mathbf{b}}{3}=\frac{5}{3}\mathbf{a}-\frac{2}{3}\mathbf{b}$

(ii) $\mathbf{c}\cdot\mathbf{b}=0$

$\frac{5\mathbf{a}-2\mathbf{b}}{3}\cdot\mathbf{b}=0$

$\frac{5\mathbf{a}\cdot\mathbf{b}}{3}=\frac{2\mathbf{b}\cdot\mathbf{b}}{3}$

$\mathbf{a}\cdot\mathbf{b}=\frac{2}{5}\lvert\mathbf{b}\rvert^2$



iii Area = 2x Area OBC
= $|\overrightarrow{OB}|\times|\overrightarrow{OC}|$

Commented [LMH21]: Things to note
This definite integral is the area under the curve of $y=x|2-x|$ from $x=0$ to $x=p$. Notice that:
when $0\le x<2$, $y=x\underbrace{|2-x|}_{\text{positive}}=x\underbrace{(2-x)}_{\text{no change}}$
when $2<x\le p$, $y=x\underbrace{|2-x|}_{\text{negative}}=x(\underbrace{x-2}_{\text{changed to positive}})$
Therefore, it is necessary to split the integral as such.

Commented [LMH22]: Careless Mistake
It is especially concerning to see that a large **number of students cannot calculate this correctly** (e.g. getting the minus sign wrong), even though they got the integration right. Clearly, algebraic manipulation skills must be strengthened.

Commented [KSM23]: Interpretation of question
Point C is on BA produced and not on AB produced.

Commented [KSM24]: Interpretation of question
Student did not realize that *B, O* and *P* are collinear.
Recommendation
Draw a diagram to illustrate the relationship of the points will help to get the correct solution, for (iii) and (iv).

[Turn Over

$=\left|\mathbf{b}\times\left(\frac{5\mathbf{a}}{3}-\frac{2\mathbf{b}}{3}\right)\right|$

Area $=\frac{1}{2}\left|\overrightarrow{BP}\times\overrightarrow{BC}\right|$

$=\frac{1}{2}\left|\left[\mathbf{b}-(-\mathbf{b})\right]\times\left[\frac{5\mathbf{a}-2\mathbf{b}}{3}-\mathbf{b}\right]\right|$

$=\frac{1}{2}\left|2\mathbf{b}\times\left(\frac{5\mathbf{a}}{3}-\frac{5\mathbf{b}}{3}\right)\right|$

$=\frac{5}{3}\left|\mathbf{b}\times(\mathbf{a}-\mathbf{b})\right|$

$=\frac{5}{3}\left|\mathbf{b}\times\mathbf{a}\right|(\because \mathbf{b}\times\mathbf{b}=\mathbf{0})$

$=\frac{5}{3}\left|-\mathbf{a}\times\mathbf{b}\right|=\frac{5}{3}\left|\mathbf{a}\times\mathbf{b}\right|(\because \mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a})$

OR

Area $=\frac{1}{2}\left|\overrightarrow{PC}\times\overrightarrow{BC}\right|$

$=\frac{1}{2}\left|\left[\frac{5\mathbf{a}-2\mathbf{b}}{3}-(-\mathbf{b})\right]\times\left[\frac{5\mathbf{a}-2\mathbf{b}}{3}-\mathbf{b}\right]\right|$

$=\frac{1}{2}\left|\frac{5\mathbf{a}+\mathbf{b}}{3}\times\frac{5}{3}(\mathbf{a}-\mathbf{b})\right|$

$=\frac{5}{18}\left|5\mathbf{a}\times\mathbf{a}-5\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}\right|$

$=\frac{5}{18}\left|-5\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{b}\right|$ ($\because \mathbf{a}\times\mathbf{a}=\mathbf{0}$ and $\mathbf{b}\times\mathbf{b}=\mathbf{0}$)

$=\frac{5}{3}\left|\mathbf{a}\times\mathbf{b}\right|$

(iv) Since F is on the line AB,

$\overrightarrow{OF}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$ for some $\lambda\in\mathbb{R}$

$\overrightarrow{PF}\cdot\overrightarrow{AB}=0$

$[\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})-(-\mathbf{b})]\cdot(\mathbf{b}-\mathbf{a})=0$

$[(1-\lambda)\mathbf{a}+(1+\lambda)\mathbf{b}]\cdot(\mathbf{b}-\mathbf{a})=0$

$(\lambda-1)|\mathbf{a}|^2+(1+\lambda)|\mathbf{b}|^2+(1-\lambda)\mathbf{a}\cdot\mathbf{b}-(1+\lambda)\mathbf{a}\cdot\mathbf{b}=0$

Commented [KSM25]: Misconception $|\mathbf{b}\times\mathbf{a}|=-|\mathbf{a}\times\mathbf{b}|$

Commented [KSM26]: Wrong interpretation of question leads to wrong position vector of P is form, hence wrong vector BP. Should be $\overrightarrow{OP}=-\mathbf{b}$.

Commented [KSM27]: Recommendation Students need to strengthen on algebraic skills.

$(\lambda-1)\frac{7}{25}|\mathbf{b}|^2+(1+\lambda)|\mathbf{b}|^2-2\lambda\left(\frac{2}{5}|\mathbf{b}|^2\right)=0$

$\left(\frac{7}{25}\lambda-\frac{7}{25}+1+\lambda-\frac{4}{5}\lambda\right)|\mathbf{b}|^2=0$

$\frac{12}{25}\lambda+\frac{18}{25}=0$ (since b is non zero)

$\lambda=-\frac{3}{2}$

$\overrightarrow{OF}=\mathbf{a}-\frac{3}{2}(\mathbf{b}-\mathbf{a})=\frac{5\mathbf{a}-3\mathbf{b}}{2}$

9 The curve C has equation

$$y=\frac{4x^2+px-q}{x^2-s},\quad x\in\mathbb{R},\ x^2\neq s$$

where p, q and s are non-zero constants.

It is given that C passes through the point $\left(0,\frac{1}{2}\right)$ and has a vertical asymptote $x=2$.

(i) State the value of s and show that the value of q is 2. [2]

(ii) It is given further that the line $y=1$ is a tangent to C and it does not meet the curve again. Find the exact value of p if p is a negative real value. [3]

(b) It is now given instead that $p=4$, $q=-1$ and $s=1$.

(i) Sketch the curve C, showing clearly the coordinates of any turning point(s), equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(ii) Find the equation of the additional curve that needs to be added to the curve sketched in **(b)(i)** to determine the number of distinct real roots for the equation $10(x+2)^2=3\left(10-\left(\frac{(4x^2+4x+1)}{(x^2-1)}\right)^2\right)$. [2]

Solution

(ai) Since $x=2$ is a vertical asymptote, $s=4$

At the point $(0,\frac{1}{2})$ $\frac{4(0)^2+p(0)-q}{(0)^2-4}=\frac{1}{2}$

$\therefore q=2$

(ii) When $y=1$, $1=\frac{4x^2+px-2}{x^2-4}$

$x^2-4=4x^2+px-2$

$3x^2+px+2=0$

Since $y=1$, is tangent, discriminant $=0$

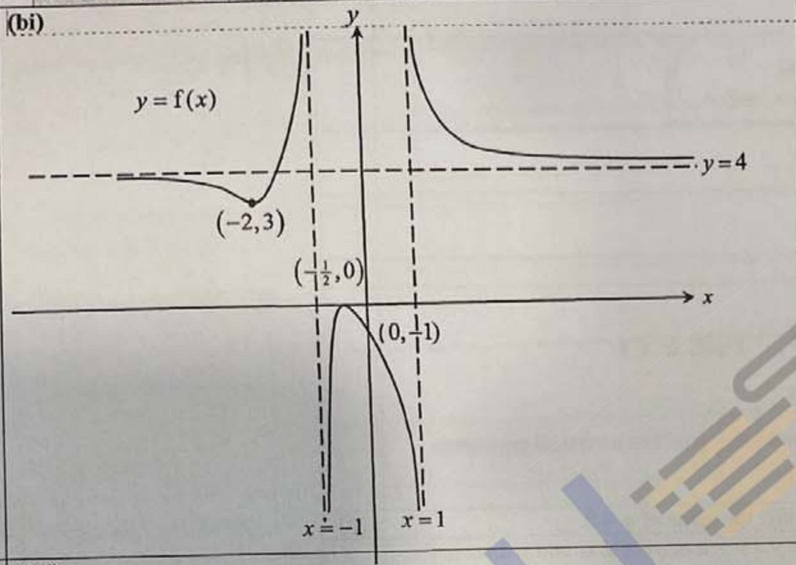
$p^2-4(3)(2)=0$

Commented [SH28]: Question reading
2. Majority saw the word ' tangent ' in the problem and went ahead to solve it by differentiating the function *C*
3. Students failed to understand that there is only 1 intersection between Curve C and the line y=1
4. If they have understood it, then they would be able to then use $b^2-4ac=0$ instead of any other inequality.

[Turn Over

$p=-2\sqrt{6}\ (\because p<0)$

(bi)



(bii)

$$10(x+2)^2=3\left(10-\left(\frac{4x^2+4x+1}{x^2-1}\right)^2\right)$$

$$10(x+2)^2+3\left(\frac{4x^2+4x+1}{x^2-1}\right)^2=30$$

$$\frac{(x+2)^2}{3}+\frac{\left(\frac{4x^2+4x+1}{x^2-1}\right)^2}{10}=1$$

The equation of the additional curve is $\frac{(x+2)^2}{3}+\frac{y^2}{10}=1$

Commented [SH29]: Presentation
1. Majority of students were penalized here for 2 components.
   1. Answered left as $-\sqrt{24}$ not in the simplified form
   2. Those who simplified gave $-2\sqrt{3}$ due to carelessness.

Commented [SH30]: Presentation
2. For $x<-1$, Graph was not drawn correctly, Many had y = 4 as horizontal asymptote and still didn't draw the graph asymptotic to C.
3. Many didn't capture the min point (-2, 3) in their sketch.

Commented [SH31]: Question Reading / Presentation
1. Find the equation of the additional curve not to determine the number of roots.
2. Majority of students didn't see the $\left(\frac{4x^2+4x+1}{x^2-1}\right)^2$ as y^2. If they had been a little more cautious, students would have been able to recognize the additional curve as one of the special conics curve.

10 (a)

Fig. 1 **Fig. 2**

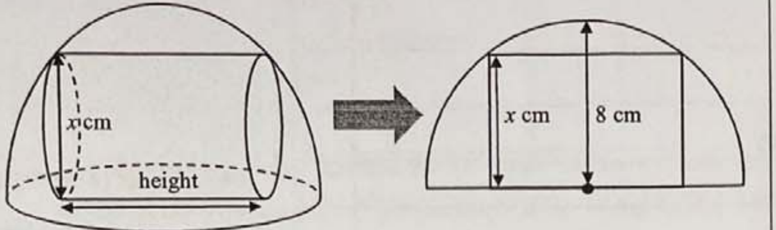


Fig. 1 shows the cylindrical-shaped water pipe, with negligible thickness and open on both ends, inscribed in a hemisphere with fixed radius 8 cm. The cross sectional view of the pipe and the hemisphere is shown in Fig. 2.

(i) If the diameter of the pipe is x cm, show that the curved surface area, S, of the pipe is $2\pi x\sqrt{64-x^2}$. [2]

(ii) It is given that as x varies, the maximum value of S occurs when the ratio of the diameter of the pipe to its height is $1:k$. Find the exact value of k and the exact maximum value of S. [6]

(b) A particle is moving on the curve with equation

$$y = 2\sin^{-1}(3x), \quad -\frac{1}{3} \le x \le \frac{1}{3},$$

where (x, y) is the coordinate of the particle at time t relative to a fixed point O. The x and y values represent the horizontal and vertical displacements.

When the $y-$ordinate of the particle is $\frac{\pi}{3}$, the rate at which the $y-$ordinate is decreasing with time t is 2 units per second. At this instant, find the exact rate at which the x - ordinate of the particle changes with time. [4]

> **Commented [KSM32]:** Interpretation of Question/Qn reading
> Quite many disregard the negative sign that depicts the fact that as t increases, y decreases. Thus dy/dt = -2.

Solution

(ai) Height of cylinder $= \sqrt{8^2 - x^2} \times 2 = 2\sqrt{8^2 - x^2}$

$S = 2\pi rh$

$= 2\pi \times \left(\frac{x}{2}\right) \times 2\sqrt{64-x^2}$

$S = 2\pi x\sqrt{64-x^2}$ c (shown)

> **Commented [KSM33]:** Things to note:
> This is a 'show' question. You need to explain your steps clearly (specify r, label which side is h in your diagram, etc.)

(ii) $\frac{\mathrm{d}S}{\mathrm{d}x} = 2\pi\sqrt{64-x^2} + 2\pi x\left(\frac{-2x}{2\sqrt{64-x^2}}\right)$

$= 2\pi\sqrt{64-x^2} - \frac{2\pi x^2}{\sqrt{64-x^2}}$

$= \frac{4\pi(32-x^2)}{\sqrt{64-x^2}}$

> **Commented [KSM34]:** Careless Mistakes
> Many simplifies wrongly in the midst of simplifying, thus resulting in the subsequent works all gone to waste. You are reminded to check the essential steps before proceeding, esp. when that affects all other answers.

For stationary point,

[Turn Over

$\frac{dS}{dx}=0$

$\frac{4\pi(32-x^2)}{\sqrt{64-x^2}}=0$

$x=\sqrt{32}=4\sqrt{2}$ (since diameter $x>0$)

$x:2\sqrt{64-x^2}=1:k$

$\sqrt{32}:2\sqrt{32}=1:k$

$k=2$

First derivative test:

x	$\sqrt{32}^-$	$\sqrt{32}$	$\sqrt{32}^+$
$\frac{dS}{dx}=\frac{4\pi(32-x^2)}{\sqrt{64-x^2}}$	$\frac{(+)}{(+)}=(+)$	0	$\frac{(-)}{(+)}=(-)$
Shape	/	—	\

$\therefore$ S is maximum when $x=4\sqrt{2}$.

Second derivative test:

$\frac{d^2S}{dx^2}=\frac{d\left(\frac{4\pi(32-x^2)}{\sqrt{64-x^2}}\right)}{dx}=\frac{4\pi[\sqrt{64-x^2}(-2x)-(32-x^2)\frac{1}{2}(64-x^2)^{-\frac{1}{2}}(-2x)]}{64-x^2}$

$=\frac{4\pi(-x)[2(64-x^2)-(32-x^2)]}{(64-x^2)^{3/2}}=\frac{4\pi x(x^2-96)}{(64-x^2)^{3/2}}$

When $x=4\sqrt{2}$, $\frac{d^2S}{dx^2}=-8\pi<0\Rightarrow$ S is maximum when $x=4\sqrt{2}$.

Max $S=2\pi\left(4\sqrt{2}\right)\sqrt{64-32}=64\pi$ cm^2

(b) $y=2\sin^{-1}(3x)$, $-\frac{1}{3}\le x\le\frac{1}{3}$.

$\frac{dy}{dx}=\frac{2(3)}{\sqrt{1-9x^2}}=\frac{6}{\sqrt{1-9x^2}}$

$y=2\sin^{-1}(3x)\Rightarrow\sin\frac{y}{2}=(3x)$

When $y=\frac{\pi}{3}$, $3x=\sin\frac{\pi}{6}=\frac{1}{2}$

$x=\frac{1}{6}$

$\frac{dx}{dt}=\frac{dx}{dy}\times\frac{dy}{dt}$

$=\frac{\sqrt{1-9x^2}}{6}\times(-2)$

Commented [KSM35]: Careless Mistakes/Qn Reading
Question asked for diameter: height of cylinder. Many did not double the $\sqrt{64-x^2}$.

Commented [KSM36]: Misconception
1. A few use values quite far from $\sqrt{32}$. You could use values eg. within ±0.1 from $\sqrt{32}$.

Commented [KSM37]: Things to note:
Majority left the dS/dx expression as a difference of two terms, which does not explain how you derive the signs of dS/dx in the close neighborhood of $\sqrt{32}$. Here, you should suggest sample value of $\sqrt{32}^-$ & $\sqrt{32}^+$ and compute (write down) the corresponding values of dS/dx.
Recommended:
You are advised to simplify the dS/dx expression to this concise form (eg. single fraction) to explain clearly how you derive the overall positive and negative signs.

Commented [KSM38]: Note: You should provide the value of 2nd derivative at $x=\sqrt{32}$.
Recommended:
You could use GC to get this value of -25.1 directly using
Alpha Windows → choose 'nDeriv'
→ Key in the dS/dx expression, etc.
→ Enter for answer

Commented [KSM39]: Things to note:
Chain rule applies. Many did not differentiate 3x, or did not adapt the formula given in MF26 (you should replace x by 3x in the denominator).

Commented [KSM40]: Question reading
Exact answer is required – you are reminded to leave x and dx/dt in exact form.

	$= \dfrac{\sqrt{1-9\left(\frac{1}{6}\right)^2}}{6} \times (-2)$	
	$\dfrac{dx}{dt} = -\dfrac{\sqrt{3}}{2} \times \dfrac{1}{6} \times (-2) = -\dfrac{\sqrt{3}}{6}$ units/s	
11	**(a)** Ivy took a \$40 000 tuition fee loan for her 4-year university course that commences on 1st January 2021. The loan is interest-free during the period of study. Immediately after graduation, interest is charged at 4% per annum of the outstanding amount owe at the end of each year. The maximum loan repayment period is at most 15 years upon graduation. Ivy is planning to pay \$550 every month upon graduation.	
	(i) Show that the amount she owes at the end of the n years after graduation is $\$171600 - 131600(1.04)^n$.	[3]
	(ii) Will she be able to finish repaying the loan by the end of 2030? Justify your answer clearly.	[2]
	(iii) Find the minimum monthly repayment Ivy should make if she intends to utilize fully the loan repayment period.	[2]
	(b) To save for her tuition fee loan repayment, Ivy wishes to start a new savings plan on the first day of November 2021. In this plan, she needs to invest \$200 into the account on the first day of each month. Every \$200 invested earns a fixed interest of d % of \$200 at the end of each month until a withdrawal is made from the account. The interest is added to the account and does not accumulate further interest.	
	(i) How much interest, in terms of d, will the first \$200 deposited earn at the end of 2022?	[2]
	(ii) Find the least value of d such that the total amount in the account exceed \$10 000 at the end of 36 months.	[3]

Commented [KW41]: Question Reading
The loan amount remains at \$40 000 till end of 2024.

Commented [KW42]: Question Reading
Monthly repayment starts in January 2025.

Commented [KW43]: Presentation
Clear working is required for 'Show' question.

Commented [KW44]: Presentation
Credit is not given for those with correct answer but no justification.

Commented [KW45]: Question Reading
Students generally have difficulty understanding how the interest is computed. Many thought that it follows a geometric series which is incorrect.

Commented [KW46]: Question Reading
Here, we are only interested in the \$200 Ivy invested on 1 November 2021 and the interest it earned over 14 months.

[Turn Over

Solution

(a)(i)

Year / n		Outstanding amount at the end of year
2025	1	$\left[40000-12(550)\right]1.04$ $=40000(1.04)-12(550)(1.04)$
2026	2	$\left[40000(1.04)-12(550)(1.04)-12(550)\right]1.04$ $=40000(1.04)^2-12(550)(1.04)^2-12(550)(1.04)$
2027	3	$\left[40000(1.04)^2-12(550)(1.04)^2-12(550)(1.04)-12(550)\right]1.04$ $=40000(1.04)^3-12(550)(1.04)^3-12(550)(1.04)^2-12(550)(1.04)$
		…

Students should use the information in the question to craft out the expression she owes at the end for first 2 to 3 years.

Amt owe at the end of n years =

$40000(1.04)^n-12(550)(1.04)^n-12(550)(1.04)^{n-1}-\cdots-12(550)(1.04)$

$=40000(1.04)^n-12(550)\left[1.04+\cdots+1.04^n\right]$

$=40000(1.04)^n-171600(1.04^n-1)$

$=171600-131600(1.04)^n$

(ii) At the end of 2030, $n=6$

Outstanding amount at end of 2030

$=171600-131600(1.04)^6$

$=5084.02>0$

She will not be able to finish repaying the loan by the end of 2030.

Commented [KW47]: Question Reading
Some students did not take into account the monthly repayments made in the year 2025. Some others computed interest before taking into account the monthly repayments made for that year which is incorrect.

Commented [KW48]: Presentation
1.This step is crucial as it shows the geometric series involved.
2.Some students did not write the last term of the series which suggest an infinite series and is incorrect.
Misconception
Some students who combined the first two terms were not aware that the number of terms in the geometric series is n-1 instead of n.

Commented [KW49]: Misconception
The formula to use is $S_n=\dfrac{a(r^n-1)}{r-1}$. Some students forgot to include a in their working.

Commented [KW50]: Misconception
Many students mistook the number of years from the start of 2025 to the end of 2030 to be 5. Note that the year 2025 should be included in the count as well.

Commented [KW51]: Misconception
Some students did not understand the significance of this amount as seen from their justification.

(iii) Let m be the monthly loan repayment.

To utilise fully the loan repayment period, $n = 15$

$$40000(1.04)^{15} - 12m\left[\frac{1.04(1.04^{15}-1)}{1.04-1}\right] \le 0$$

$$m \ge \frac{40000(1.04)^{15}}{12}\left[\frac{0.04}{1.04(1.04^{15}-1)}\right] = 288.27276$$

$\therefore$ Minimum monthly repayment = \$288.28

(b)(i)

n	Amount at end of month
1	$200+\left(\frac{d}{100}\right)(200) = 200+2d$
2	$200+2d+2d = 200+2d(2)$
3	$200+2d(3)$
	. .
n	$200+2d(n)$

At end of 2022, $n = 14$

Interest $= 2d(14)$

$= 28d$

(b)(ii)

n	Amount at start of month	Amount at end of month
1	200	$200+2d$
2	$200+2d+200$ $=2(200)+2d$	$2(200)+2d+2d(2)$ $=2(200)+2d(1+2)$
3	$2(200)+2d(1+2)+200$ $=3(200)+2d(1+2)$	$3(200)+2d(1+2)+2d(3)$ $=3(200)+2d(1+2+3)$
..	…	…
n	…	$n(200)+2d(1+2+\cdots+n)$ $=200n+2d\left[\frac{n}{2}(1+n)\right]$ $=200n+dn(1+n)$

$200(36)+d(36)(1+36) > 10000$

$d > 2.1021$

$\therefore$ Least $d = 2.11$ (3sf)

Commented [KW52]: Presentation
Some students did not define the variable used. Note that the definition of this variable affects the working that follows and the answer.

Commented [KW53]: Misconception
Some students used the expression given in (a)(i) which is incorrect as the monthly repayment for that expression is fixed at \$550.

Commented [KW54]: Things to note
The degree of accuracy for money is the nearest cent unless otherwise specified.
Question Reading
Keyword is 'minimum' so answer needs to be rounded up.

Commented [KW55]: Recommendation
Students should use a table to help them see the interest earned at the end of each month.

Commented [KW56]: Question Reading
Interest is d% of \$200 and not d.
Presentation
1. d% should be written as $\frac{d}{100}$ and not 0.0d.
2. 0.01d is not d%.

Commented [KW57]: Recommendation
Students should use a table to help them see the interest earned at the end of each month and the arithmetic series involved.

Commented [KW58]: Question Reading
The question wants the amount to 'exceed \$10 000' so the inequality sign '>' should be used.

Commented [KW59]: Question Reading
Keyword is 'least' so answer needs to be rounded up.

[Turn Over

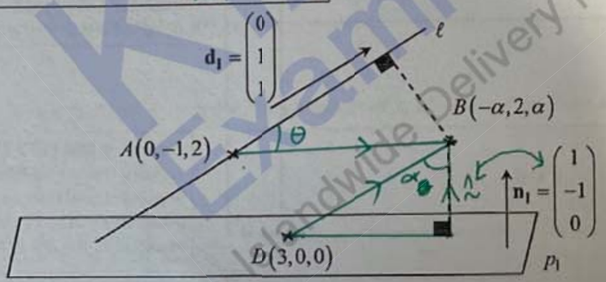
12	Relative to the origin O, a point A has position vector $-\mathbf{j}+2\mathbf{k}$. The plane p_1 has equation $\mathbf{r}.\begin{pmatrix}1\\-1\\0\end{pmatrix}=3$.	
	(i) Find the position vector of the foot of perpendicular from point A to the plane p_1.	[4]
	The line l has equation $\mathbf{r}=\begin{pmatrix}0\\-1\\2\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}, \mu\in\mathbb{R}$.	
	(ii) Find the acute angle between the plane p_1 and the line l.	[2]
	(iii) The point $B(-\alpha,2,\alpha)$ is equidistant from the plane p_1 and the line l. Find the possible values of α.	[4]
	The plane p_2 has equation $\mathbf{r}.\begin{pmatrix}1\\1\\-1\end{pmatrix}=5-\beta,\ \beta\neq 8$.	
	(iv) Show that the point $C(4,1,\beta)$ lies on both p_1 and p_2. Hence find the vector equation of the line of intersection between the planes p_1 and p_2.	[3]
	Solution	
	(i) Let F be the foot of perpendicular from A to the plane p_1.	
	$\ell_{AF}:\mathbf{r}=\begin{pmatrix}0\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}, \lambda\in\mathbb{R}$	
	Since F lies on line AF	
	$\overrightarrow{OF}=\begin{pmatrix}0\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$ for some $\lambda\in\mathbb{R}$	
	As F is also on the plane p_1,	
	$\begin{pmatrix}\lambda\\-1-\lambda\\2\end{pmatrix}.\begin{pmatrix}1\\-1\\0\end{pmatrix}=3$	
	$\lambda+1+\lambda=3$ $\lambda=1$	

Commented [LT60]: Misconception
Incorrect to interpret "B equidistant from plan and line" as "having B to lie along the angle bisector of the angle between the line and the plane".

Commented [LT61]: Notation
Incorrect to write "Since $\overrightarrow{OF}$ lies on the line". Also do note that points are represented using Capital letter.

Commented [LT62]: Misconception
Incorrect to have written as $\overrightarrow{AF}=\begin{pmatrix}0\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$ because this should be a vector that is parallel to the normal vector.
It should written as $\overrightarrow{AF}=\lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

Commented [LT63]: Presentation
Need to have this phrase to indicate that we are narrowing down to a particular point on the line.

	$\overrightarrow{OF}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$	
	Method 2 (strongly not recommended) Let point D be $(3,0,0)$ $\overrightarrow{FA}=\left(\overrightarrow{DA}\cdot\hat{\mathbf{n}}_1\right)\hat{\mathbf{n}}_1$ $=\left(\dfrac{\begin{pmatrix}-3\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\sqrt{2}}\right)\dfrac{\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\sqrt{2}}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$	
	$\overrightarrow{OF}=\overrightarrow{OA}-\overrightarrow{FA}=\begin{pmatrix}0\\-1\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$	
	(ii) Let θ be the acute angle between the plane p_1 and the line l.	
	$\sin\theta=\dfrac{\left\lvert\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}\right\rvert}{\sqrt{2}\sqrt{2}}$	
	$\theta=30^\circ$	
	(iii) Let point D be $(3,0,0)$	
	dist between pt B to plane p_1 = dist between pt B to line l	
	$\dfrac{\lvert\overrightarrow{BD}\cdot \underset{\sim}{n}\rvert}{\lvert\underset{\sim}{n}\rvert}=\dfrac{\lvert\overrightarrow{BA}\times \underset{\sim}{d_1}\rvert}{\lvert\underset{\sim}{d_1}\rvert}$	
		

Commented [LT64]: Misconception
A number of students who find foot of perpendicular using projection vector method scored poorly because the direction of the vector obtained is in the wrong direction. It is not vector $\overrightarrow{AF}$.
We strongly discourage the use of this method to find foot of perpendicular.

Commented [LT65]: Presentation
Some did not define the variable used in the question.

Commented [LT66]: Misconception
There is a need to have a modulus sign in this formula in order to get the acute angle because the dot product may yield a negative outcome.

For finding angle between line and plane, the formula is not $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}\right\rvert}{\sqrt{2}\sqrt{2}}$.

Commented [LT67]: Accuracy
If the final answer is an exact value, do not perform rounding off and leave it to 1 decimal place or 3 significant figures.

Commented [LT68]: Notation
The coordinates of all points are expressed in row form. Column vectors will be written in a column form.

[Turn Over

$$\frac{\begin{pmatrix}3+\alpha\\-2\\-\alpha\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\sqrt{2}} = \frac{\left|\begin{pmatrix}\alpha\\-3\\2-\alpha\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}\right|}{\sqrt{2}}$$

$$|3+\alpha+2| = \left|\begin{pmatrix}\alpha-5\\-\alpha\\\alpha\end{pmatrix}\right|$$

$$|\alpha+5| = \sqrt{2\alpha^2+(\alpha-5)^2}$$

$\alpha^2+10\alpha+25=3\alpha^2-10\alpha+25$

$\alpha(\alpha-10)=0$

$\therefore \alpha=0$ or 10

(iv) $\begin{pmatrix}4\\1\\\beta\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=4(1)+1(-1)+\beta$

Hence C lies on both p_1 and p_2.

$\begin{pmatrix}1\\-1\\0\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$

Eqn of the line of intersection is $\mathbf{r}=\begin{pmatrix}4\\1\\\beta\end{pmatrix}+\gamma\begin{pmatrix}1\\1\\2\end{pmatrix}, \gamma\in\mathbb{R}$

Commented [LT69]: Misconception
Some used the wrong formula in the evaluation of the length.
It is important to have the modulus sign whenever one is finding length because the dot product may yield a negative output.
Some wrong expressions are:

dist from B to plane $=\frac{\begin{pmatrix}3+\alpha\\-2\\-\alpha\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\sqrt{2}}$

dist from B to plane $=\frac{\begin{pmatrix}-\alpha\\2\\\alpha\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\sqrt{2}}$

dist from B to plane $=\begin{pmatrix}3+\alpha\\-2\\-\alpha\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}$

dist between B to line l $=\frac{\begin{pmatrix}\alpha\\-3\\2-\alpha\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}}{\sqrt{2}}$

Commented [LT70]: Misconception
$|-\alpha-5|\neq\alpha+5$
This is only true if $\alpha\geq-5$.

Commented [LT71]: Presentation
As it is a "show" question type, it is therefore important to be explicit on how you perform the dot product.

Commented [LT72]: Question Reading
This is a "Hence" question type and so it is important to show how the information in the previous part can be used to find the vector equation of the line of intersection.

End of Paper

CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC1 Promotional Examination

CANDIDATE NAME			
CLASS		INDEX NUMBER	

MATHEMATICS 9758/01

Paper 1 **4 October 2021**

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Write your answers in the spaces provided in the Question Paper.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

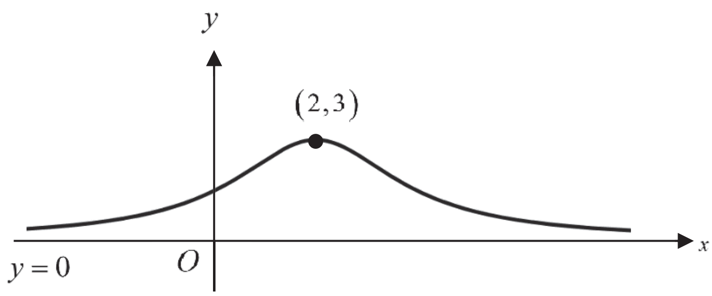
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**Total**
Marks													
Total	**3**	**4**	**7**	**8**	**9**	**8**	**9**	**7**	**11**	**12**	**11**	**11**	**100**

This document consists of **24** printed pages and **2** blank pages.

1.



The diagram shows the curve $y = \mathrm{f}(x)$. The curve has an asymptote $y = 0$ and a maximum point at $(2,3)$. It is given that f is concave downwards for $1 \le x \le 3$.

Sketch the graph of $y = \mathrm{f}'(x)$, stating the equations of any asymptotes, the $x-$coordinates of any stationary points and any points of intersection with the $x-$axis. [3]

2. In a convergent geometric series that only consists of positive terms, the sum of the first four terms is 80 times the sum of all the remaining terms. Find the common ratio. [4]

3. Differentiate the following expressions with respect to *x*, leaving your answers in terms of *x*,

(a) $\cos^{-1}\sqrt{x^3}$, giving your answer in non-trigonometric form, [2]

(b) $\operatorname{cosec}\left(\ln\left(x^2+3\right)\right)$, [2]

(c) $\left(x^2+1\right)^x$. [3]

4. The points $A(1,0,-2)$, $B(3,-1,-2)$ and $C(-3,7,0)$ lie on plane p_1. Another plane p_2 has equation $3x - y + 2z = 3$.

(i) Find a vector equation of plane p_1 in the form $\mathbf{r}\cdot\mathbf{n} = d$. [3]

(ii) Find the acute angle between p_1 and p_2. [2]

The equation of plane p_3 is given to be $-9x + 3y - 6z = 7$.

(iii) Find the shortest distance between p_2 and p_3. [3]

5. **(a)** **(i)** Express $\dfrac{2r^2+1}{r^2-1}$ in the form $A + \dfrac{B}{r-1} + \dfrac{C}{r+1}$, where *A*, *B* and *C* are constants to be found. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{n} \frac{2r^2+1}{r^2-1}$ in terms of *n*. [4]

(b) Express $\displaystyle\sum_{r=1}^{2n}\left[(-1)^{r+1}2^r\right]$ in the form $\dfrac{p}{q}\left[s\left(4^n\right)+t\right]$, where *p*, *q* ,*s* and *t* are integers to be determined. [3]

6. The curve C is defined by the parametric equations

$$x = \theta + \frac{1}{2}\sin 2\theta,\ y = 2\tan\theta \qquad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(i) Show that $\frac{dy}{dx} = \frac{1}{\cos^k \theta}$, where k is an integer to be determined. [3]

The lines T and N are the tangent and normal to C at the point where $\theta = \frac{\pi}{4}$ respectively.

(ii) Find the equations of T and N, leaving your answers in exact form. [3]

(iii) Find the area enclosed by T, N and the y-axis. [2]

7. It is given that $y = \sin\left[\ln(1+2x)\right]$.

(i) Show that $(1+2x)^2 \frac{d^2y}{dx^2} + 2(1+2x)\frac{dy}{dx} + ky = 0$, where k is a constant to be found.

Hence, find the first three non-zero terms of the Maclaurin expansion for y. [6]

(ii) Using standard series from the List of Formulae (MF26), verify the correctness of your result from part **(i)** up to and including the term in x^3. [2]

Explain why the expansion is not valid when $x = -\frac{1}{2}$. [1]

8 **(i)** The arithmetic progression is grouped into sets of integers, such that the n^{th} set contains n integers as shown.

$$\{\ 1\ \}, \{\ 3, 5\ \}, \{\ 7, 9, 11\ \}, \{\ 13, 15, 17, 19\ \}, \ldots \ldots$$

(a) Find the total number of terms in the first n sets, and hence show that the last term of the n^{th} set is $n^2 + n - 1$. [2]

(b) Find the first term of the n^{th} set. [2]

(c) Show that the sum of the terms in the n^{th} set is n^3. [1]

(ii) Hence, prove that $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$. [2]

9. With reference to the origin O, the points A and B have position vectors $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 5\mathbf{k}$ respectively.

(i) Find a vector equation of the line l_1 that passes through point A and is parallel to the vector $\mathbf{a}$. [1]

(ii) Find the exact length of projection of $\mathbf{b}$ on l_1. Hence find d, the exact perpendicular distance from the point B to l_1. [4]

(iii) Using the value of d found in part **(ii)**, find the position vector of the point C, the foot of perpendicular from the point B to l_1. [3]

(iv) The line l_2 passes through point B and is parallel to vector $\mathbf{b}$. Find a cartesian equation of l_3 which is the reflection of l_2 in l_1. [3]

10. The curves C_1 and C_2 have equations $y=\dfrac{2x^2+9}{x^2-4}$ and $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$ respectively.

(i) Find the equations of the asymptotes of the curve C_1. [3]

(ii) Sketch C_1 and C_2 on the same diagram, stating the equations of any asymptotes, coordinates of any points where C_1 or C_2 crosses the axes and any turning points. [5]

(iii) Find the x-coordinates of the points where the two curves intersect. [2]

(iv) Hence solve the inequality $-5\sqrt{1-\dfrac{1}{9}x^2}\le\dfrac{2x^2+9}{x^2-4}$. [2]

11. Sigmoid functions are used to model many natural processes such as population growth of virus. One example of a Sigmoid function f is given by

$$\mathrm{f}: x\mapsto\frac{1}{1+\mathrm{e}^{-x}},\quad x\in\mathbb{R}.$$

(i) Sketch the graph of $y=\mathrm{f}(x)$, indicating clearly the equation(s) of any asymptote(s) and the coordinates of any points where the curve crosses the axes. [2]

(ii) Find $\mathrm{f}^{-1}(x)$ in similar form. [3]

Another function g is given by $\mathrm{g}: x\mapsto 3x-1,\ x\in\mathbb{R},\ 0\le x\le 2$.

(iii) Show that fg exists and find the range of fg, expressing your answer in terms of e. [4]

(iv) Describe a sequence of transformations which transform the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \mathrm{fg}(x)$. [2]

12. **(i)** Amanda wants to make a mask holder. To do so, she cuts out 4 identical equilateral triangles of sides 1 cm and a rectangular strip $\dfrac{x-1}{16}$ cm by $\dfrac{2y-1}{4}$ cm from a rectangle of sides x cm by y cm (see Diagram A). The rectangular strip has an area of 5.5 cm^2.

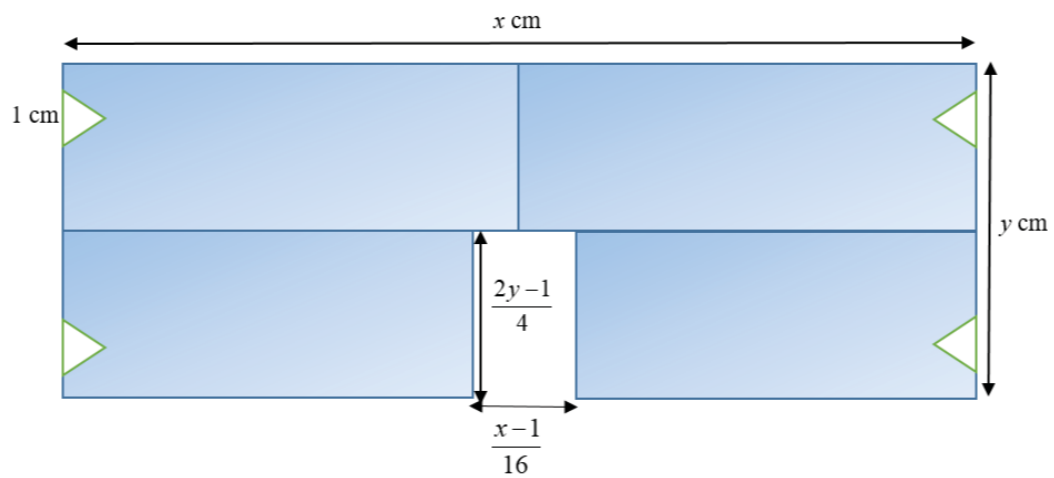


Diagram A

Show that A, the area of the mask holder is given by

$$A = \frac{176x}{x-1} + \frac{x}{2} - \sqrt{3} - \frac{11}{2}$$

Using differentiation, find the exact value of x such that area of the mask holder is a minimum. Hence determine the minimum area of the mask holder. [9]

(ii) The area of the rectangular strip is kept at 5.5 cm^2. Beth suggests that if the 4 identical triangle cut-outs are isosceles instead, the area of the mask holder could be made smaller as compared to the area found in part **(i)(b).**

For each of the triangles, let θ be the angle between two edges of length 1 cm each (see diagram B). Determine if Beth's suggestion is correct.

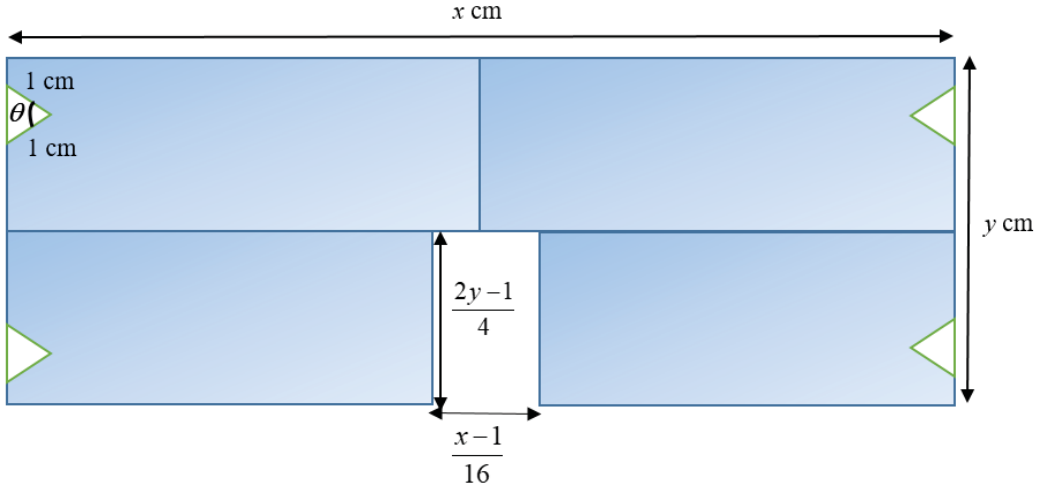
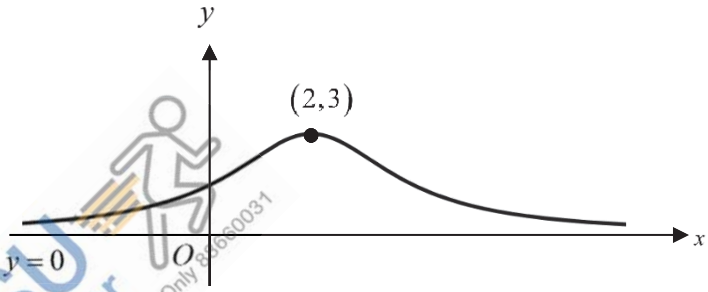


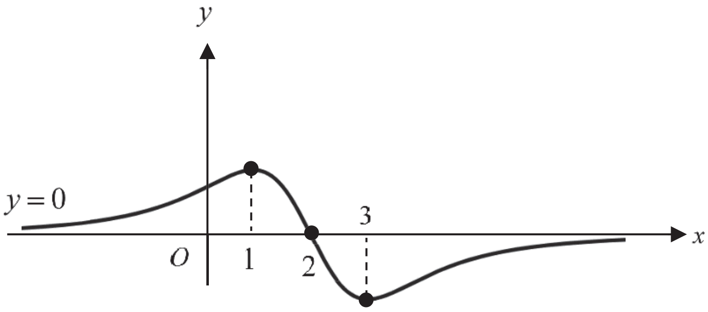
Diagram B [2]

End of Paper

2021 JC1 H2 Math PROMOS Solutions

1.	 The diagram shows the curve $y = \mathrm{f}(x)$. The curve has an asymptote $y = 0$ and a maximum point at $(2,3)$. It is given that f is concave downwards for $1 \le x \le 3$.	
	Sketch the graph of $y = \mathrm{f}'(x)$, stating the equations of any asymptotes, the $x-$coordinates of any stationary points and any points of intersection with the $x-$axis.	[3]

<u>Suggested Solution:</u>



2.	In a convergent geometric series that only consists of positive terms, the sum of the first four terms is 80 times the sum of all the remaining terms. Find the common ratio.	[4]

Suggested Solution:

Let *a* be the first term of the infinite geometric progression (GP), and *r* be the common ratio.
$\because$ The sum of the first four terms is 80 times the sum of all the remaining terms,

$$S_4 = 80\left[S_\infty - S_4\right]$$

$$\therefore \quad \frac{a(1-r^4)}{1-r} = 80\frac{ar^4}{1-r}$$

$$1-r^4 = 80r^4 \quad (\because a \neq 0)$$

$$1 = 81r^4$$

$$r^4 = \frac{1}{81}$$

$$r = \pm\frac{1}{3}$$

$\because$ The terms of the geometric series are all positive,
common ratio *r* is $\frac{1}{3}$.

3.	Differentiate the following expressions with respect to x, leaving your answers in terms of x,		
	(a)	$\cos^{-1}\sqrt{x^3}$, giving your answer in non-trigonometric form,	[2]
	(b)	$\operatorname{cosec}\left(\ln\left(x^2+3\right)\right)$,	[2]
	(c)	$\left(x^2+1\right)^x$.	[3]

Suggested Solution:

	(a)	$\cos^{-1}\sqrt{x^3}$, giving your answer in non-trigonometric form,	[2]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\sqrt{x^3}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\left(x^{\frac{3}{2}}\right)\right)$$

$$=-\frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}}\left(\frac{3}{2}x^{\frac{1}{2}}\right)$$

$$=-\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

Alternative Method 1

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\sqrt{x^3}\right)=-\frac{1}{\sqrt{1-\left(\sqrt{x^3}\right)^2}}\left(\frac{1}{2}\frac{1}{\sqrt{x^3}}\left(3x^2\right)\right)$$

$$=-\frac{3x^2}{2\sqrt{x^3}\sqrt{1-x^3}}$$

$$=-\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

Alternative Method 2

Let $y = \cos^{-1}\sqrt{x^3}$

$\cos y = x^{\frac{3}{2}}$

Differentiating implicitly w.r.t. x,

$$-\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}\sqrt{x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3\sqrt{x}}{2\sin y}$$

$$= -\frac{3\sqrt{x}}{2\sqrt{\sin^2 y}} = -\frac{3\sqrt{x}}{2\sqrt{1-\cos^2 y}}$$

$$= -\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

3.	Differentiate the following expressions with respect to x, leaving your answers in terms of x,		
	(b)	$\operatorname{cosec}\left(\ln\left(x^2+3\right)\right)$.	[2]

Suggested Mark Scheme:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{cosec}\left(\ln(x^2+3)\right)\right]$$

$$=-\operatorname{cosec}\left(\ln(x^2+3)\right)\cot\left(\ln(x^2+3)\right)\cdot\left(\frac{1}{x^2+3}(2x)\right)$$

$$=-\frac{2x\operatorname{cosec}\left(\ln(x^2+3)\right)\cot\left(\ln(x^2+3)\right)}{x^2+3}$$

Alternative Method

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{cosec}\left(\ln(x^2+3)\right)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sin\left(\ln(x^2+3)\right)}\right]$$

$$=-\frac{1}{\left[\sin\left(\ln(x^2+3)\right)\right]^2}\cdot\left[\cos\left(\ln(x^2+3)\right)\cdot\left(\frac{1}{x^2+3}(2x)\right)\right]$$

$$=-\frac{2x\cos\left(\ln(x^2+3)\right)}{(x^2+3)\sin^2\left(\ln(x^2+3)\right)}$$

3.	Differentiate the following expressions with respect to x, leaving your answers in terms of x,		
	(c)	$\left(x^2+1\right)^x$.	[3]

Suggested Mark Scheme:

Let $y=(x^2+1)^x$

$\ln y=\ln(x^2+1)^x$

$\ln y=x\ln(x^2+1)$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=x\left(\frac{2x}{x^2+1}\right)+(1)\ln(x^2+1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\left[\frac{2x^2}{x^2+1}+\ln(x^2+1)\right]$$

$$=(x^2+1)^x\left[\frac{2x^2}{x^2+1}+\ln(x^2+1)\right]$$

Alternative Method

Let $y=(x^2+1)^x$

$=\left(\mathrm{e}^{\ln(x^2+1)}\right)^x$

$=\mathrm{e}^{x\ln(x^2+1)}$

OR

$y=(x^2+1)^x$

$=\mathrm{e}^{\ln\left((x^2+1)^x\right)}$

$=\mathrm{e}^{x\ln(x^2+1)}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{x\ln(x^2+1)}\left[x\left(\frac{2x}{x^2+1}\right)+(1)\ln(x^2+1)\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{x\ln(x^2+1)}\left[\frac{2x^2}{x^2+1}+\ln(x^2+1)\right]$$

4.	The points $A(1,0,-2)$, $B(3,-1,-2)$ and $C(-3,7,0)$ lie on plane p_1. Another plane p_2 has equation $3x-y+2z=3$.		
	(i)	Find a vector equation of plane p_1 in the form $\mathbf{r}\cdot\mathbf{n}=d$.	[3]
	(ii)	Find the acute angle between p_1 and p_2.	[2]
	The equation of plane p_3 is given to be $-9x+3y-6z=7$.		
	(iii)	Find the shortest distance between p_2 and p_3.	[3]

Suggested Solution:

(i)

normal to plane p_1

$$= \overrightarrow{AB}\times\overrightarrow{AC}$$

$$= \begin{pmatrix} 3-1 \\ -1-0 \\ -2-(-2) \end{pmatrix} \times \begin{pmatrix} -3-1 \\ 7-0 \\ 0-(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -4 \\ 7 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -4 \\ 10 \end{pmatrix}$$

$$= -2\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$$

OR

normal to plane p_1

$$= \overrightarrow{AB}\times\overrightarrow{BC}$$

$$= \begin{pmatrix} 3-1 \\ -1-0 \\ -2-(-2) \end{pmatrix} \times \begin{pmatrix} -3-3 \\ 7-(-1) \\ 0-(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -4 \\ 10 \end{pmatrix}$$

$$= -2\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$$

OR any other correct combinations

Substitute A or B or C into $d = \underset{\sim}{r}\cdot\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$

$$d = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 11$$

Equation of plane p_1:

$$\underset{\sim}{r}\cdot\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 11$$

(ii)

Equation of plane $p_2: \underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 3$

acute angle between p_1 and p_2

$$= \cos^{-1} \frac{\left| \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right|}{\sqrt{30}\sqrt{14}}$$

$= 1.12 \quad \text{OR} \quad 64.0^\circ$

(iii)

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 3$ and $\begin{pmatrix} 0 \\ 0 \\ -\frac{7}{6} \end{pmatrix} \cdot \begin{pmatrix} -9 \\ 3 \\ -6 \end{pmatrix} = 7$

Hence $D(1,0,0)$ and $E\left(0,0,-\frac{7}{6}\right)$ lie on planes p_2 and p_3 respectively.

shortest distance between p_2 and p_3

$$= \frac{\left| \overrightarrow{DE} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right|}{\sqrt{14}}$$

$$= \frac{\left| \begin{pmatrix} 0-1 \\ 0-0 \\ -\frac{7}{6}-0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right|}{\sqrt{14}}$$

$$= \frac{\left| \begin{pmatrix} -1 \\ 0 \\ -\frac{7}{6} \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right|}{\sqrt{14}}$$

$$= \frac{16}{3\sqrt{14}}$$

$= 1.43$ units

Alternative Method

$$p_2: \underset{\sim}{r}\cdot\begin{pmatrix}3\\-1\\2\end{pmatrix}=3 \Rightarrow \underset{\sim}{r}\cdot\frac{1}{\sqrt{14}}\begin{pmatrix}3\\-1\\2\end{pmatrix}=\frac{3}{\sqrt{14}}$$

$$p_3: \underset{\sim}{r}\cdot\begin{pmatrix}-9\\3\\-6\end{pmatrix}=7 \Rightarrow \underset{\sim}{r}\cdot\frac{1}{3\sqrt{14}}\begin{pmatrix}-3\\1\\-2\end{pmatrix}=\frac{7}{3\sqrt{14}}$$

Hence, perpendicular distance $=\frac{3}{\sqrt{14}}+\frac{7}{3\sqrt{14}}$

$$=\frac{16}{3\sqrt{14}}$$

Alternative Method

$E(1,0,0)$ lies on p_2 and let F be the foot of perpendicular from $(1,0,0)$ to p_3.

$$\overrightarrow{OF}=\begin{pmatrix}1+3\lambda\\-\lambda\\2\lambda\end{pmatrix}$$

$$\begin{pmatrix}1+3\lambda\\-\lambda\\2\lambda\end{pmatrix}\cdot\begin{pmatrix}-9\\3\\-6\end{pmatrix}=7$$

$$\lambda=-\frac{8}{21}$$

Shortest distance

$$=\left|\overrightarrow{EF}\right|$$

$$=\left|\begin{pmatrix}3\lambda\\-\lambda\\2\lambda\end{pmatrix}\right|$$

$$=\left|-\frac{8}{21}\begin{pmatrix}3\\-1\\2\end{pmatrix}\right|$$

$$=\frac{8}{21}\sqrt{3^2+(-1)^2+2^2}$$

$$=\frac{8}{21}\sqrt{14}$$

5.	**(a)**	**(i)**	Express $\frac{2r^2+1}{r^2-1}$ in the form $A+\frac{B}{r-1}+\frac{C}{r+1}$, where A, B and C are constants to be found.	[2]
		(ii)	Hence find $\sum_{r=2}^{n}\frac{2r^2+1}{r^2-1}$ in terms of n.	[4]
	(b)		Express $\sum_{r=1}^{2n}\left[(-1)^{r+1}2^r\right]$ in the form $\frac{p}{q}\left[s\left(4^n\right)+t\right]$, where p, q, s and t are integers to be determined.	[3]

Suggested Solution:

(ai) $\frac{2r^2+1}{r^2-1}=A+\frac{B}{r-1}+\frac{C}{r+1}$

$2r^2+1=A(r-1)(r+1)+B(r+1)+C(r-1)$

Comparing coefficients, $2=A$

$0=B+C$

$1=-A+B-C$

Solving, $A=2, B=\frac{3}{2}$ and $C=-\frac{3}{2}$

Hence, $\frac{2r^2+1}{r^2-1}=2+\frac{3}{2(r-1)}-\frac{3}{2(r+1)}$

(aii) $\sum_{r=2}^{n}\frac{2r^2+1}{r^2-1}=\sum_{r=2}^{n}\left[2+\frac{3}{2(r-1)}-\frac{3}{2(r+1)}\right]$

$=\sum_{r=2}^{n}2+\frac{3}{2}\sum_{r=2}^{n}\left[\frac{1}{(r-1)}-\frac{1}{(r+1)}\right]$

$$=(n-2+1)(2)+\frac{3}{2}\left[\frac{1}{1}-\frac{1}{3}+\right.$$
$$\frac{1}{2}-\frac{1}{4}+$$
$$\frac{1}{3}-\frac{1}{5}+$$
$$\frac{1}{4}-\frac{1}{6}+$$
$$\vdots$$
$$\frac{1}{n-3}-\frac{1}{n-1}+$$
$$\frac{1}{n-2}-\frac{1}{n}+$$
$$\left.\frac{1}{n-1}-\frac{1}{n+1}\right]$$

$$= 2(n-1) + \frac{3}{2}\left[1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right]$$

$$= 2(n-1) + \frac{3}{2}\left[\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right]$$

(b) $\sum_{r=1}^{2n}\left[(-1)^{r+1} 2^r\right] = 2^1 - 2^2 + 2^3 - 2^4 + ... + 2^{2n-1} - 2^{2n}$

$$= \left(2^1 + 2^3 + ... + 2^{2n-1}\right) - \left(2^2 + 2^4 + ... 2^{2n}\right)$$

$$= \frac{2\left(4^n - 1\right)}{4-1} - \frac{2^2\left(4^n - 1\right)}{4-1}$$

$$= \tfrac{2}{3}\left(4^n - 1 - 2\left(4^n\right) + 2\right)$$

$$= \tfrac{2}{3}\left(1 - 4^n\right)$$

Alternatively,

$\sum_{r=1}^{2n}\left[(-1)^{r+1} 2^r\right] = -\sum_{r=1}^{2n}\left[(-2)^r\right]$ which is GP with first term = -2 and common ratio = -2

$$= -\left(\frac{(-2)((-2)^{2n} - 1}{(-2) - 1}\right)$$

$$= \frac{2}{3}\left(1 - 4^n\right)$$

6.	The curve C is defined by the parametric equations $$x=\theta+\frac{1}{2}\sin 2\theta,\ y=2\tan\theta \qquad \text{where } -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$		
	(i)	Show that $\frac{dy}{dx}=\frac{1}{\cos^k\theta}$, where k is an integer to be determined.	[3]
	The lines T and N are the tangent and normal to C at the point where $\theta=\frac{\pi}{4}$ respectively.		
	(ii)	Find the equations of T and N, leaving your answers in exact form.	[3]
	(iii)	Find the area enclosed by T, N and the y-axis.	[2]

<u>Suggested Solution:</u>

1. **(i)** $\frac{dx}{d\theta}=1+\frac{1}{2}(2)\cos 2\theta,\ \frac{dy}{d\theta}=2\sec^2\theta$

$$\frac{dy}{dx}=\frac{2\sec^2\theta}{1+\cos 2\theta}$$
$$=\frac{2\sec^2\theta}{2\cos^2\theta}$$
$$=\frac{1}{\cos^4\theta}$$

So, $k=4$

(ii) When $\theta=\frac{\pi}{4},\ x=\frac{\pi}{4}+\frac{1}{2},\ y=2,\ \frac{dy}{dx}=4$

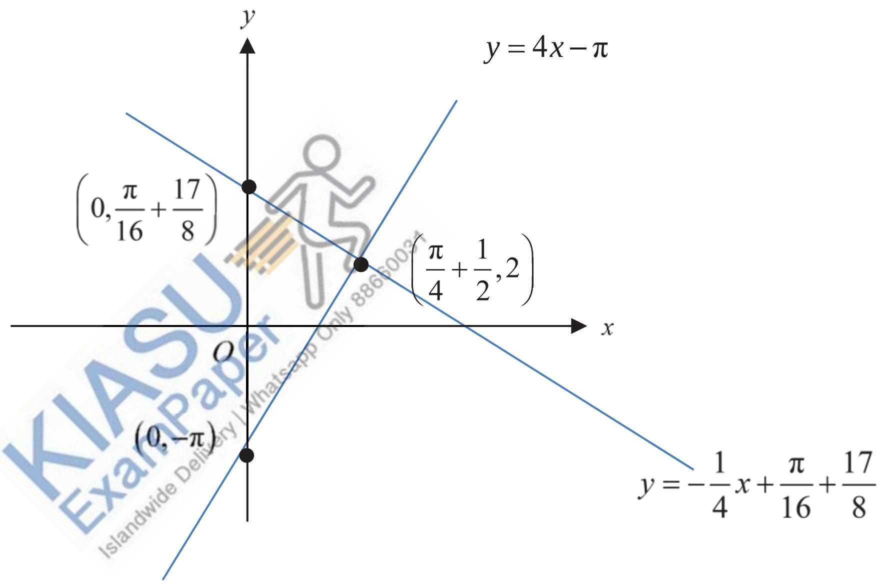
Equation of tangent : $y-2=4\left[x-\left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$

$y=4x-\pi$

Equation of normal : $y-2=-\frac{1}{4}\left[x-\left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$

$y=-\frac{1}{4}x+\frac{\pi}{16}+\frac{17}{8}$

(iii)



Coordinates of y-intercepts are $\left(0, \frac{\pi}{16} + \frac{17}{8}\right)$ and $(0, -\pi)$

Area enclosed $= \frac{1}{2}\left(\frac{\pi}{16} + \frac{17}{8} + \pi\right)\left(\frac{\pi}{4} + \frac{1}{2}\right) = 3.51$

Alternative:

Area enclosed $= \frac{1}{2}\begin{vmatrix} 0 & \frac{\pi}{4} + \frac{1}{2} & 0 & 0 \\ -\pi & 2 & \frac{\pi}{16} + \frac{17}{8} & -\pi \end{vmatrix}$

$= 3.51$ square units

7.	It is given that $y=\sin\left[\ln(1+2x)\right]$.		
	(i)	Show that $(1+2x)^2\frac{d^2y}{dx^2}+2(1+2x)\frac{dy}{dx}+ky=0$, where k is a constant to be found. Hence, find the first three non-zero terms of the Maclaurin expansion for y.	[6]
	(ii)	Using standard series from the List of Formulae (MF26), verify the correctness of your result from part (i) up to and including the term in x^3. Explain why the expansion is not valid when $x=-\frac{1}{2}$.	[3]

Suggested Solution:

(i)

$y=\sin\left[\ln(1+2x)\right]$

Differentiating with respect to x,

$\frac{dy}{dx}=\cos\left[\ln(1+2x)\right].\frac{2}{1+2x}$

$(1+2x)\frac{dy}{dx}=2\cos\left[\ln(1+2x)\right]$

Differentiating with respect to x,

$(1+2x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=-2\sin\left[\ln(1+2x)\right].\frac{2}{1+2x}$

$(1+2x)^2\frac{d^2y}{dx^2}+2(1+2x)\frac{dy}{dx}+4y=0$ (Shown)

So $k=4$

Differentiating with respect to x,

$(1+2x)^2\frac{d^3y}{dx^3}+4(1+2x)\frac{d^2y}{dx^2}+2(1+2x)\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4\frac{dy}{dx}=0$

$(1+2x)^2\frac{d^3y}{dx^3}+6(1+2x)\frac{d^2y}{dx^2}+8\frac{dy}{dx}=0$

When $x=0$,

$y=0$

$\frac{dy}{dx}=2$

$\frac{d^2y}{dx^2}+2(2)=0\Rightarrow\frac{d^2y}{dx^2}=-4$

$\frac{d^3y}{dx^3}+6(-4)+8(2)=0\Rightarrow\frac{d^3y}{dx^3}=8$

$\therefore y=0+2x-\frac{4}{2!}x^2+\frac{8}{3!}x^3+\ldots$

$= 2x - 2x^2 + \frac{4}{3}x^3 + \ldots$ (up to x^3)

(ii)
Using standard series from MF26,

$y = \sin\left[\ln(1+2x)\right]$

$= \sin\left[2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \ldots\right]$

$= \sin\left[2x - 2x^2 + \frac{8}{3}x^3 + \ldots\right]$

$= \left[2x - 2x^2 + \frac{8}{3}x^3 + \ldots\right] - \frac{1}{3!}\left[2x - 2x^2 + \frac{8}{3}x^3 + \ldots\right]^3 + \ldots$

$= 2x - 2x^2 + \frac{8}{3}x^3 + \ldots - \frac{1}{6}(2x)^3 + \ldots$

$= 2x - 2x^2 + \frac{8}{3}x^3 + \ldots - \frac{4}{3}x^3 + \ldots$

$= 2x - 2x^2 + \frac{4}{3}x^3 + \ldots$

Hence correct up to x^3 term.

When $x = -\frac{1}{2}$, $\ln(1+2x)$ is undefined.

Or,

Since expansion for $\ln(1+2x)$ is valid only for $-1 < 2x \le 1 \Rightarrow -\frac{1}{2} < x \le \frac{1}{2}$

8	**(i)**	The arithmetic progression is grouped into sets of integers, such that the n^{th} set contains n integers as shown. $\{1\},\{3, 5\},\{7, 9, 11\},\{13, 15, 17, 19\},\ldots\ldots$		
		(a)	Find the total number of terms in the first n sets, and hence show that the last term of the n^{th} set is n^2+n-1.	[2]
		(b)	Find the first term of the n^{th} set.	[2]
		(c)	Show that the sum of the terms in the n^{th} set is n^3.	[1]
	(ii)	Hence, prove that $1^3+2^3+3^3+\ldots\ldots+n^3=\frac{1}{4}n^2(n+1)^2$.		[2]

Suggested Solution:

		(a)	Find the total number of terms in the first n sets, and hence show that the last term of the n^{th} set is n^2+n-1.	[2]

$$\underbrace{\{1\}}_{\substack{1^{st}\text{ set}\\(1\text{ term})}},\underbrace{\{3, 5\}}_{\substack{2^{nd}\text{ set}\\(2\text{ terms})}},\underbrace{\{7, 9, 11\}}_{\substack{3^{rd}\text{ set}\\(3\text{ terms})}},\underbrace{\{13, 15, 17, 19\}}_{\substack{4^{th}\text{ set}\\(4\text{ terms})}},\ldots\ldots,\underbrace{\{\ldots\ldots\}}_{\substack{n^{th}\text{ set}\\(n\text{ terms})}}$$

Total number of terms in the first n sets

$=1+2+3+\ldots+n$

$=\frac{n}{2}(1+n)$

OR

Total number of terms in the first n sets

$=1+2+3+\ldots+n$

$=\frac{n}{2}\big(2(1)+(n-1)(1)\big)$

$=\frac{n}{2}(1+n)$

Last term of the n^{th} set

$=\left[\frac{n}{2}(1+n)\right]^{th}$ term of the arithmetic progression (AP),

$=1+\left[\frac{n}{2}(1+n)-1\right](2)$

$=1+n(1+n)-2$

$=n^2+n-1$

Alternatively :

Let u_k denote the k^{th} term of the AP.

$\Rightarrow u_k=1+(k-1)(2)$

$\quad=2k-1$

Last term of the n^{th} set, $u_{\frac{n}{2}(1+n)}=2\left[\frac{n}{2}(1+n)\right]-1$

$=n(1+n)-1$

$=n^2+n-1$

8	(i)	(b)	Find the first term of the n^{th} set.	[2]

Method 1a :

Last term of the $(n-1)^{th}$ set

$= (n-1)^2 + (n-1) - 1$

$= n^2 - 2n + 1 + n - 2$

$= n^2 - n - 1$

First term of the n^{th} set
= Last term of the $(n-1)^{th}$ set + 2

$= n^2 - n + 1$

Method 1b :

First term of the $(n+1)^{th}$ set
= Last term of the n^{th} set + 2

$= (n^2 + n - 1) + 2$

$= n^2 + n + 1$

First term of the n^{th} set, i.e. $[(n-1)+1]^{th}$ set

$= (n-1)^2 + (n-1) + 1$

$= n^2 - 2n + 1 + n$

$= n^2 - n + 1$

Alternative Method 2a :

First term of the n^{th} set
= Last term of the n^{th} set − $(n-1)$(common difference of A.P.)

$= (n^2 + n - 1) - (n-1)(2)$

$= n^2 - n + 1$

Alternative Method 2b :

Let a denote the first term of the n^{th} set.

$\underbrace{n^2 + n - 1}_{n^{th}\text{ (last) term of the set}} = a + (n-1)(2)$

$\Rightarrow a = n^2 + n - 1 - 2(n-1)$

$= n^2 - n + 1$

Alternative Method 3 :

Total number of terms in the first $(n-1)$ sets

$$= 1 + 2 + 3 + \ldots + (n-1)$$

$$= \frac{n-1}{2}\left(1 + (n-1)\right) = \frac{n(n-1)}{2}$$

First term of the n^{th} set

$$= \left[\frac{n}{2}(n-1) + 1\right]^{\text{th}} \text{ term of the arithmetic progression (AP),}$$

$$= 1 + \left[\frac{n}{2}(n-1) + 1 - 1\right](2)$$

$$= 1 + n(n-1)$$

$$= n^2 - n + 1$$

Alternative Method 4 : (Pattern observation – Guess and extend)

$1^2=1$; -1 , $2^2=4$; -2 , $3^2=9$; -3 , $4^2=16$

$\{\ 1\ \}, \{\ 3, 5\ \}, \{\ 7, 9, 11\ \}, \{\ 13, 15, 17, 19\ \}, \ldots\ \ldots$

1^{st} set, 2^{nd} set, 3^{rd} set, 4^{th} set

Set	First term of set
1	$1 = 1^2 - 0$
2	$3 = 2^2 - 1$
3	$7 = 3^2 - 2$
4	$13 = 4^2 - 3$
…	…
n	$n^2 - (n-1)$

$\therefore$ Last term of n^{th} set $= n^2 - n + 1$

8	(i)	(c)	Show that the sum of the terms in the n^{th} set is n^3.	[1]

Sum of the terms in the n^{th} set

$$= \underbrace{(n^2 - n + 1) + \ \ldots\ \ldots\ + (n^2 + n - 1)}_{\text{Terms of the } n^{th} \text{ set: } n \text{ terms in A.P.}}$$

$$= \frac{n}{2}\left[(n^2 - n + 1) + (n^2 + n - 1)\right]$$

$$= \frac{n}{2}\left(2n^2\right)$$

$$= n^3 \qquad \text{(shown)}$$

Alternative Method

Sum of the terms in the n^{th} set

$$= \underbrace{(n^2 - n + 1) + \ \dots\dots \ + (n^2 + n - 1)}_{\text{Terms of the } n^{th} \text{ set: } n \text{ terms in A.P.}}$$

$$= \frac{n}{2}\left[2(n^2 - n + 1) + (n-1)(2)\right]$$

$$= \frac{n}{2}\left(2n^2 - 2n + 2 + 2n - 2\right)$$

$$= n^3 \qquad \text{(shown)}$$

8	**(ii)**	Hence, prove that $1^3 + 2^3 + 3^3 + \dots\dots + n^3 = \frac{1}{4}n^2(n+1)^2$.	[2]

$$\text{LHS} = 1^3 + 2^3 + \dots\dots + n^3$$

$$= \begin{pmatrix}\text{Sum of terms}\\ \text{in the } 1^{\text{st}} \text{ set}\end{pmatrix} + \begin{pmatrix}\text{Sum of terms}\\ \text{in the } 2^{\text{nd}} \text{ set}\end{pmatrix} + \dots\dots + \begin{pmatrix}\text{Sum of terms}\\ \text{in the } n^{\text{th}} \text{ set}\end{pmatrix}, \quad \text{sum of all terms in the first } n \text{ sets}$$

$$= (1) + (3+5) + \dots\dots + \left[(n^2 - n + 1) + \dots + (n^2 + n - 1)\right]$$

$$= \underbrace{1 + 3 + 5 + \ \dots\dots \ + \dots\dots + \ \dots + (n^2 + n - 1)}_{\text{Sum of the first } \left[\frac{n}{2}(1+n)\right] \text{ terms of the A.P.}}$$

$$= ^{*}\ \frac{\left[\frac{n}{2}(1+n)\right]}{2}\left[1 + \underbrace{(n^2 + n - 1)}_{\text{last term of first } n \text{ sets}}\right]$$

$$= \frac{1}{4}n(n+1)\left[n^2 + n\right]$$

$$= \frac{1}{4}n^2(n+1)^2 \qquad = \text{RHS} \quad \text{(shown)}$$

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* OR Alternatively :

$$\dots = \frac{\left[\frac{n}{2}(1+n)\right]}{2}\left[2(1) + \left[\frac{n}{2}(1+n) - 1\right](2)\right]$$

$$= \frac{1}{4}n(n+1)\left[n(1+n)\right]$$

$$= \frac{1}{4}n^2(n+1)^2 \qquad = \text{RHS} \quad \text{(shown)}$$

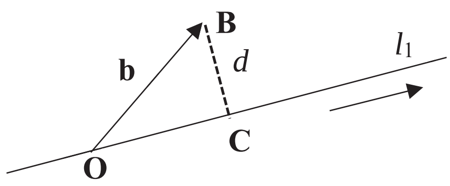
9.	With reference to the origin O, the points A and B have position vectors $\mathbf{a}=-\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{b}=2\mathbf{j}+5\mathbf{k}$ respectively.		
	(i)	Find a vector equation of the line l_1 that passes through point A and is parallel to the vector **a**.	[1]
	(ii)	Find the exact length of projection of **b** on l_1. Hence find d, the exact perpendicular distance from the point B to l_1.	[4]
	(iii)	Using the value of d found in part **(ii)**, find the position vector of the point C, the foot of perpendicular from the point B to l_1.	[3]
	(iv)	The line l_2 passes through point B and is parallel to vector **b**. Find a cartesian equation of l_3 which is the reflection of l_2 in l_1.	[3]

Suggested Solution:

(i) $l_1 : \mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + k\begin{pmatrix}-1\\1\\2\end{pmatrix} = \lambda\begin{pmatrix}-1\\1\\2\end{pmatrix},\ \lambda \in \mathbb{R}$

$l_1 : \mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + k\begin{pmatrix}-1\\1\\2\end{pmatrix},\ k \in \mathbb{R}$ or AEF

(ii) length of projection of **b** on l_1 = $\left|\begin{pmatrix}0\\2\\5\end{pmatrix}\cdot\dfrac{\begin{pmatrix}-1\\1\\2\end{pmatrix}}{\sqrt{(-1)^2+(1)^2+(2)^2}}\right|$



$= \dfrac{12}{\sqrt{6}} = 2\sqrt{6}$

$|\mathbf{b}| = \left|\begin{pmatrix}0\\2\\5\end{pmatrix}\right| = \sqrt{29}$

By Pythagoras' Theorem,

d, Perpendicular distance from B to l_1 = $\sqrt{|\mathrm{b}|^2 - \left(2\sqrt{6}\right)^2}$

$= \sqrt{29-4(6)} = \sqrt{5}$

Alternatively,
Let C be foot of perpendicular from B to l_1, using length of projection of **b** on l_1 and O lies on l_1,

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= 2\sqrt{6}\,\hat{\underset{\sim}{a}} - \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

$$= 2\sqrt{6} \times \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\left|\overrightarrow{BC}\right| = \left|\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right| = \sqrt{5}$$

(iii)
Since C lies on l_1,

$$\mathbf{c} = \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$\overrightarrow{BC} = \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -\lambda \\ \lambda - 2 \\ 2\lambda - 5 \end{pmatrix}$$

$$\left|\overrightarrow{BC}\right| = \left|\begin{pmatrix} -\lambda \\ \lambda - 2 \\ 2\lambda - 5 \end{pmatrix}\right| = \sqrt{5}$$

$$\sqrt{(-\lambda)^2 + (\lambda - 2)^2 + (2\lambda - 5)^2} = \sqrt{5}$$

$$6\lambda^2 - 24\lambda + 24 = 0$$

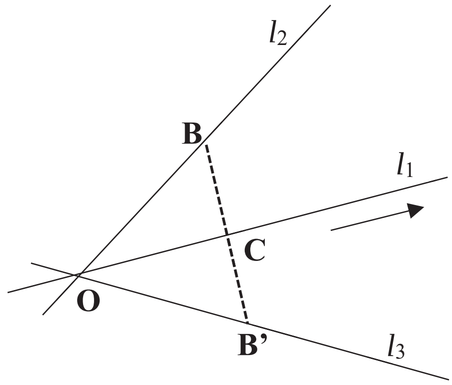
$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2$$

$$\mathbf{c} = 2\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$

(vi) $l_2 : \mathbf{r} = \mu \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix},\ \mu \in \mathbb{R}$



Let B' be the point of reflection of B about the line l_1.
By mid-point theorem,

$$\overrightarrow{OC} = \frac{\overrightarrow{OB} + \overrightarrow{OB'}}{2}$$

$$\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} + \overrightarrow{OB'}\right]$$

$$\overrightarrow{OB'} = 2\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}$$

Since l_3 parallel to $\overrightarrow{OB'}$ and the origin O lies on l_3,

$$l_3 : \mathbf{r} = \gamma \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix},\ \gamma \in \mathbb{R}$$

So Cartesian equation of l_3 is : $-\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ or $\frac{-x-4}{4} = \frac{y-2}{2} = \frac{z-3}{3}$ or AEF

10.	The curves C_1 and C_2 have equations $y=\dfrac{2x^2+9}{x^2-4}$ and $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$ respectively.		
	(i)	Find the equations of the asymptotes of the curve C_1.	[3]
	(ii)	Sketch C_1 and C_2 on the same diagram, stating the equations of any asymptotes, coordinates of any points where C_1 or C_2 crosses the axes and any turning points.	[5]
	(iii)	Find the x-coordinates of the points where the two curves intersect.	[2]
	(iv)	Hence solve the inequality $-5\sqrt{1-\frac{1}{9}x^2}\le\dfrac{2x^2+9}{x^2-4}$.	[2]

Suggested Solution:

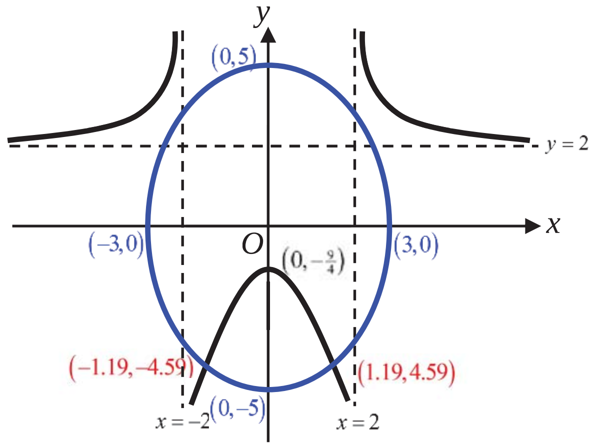
(i)

$$y=\frac{2x^2+9}{x^2-4}$$
$$=2+\frac{17}{(x+2)(x-2)}$$

$x=-2,\ x=2\ ,\ y=2$

(ii) $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$

$$\frac{y^2}{25}=1-\frac{x^2}{9}$$
$$y^2=25\left(1-\frac{x^2}{9}\right)$$
$$y=\pm5\sqrt{1-\frac{x^2}{9}}$$



(iii) From the graph,
Intersections $(-1.19,-4.59)$ and $(1.19,4.59)$
The x-coordinates are -1.19 and 1.19

(iv)

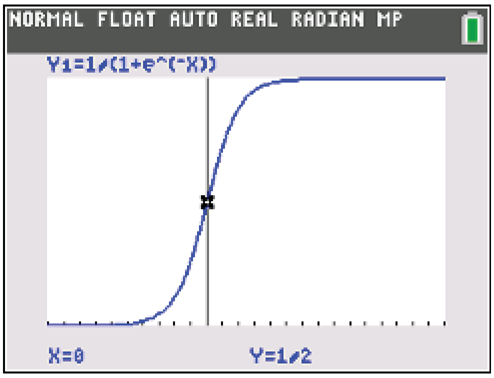
$-3\le x<-2$ or $-1.19\le x\le1.19$ or $2<x\le3$
$-1.19\le x\le1.19$; $\quad -3\le x<-2\,,2<x\le3$

11.	Sigmoid functions are used to model many natural processes such as population growth of virus. One example of a Sigmoid function f is given by $$\mathrm{f}: x \mapsto \frac{1}{1+\mathrm{e}^{-x}}, \quad x \in \mathbb{R}$$		
	(i)	Sketch the graph of $y=\mathrm{f}(x)$, indicating clearly the equation(s) of any asymptote(s) and the coordinates of any points where the curve crosses the axes.	[2]
	(ii)	Find $\mathrm{f}^{-1}(x)$ in similar form.	[3]
	Another function g is given by $\mathrm{g}: x \mapsto 3x-1, \ x \in \mathbb{R}, \ 0 \le x \le 2$.		
	(iii)	Show that fg exists and find the range of fg, expressing your answer in terms of e.	[4]
	(iv)	Describe a sequence of transformations which transform the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \mathrm{fg}(x)$.	[2]

Suggested Solution:

Solution:

(i)



(ii) Let $y=\dfrac{1}{1+e^{-x}}$

$$e^{-x}=\frac{1}{y}-1$$

Taking ln on both sides,

$$-x=\ln\left(\frac{1}{y}-1\right)$$

$$x=-\ln\left(\frac{1}{y}-1\right) \qquad \text{or equivalent}$$

$$\mathrm{f}^{-1}: x \to -\ln\left(\frac{1}{x}-1\right), x \in (0,1)$$

(iii) $R_g = [-1, 5]$ This is clearly a subset of $\mathbb{R} = D_f$

Alternatively,
If student can state that since $D_f = \mathbb{R}$, so definitely $R_g \subseteq D_f$ so composite function fg exists.
(i.e. do not need to find R_g.)

$D_{fg} = D_g = [0,2]$.
$fg(0) = f(-1) = \dfrac{1}{1+e}$ and $fg(2) = f(5) = \dfrac{1}{1+e^{-5}} = \dfrac{e^5}{e^5+1}$

Thus $R_{fg} = [\dfrac{1}{e+1}, \dfrac{1}{1+e^{-5}}]$.

(iv) First, translate f(x) 1 unit in the positive x-axis direction followed by a scaling parallel to the x-axis with factor 1/3 .
Alternatively,
Scale parallel to the x-axis with scale factor 1/3 followed by a translation of 1/3 units in the positive x-axis direction.

12. **(i)** Amanda wants to make a mask holder. To do so, she cuts out 4 identical equilateral triangles of sides 1 cm and a rectangular strip $\frac{x-1}{16}$ cm by $\frac{2y-1}{4}$ cm from a rectangle of sides x cm by y cm (see Diagram A). The rectangular strip has an area of 5.5 cm^2.

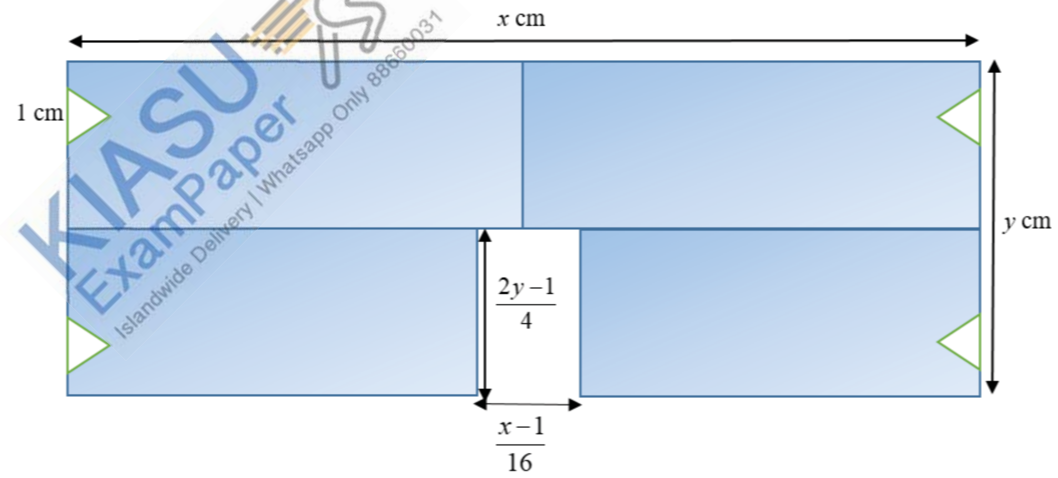
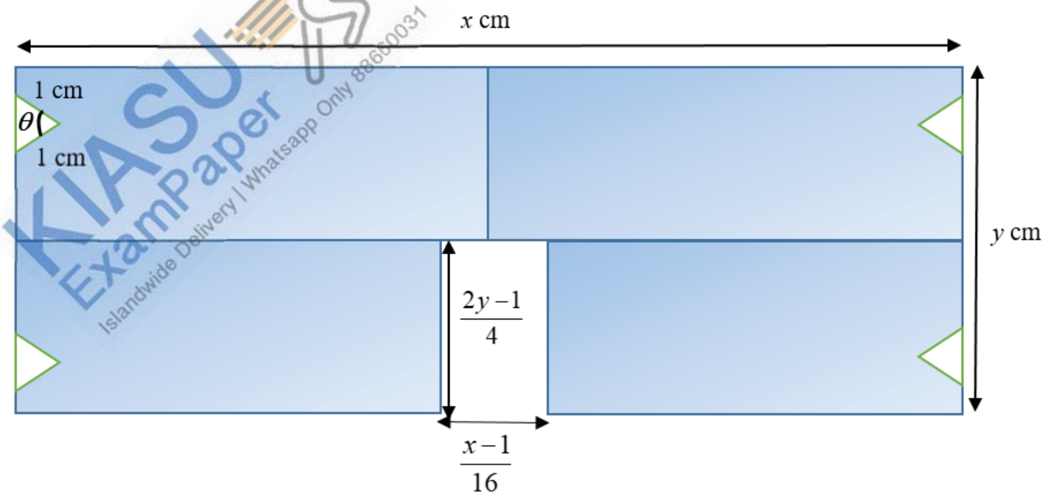


Diagram A

Show that A, the area of the mask holder is given by

$$A = \frac{176x}{x-1} + \frac{x}{2} - \sqrt{3} - \frac{11}{2}$$

Using differentiation, find the exact value of x such that area of the mask holder is a minimum. Hence determine the minimum area of the mask holder. [9]

	(ii)	The area of the rectangular strip is kept at 5.5 cm^2. Beth suggests that if the 4 identical triangle cut-outs are isosceles instead, the area of the mask holder could be made smaller as compared to the area found in part **(i)(b).** For each of the triangles, let θ be the angle between two edges of length 1 cm each (see diagram B). Determine if Beth's suggestion is correct.  Diagram B	[2]

Suggested Solution:

(i)

(a)

Area of mask holder:

$$A = xy - 4\left(\frac{1}{2}\right)(1)(1)\sin 60 - 5.5$$

$$A = xy - 4\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} - 5.5 \quad \text{--- (1)}$$

Area of rectangular strip:

$$5.5 = \frac{2y-1}{4}\left(\frac{x-1}{16}\right)$$

$$11(2)(16) = (x-1)(2y-1)$$

$$y = \frac{1}{2}\left[\frac{11(2)(16)}{x-1} + 1\right] \text{---(2)}$$

Subst (2) into (1):

$$A = x\left\{\frac{1}{2}\left[\frac{11(2)(16)}{x-1} + 1\right]\right\} - \sqrt{3} - 5.5$$

$$A = \frac{176x}{x-1} + \frac{x}{2} - \sqrt{3} - \frac{11}{2} \quad \text{(shown)}$$

$$\frac{dA}{dx} = 176x(-1)(x-1)^{-2} + (x-1)^{-1}(176) + \frac{1}{2}$$

$$\frac{dA}{dx} = \frac{-176x + 176(x-1) + \frac{1}{2}(x-1)^2}{(x-1)^2}$$

$$0 = -176 + \frac{1}{2}(x-1)^2$$

$$11(16) = \frac{1}{2}(x-1)^2$$

$$11(16)(2) = (x-1)^2$$

$$x - 1 = \pm\sqrt{11(16)(2)} = \pm 4\sqrt{22}$$

$x = 4\sqrt{22} + 1$ (since $x > 0$, reject $x = -4\sqrt{22} + 1$)

To determine if $x = 4\sqrt{22} + 1$ gives a minimum area:

$$\frac{dA}{dx} = -11(16)(x-1)^{-2} + \frac{1}{2}$$

$$\frac{d^2A}{dx^2} = -11(16)(-2)(x-1)^{-3}$$

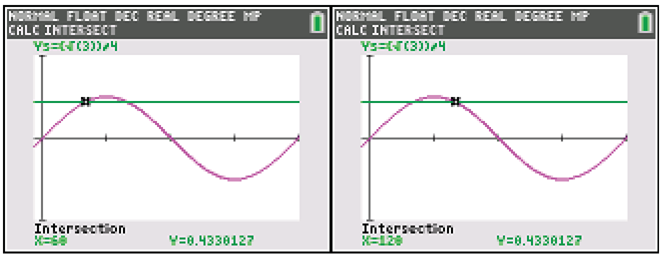
When $x = 4\sqrt{22} + 1$, $\frac{d^2A}{dx^2} = 11(16)(2)\left(4\sqrt{22}\right)^{-3} > 0$, thus $x = 4\sqrt{22} + 1$ gives a minimum value.

Thus, area $A = \frac{176\left(4\sqrt{22}+1\right)}{\left(4\sqrt{22}+1\right)-1} + \frac{\left(4\sqrt{22}+1\right)}{2} - \sqrt{3} - \frac{11}{2} = 188\text{ cm}^2$

(ii)

Area of 1 triangle = $A = \frac{1}{2}(1)(1)\sin\theta$

And when $\theta = 60°$, $A = \frac{1}{2}(1)(1)\sin 60° = \frac{\sqrt{3}}{4}$



So for any value $60° < \theta < 120°$, it will give a bigger value of A than when it is an equilateral triangle.

And so, the area of the face mask holder will be a minimum (with the area of the rectangular strip remaining at 5.5 cm^2) if the areas of the 4 triangles cut-outs are a maximum, thus Beth's suggestion is correct.

DHS JC1 Promo 2021

1. By using the substitution $u = e^x$, find $\int e^x \sin(e^x)\sin(3e^x)\,dx$. [4]

2. (i) On the same axes, sketch the graphs of $y = (x-a)^2$ and $y = |x + a^2|$, where a is a positive constant. Label all axial intercepts clearly. [3]

   (ii) Hence, or otherwise, solve the inequality $(x-a)^2 \geq |x + a^2|$, leaving your answers in terms of a. [3]

3. It is given that $x\dfrac{dy}{dx} = 2y - 8$.

   (i) Using the substitution $y = ux^2$ or otherwise, solve the differential equation giving your answer in the form $y = f(x)$. [4]

   (ii) Given that $y > 3$ when $x = 1$, show that $y > 4 - x^2$ for $x \neq 0$. [2]

4. The first three terms of an infinite series are 16, x, 9.

   (i) Find the value(s) of x, if the series is

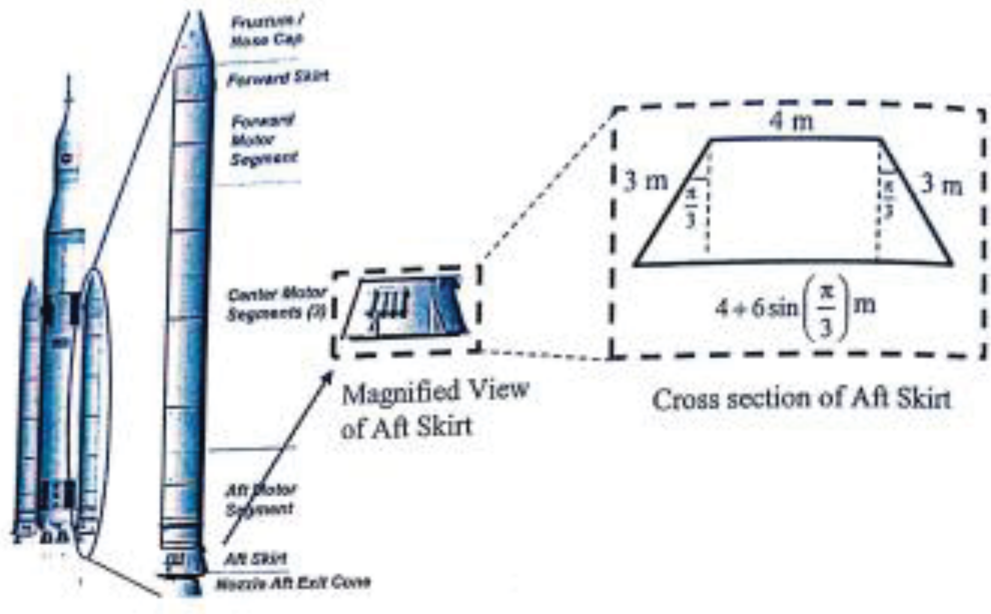
   (a) geometric,

   (b) arithmetic. [3]

   (ii) If all the terms in part **(i)(a)** are positive, calculate its sum to infinity. [2]

   (iii) Given that the sum of the first n terms in part **(i)(b)** is less than -64, find the least value of n. [3]

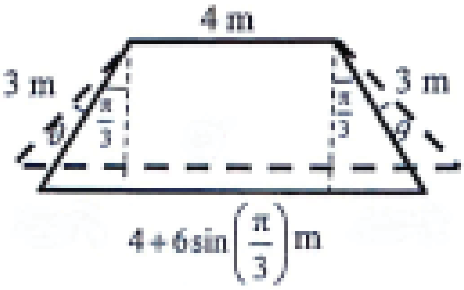
5. A famous entrepreneur, Elon Tusk, has designed a new rocket booster for his company SpaceY. A rocket booster consists of the following parts as shown in the following diagram.



Rocket Booster

The design of the Aft Skirt can be viewed using a cross sectional view with the stated dimensions (see dotted box above).

Due to the amount of heat and pressure generated from the thrust of the rocket during lift off, the sides of the Aft Skirt will tilt **outward** by a **very small angle**, θ, while keeping the slant length at 3 m (see diagram below).



Before the lift off, the diameter at the bottom of the Aft Skirt is given by $\left(4+6\sin\left(\frac{\pi}{3}\right)\right)$ m. In order for the booster to function properly during the lift off, the diameter at the bottom of the Aft Skirt must be less than 9.197 m.

Given that θ is a sufficiently small angle, find the range of values of θ. [5]

6. The functions f and g are defined by

$$f : x \mapsto x^2 - 4x + 1, \quad x \le k,$$
$$g : x \mapsto \ln(x+5), \quad x > -5.$$

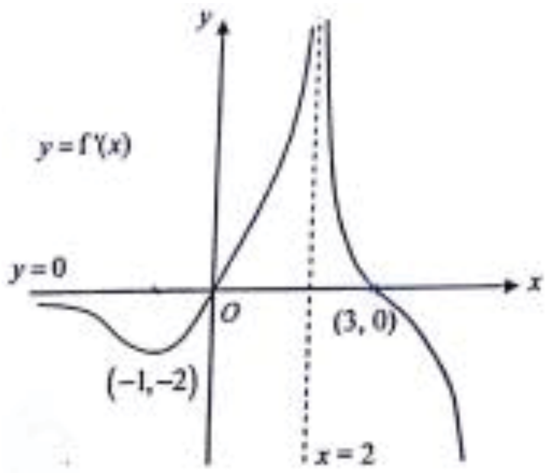
(i) Determine the largest value of k such that f^{-1} exists. Hence find $f^{-1}(x)$ and state its domain. [4]

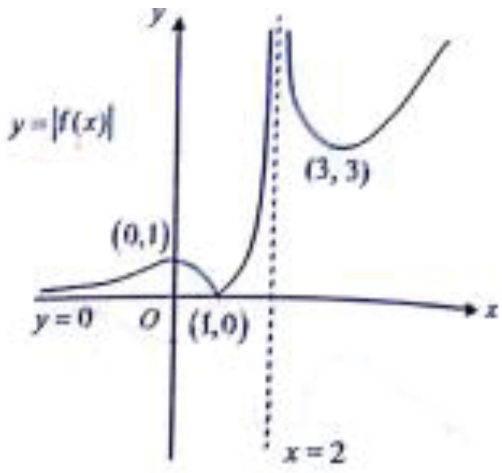
For the rest of the question, the domain of f is as found in part **(i)**.

(ii) Show that the composite function gf exists and find the exact range of gf. [3]

(iii) A function h is a decreasing function. Given that hf exists, show that hf is an increasing function. [2]

7. The graphs of $y = f'(x)$ and $y = |f(x)|$ are shown below.





(i) State the nature of all stationary points of the graph of $y = f(x)$. [2]

(ii) State the range of values of x where

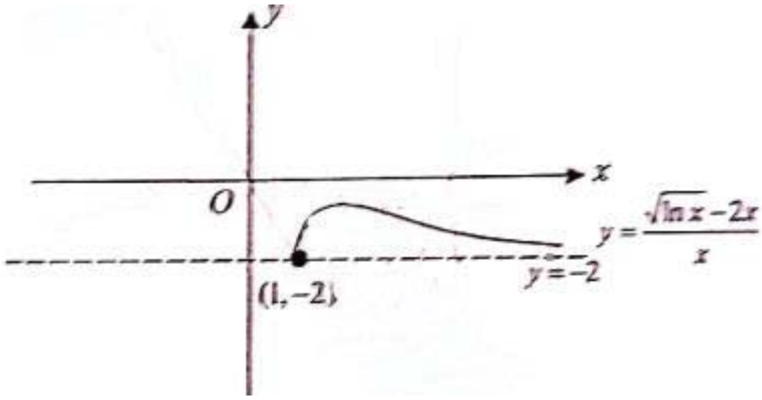
(a) f is decreasing, [2]

(b) f is concave upwards. [1]

(iii) Sketch the graph of $y = f(x)$ indicating clearly the asymptotes, coordinates of the stationary points and the intersections with the axes. [3]

8. (i) Using integration by parts, show that $\int \frac{1}{x^2}\ln x\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + \beta$, where β is an arbitrary constant. [2]

The graph of the curve C with equation $y = \frac{\sqrt{\ln x} - 2x}{x}, x \geq 1$ is as shown below. The line $y = -2$ is its horizontal asymptote.



(ii) Find the area of the region enclosed by C, the line $x = 4$ and the line $y = -2x$. [3]

(iii) The region A is enclosed by C and the lines $x = 4$ and $y = -2$. By using a suitable transformation, show that the exact volume when A is rotated about the line $y = -2$ through 2π radians is $\frac{\pi}{4}(p\ln 2 + q)$, where p and q are constants to be determined. [4]

9. A curve C has parametric equations

$$x = a(t - \sin t), \qquad y = a(1 - \cos t) \text{ for } \frac{\pi}{2} \leq t < 2\pi,$$

where a is a positive constant.

The curve C has a maximum point at $(\pi a, 2a)$.

(i) Sketch C. Label, in terms of a, the coordinates of the turning point of C and the point where C meets the x-axis. [2]

The point P on C has parameter p and the normal to C at P cuts the x-axis at $\left(\frac{3}{2}a\pi, 0\right)$.

(ii) Find, in terms of a, the exact coordinates of P. [4]

(iii) Find, in terms of a, the exact area of the region enclosed by C, the normal to C at P and the x-axis. [4]

10. (i) Find $\sum_{r=1}^{n}\left(\frac{1}{2}\right)^{r}$ in terms of n. [2]

A sequence is such that $u_0 = 3$ and $u_r = u_{r-1} + \left(\frac{1}{2}\right)^{r}$ for $r \geq 1$.

(ii) By forming equations from $r = 1$ to $r = n$ and using the method of differences, find u_n in terms of n. [3]

(iii) Show that $S = 4n - 2 + 2\left(\frac{1}{2}\right)^{n}$, where S denotes the sum of the first n terms of the sequence. [3]

(iv) Determine, with reason, whether

(a) u_n converges, [1]

(b) S converges. [1]

11. The lengths of the sides of a triangular plot of land are $(x+3)$ m, $(x+3)$ m and $(10-2x)$ m respectively.

(i) The triangle inequality states that for any triangle, the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side. Explain why $1 < x < 5$. [2]

(ii) Show that the area, A m^2, of the plot of land is given by $A = 4(5-x)\sqrt{x-1}$. [3]

(iii) Use differentiation to find the exact maximum value of A, proving that it is a maximum. [4]

(iv) A garden in the shape of a square is to be built on the land of maximum area in part **(ii)**. All four corners of the garden lie on the border of the land. Find the area of the square. [3]

12. A fish pond has a surface area of 100 m^2. Past observations on the growth of algae in similar ponds estimates that the area, A m^2, of algae present at time t days is such that the rate at which A is increasing is proportional to the product of the area of the pond covered by the algae and the area of the pond not covered by the algae.

The owner of the pond monitors the growth of algae in his pond. At $t = 2$, he noted that the algae had grown to cover 11% of the pond and was growing at a rate of 11 m^2 per day. He then decided to clear the algae at a constant rate of 11 m^2 each day, starting from $t = 3$.

(i) Show that, from $t = 3$, the differential equation relating A and t is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{1}{89}(A-11)(A-89).$$ [2]

(ii) Assuming that $A = 20$ when $t = 3$, show that $\dfrac{A-89}{A-11} = -\dfrac{23}{3}\mathrm{e}^{q(3-t)}$, where q is a positive constant to be determined. [6]

(iii) To maintain the balance of the ecosystem in the pond, the owner wishes to ensure that algae always covers an area less than 80% of the pond surface. Explain why this cannot be attained. [2]

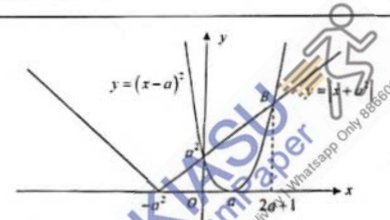
(iv) Find the value of t when the maximum possible rate of change in the area of algae occurs. [2]

(v) State one possible condition in which the model in part **(i)** is not valid. [1]

2021 Year 5 H2 Math Promotional Examination

Solution & Comments

Qn	Suggested Solution	Comments / Common Mistakes
1	$\int e^x \sin(e^x)\sin(3e^x)\,dx$ $= \int u\sin(u)\sin(3u)\,\frac{1}{u}\,du$ $= \int \sin(3u)\sin(u)\,du$ $= -\frac{1}{2}\int \cos(4u)-\cos(2u)\,du$ $= -\frac{1}{2}\left(\frac{1}{4}\sin(4e^x)-\frac{1}{2}\sin(2e^x)\right)+C$ $= -\frac{1}{8}\sin(4e^x)+\frac{1}{4}\sin(2e^x)+C$ $u = e^x,\ \frac{du}{dx} = e^x = u$ $\therefore dx = \frac{1}{u}du$	• Replace integral in terms of u • Apply factor formula rather than using integration by parts • Remember the arbitrary constant C since it is an indefinite integral • Final answer in terms of x
		Total marks : 4

Qn	Suggested Solution	Comments / Common Mistakes
2(i)		• The two graphs intersect at the same point on the y-axis, and also at B as shown.
(ii)	To find the x-coordinate of B, we will solve the following equation: $(x-a)^2 = x+a^2$ $x^2-2ax+a^2 = x+a^2$ $x(x-2a-1)=0$ $x=0$ or $x=2a+1$ Hence to solve $(x-a)^2 \geq \|x+a^2\|$ $\Rightarrow x\leq 0$ or $x \geq 2a+1$	• At B, $\|x+a^2\| = x+a^2$. • Find the x-coordinates of the pertinent intersection points, and deduce the solution based on the graph.
		Total marks : 6

1

Qn	Suggested Solution	Comments / Common Mistakes
3(i)	$y = ux^2$ $\frac{dy}{dx} = 2ux + x^2\frac{du}{dx}$ $x\frac{dy}{dx} = 2ux^2 + x^3\frac{du}{dx}$ $2ux^2 - 8 = 2ux^2 + x^3\frac{du}{dx}$ $\frac{du}{dx} = -\frac{8}{x^3}$ $u = 4x^{-2}+C$ $\frac{y}{x^2} = 4x^{-2}+C$ $y = 4+Cx^2$ where C is an arbitrary constant	• Note that u, x and y are variables in $y=ux^2$ • $\frac{dy}{dx}$ and y (in terms of u and x) need to be substituted into $x\frac{dy}{dx} = 2y-8$ to obtain an equation in u and x only • Some have missed out the arbitrary constant in the integration of $\frac{du}{dx} = -\frac{8}{x^3}$ • The final answer must be in the form of $y = f(x)$ as required by the question
(ii)	Since $y>3$ when $x=1$, $4+C>3 \Rightarrow C>-1$ For $x\neq 0, x^2>0$ $\therefore Cx^2 > -x^2$ $4+Cx^2 > 4-x^2$ $\Rightarrow y > 4-x^2$ for $x \neq 0$ (shown)	• As the condition given here is that $y>3$ when $x=1$, the general solution $y=4+Cx^2$ must be considered to show that $y>4-x^2$ for $x\neq 0$.
		Total marks: 6

Qn	Suggested Solution	Comments / Common Mistakes
4(i)	(a) $\frac{x}{16} = \frac{9}{x}$ $x^2 = 144$ $x = \pm 12$ OR $a = 16$ $ar = x$ $ar^2 = 9$ $x = \pm\sqrt{16\times 9} = \pm 12$ (b) $x - 16 = 9 - x$ $x = 12.5$	• Students should recognise that information in part (ii) may not necessarily apply to part (i). For part (i)(a), many students made the wrong assumption that x or r (common ratio) must be positive hence missed out on one solution. • Note that the question asked for values of x and not d or r.
(ii)	$S_\infty = \frac{16}{1-\frac{3}{4}} = 64$	• Students should apply $S_\infty = \frac{\text{first term}}{1-\text{ratio}}$ formula for GP sum • Some students mistook $U_1 = 9$, $U_2 = x$, $U_3 = 12$, hence taking the common ratio $r = \frac{4}{3}$ instead (then sum to infinity won't exist)

2

(iii)

Let the number of terms be n.

$\frac{n}{2}[2(16)+(n-1)(-3.5)]<-64$

$n[32-3.5n+3.5]<-128$

$3.5n^2-35.5n-128>0$

$n<-2.82$ or $n>12.96$ OR

n	S_n+64	
12	25	>0
13	-1	<0

From GC, least $n=13$

Comments / Common Mistakes:

- Common mistake observed is students applying wrong AP sum formula or wrong common difference
- Note exact answers are not required hence students can use GC to solve the inequality graphically or by table
- Students are reminded that the least value of $n=13$ is obtained from the solution to the inequality $n>12.96$, and not just derived by rounding up $n=12.96$ to the nearest integer.

Total marks: 8

Qn	Suggested Solution	Comments / Common Mistakes
5		

Suggested Solution

Diameter at the bottom of the Aft Skirt

$=4+2\times3\sin\left(\frac{\pi}{3}+\theta\right)$

$4+6\sin\left(\frac{\pi}{3}+\theta\right)<9.197$

$\sin\left(\frac{\pi}{3}+\theta\right)<0.86617$

$\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta<0.86617$

For small θ,

$\frac{\sqrt{3}}{2}\left(1-\frac{\theta^2}{2}\right)+\frac{1}{2}\theta<0.86617$

$-\frac{\sqrt{3}}{4}\theta^2+\frac{1}{2}\theta+\frac{\sqrt{3}}{2}-0.86617<0$

Using GC,

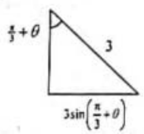
$\theta<0.000289$ or $\theta>1.154411$ (6dp)

Since $\theta>0$ and it is small,

$0<\theta<0.000289$ (6d.p)

Comments / Common Mistakes

- It was common for students to misinterpret the meaning of "diameter" in this question. Diameter refers to the whole length of the base of the skirt.
- Note that only the slant length remains unchanged at 3 m when the skirt opens.



- Most students were able to apply the compound angle formula from MF26 and considered small θ to apply the formulas: $\sin\theta\approx\theta$ and $\cos\theta\approx1-\frac{\theta^2}{2}$.
- Since the skirt opens
  1. by a small angle, rej. $\theta>1.554411$
  2. outward, $\theta>0$.

Total marks : 5

3

Qn	Suggested Solution	Comments / Common Mistakes
6(i)		

6(i) Largest $k=2$

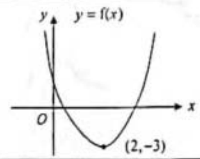
$y=x^2-4x+1$

$=(x-2)^2-3$

$x=2\pm\sqrt{y+3}$

$=2-\sqrt{y+3}$ $(\because x\le2)$

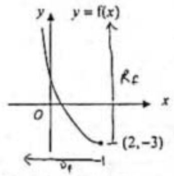
$f^{-1}(x)=2-\sqrt{x+3}, x\ge-3$

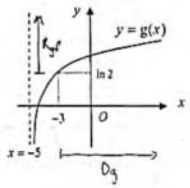


Comments:

- Remember to write "Largest"
- Since $x\le2$, we need to reject $x=2+\sqrt{y+3}$

(ii) Since $R_f=[-3,\infty)\subseteq(-5,\infty)=D_g$, thus gf exists.





Using the graphs of f and g,

$(-\infty,2]\xrightarrow{f}[-3,\infty)\xrightarrow{g}[\ln2,\infty)$

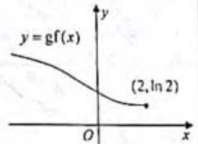
D_f $\quad R_f=D_g$ $\quad R_{gf}$

$\therefore R_{gf}=[\ln2,\infty)$

Alternative

$gf(x)=\ln(x^2-4x+6)$

From the sketch, $R_{gf}=[\ln2,\infty)$



Comments:

- When finding R_{gf}, students should show EITHER : the mapping method OR sketch the graph of $y=gf(x)$
- Note : $D_{gf}=D_f=(-\infty,2]$

(iii) If $x_2>x_1$ and f is decreasing $\Rightarrow f(x_2)<f(x_1)$

Since h is decreasing also, $\Rightarrow hf(x_2)>hf(x_1)$

$\therefore$hf is an increasing function.

Alternative

Let $u=f(x)$

$\frac{d}{dx}h[f(x)]=\frac{d}{du}[h(u)]\times\frac{du}{dx}$

$=h'(u)\times f'(x)$

Since both derivatives are negative $\Rightarrow\frac{d}{dx}h[f(x)]>0$

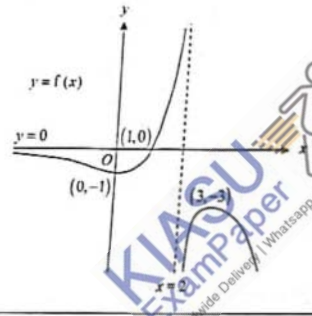
$\therefore$hf(x) is an increasing function.

Comments:

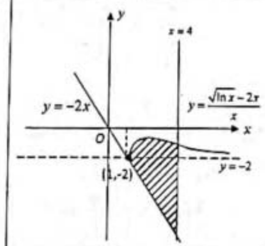
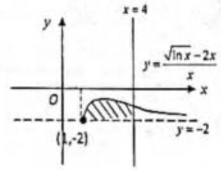
- Refer to Chapter 6, pg 2 notes to recap definition of increasing/decreasing functions
- For the Alternative method, remember to apply the chain rule correctly as shown

Total marks : 9

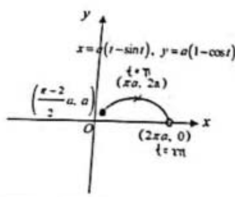
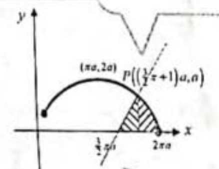
4

Qn	Suggested Solution	Comments / Common Mistakes
7(i)	Max pt = $(3,-3)$, Min pt = $(0,-1)$	• Besides stating the nature of the stationery points, the coordinates/x-values are also required.
(ii)	a) f is decreasing: $x \le 0$ or $x \ge 3$ b) f is concave upwards: $-1 \le x < 2$	Refer to the graph of $y = f'(x)$ • a) f is decreasing $\Rightarrow f'(x) < 0$ $\Rightarrow x \le 0$ or $x \ge 3$. The term **"and"** should not be used as it means union of both inequalities which is $\varnothing$ in this case. • b) f is concave upwards $\Rightarrow f''(x) > 0$ where the gradient continually increases $\Rightarrow -1 \le x < 2$
(iii)		• Remember to label the key features as required by the question. • Due to their different y-values, the minimum pt $(0,-1)$ should be **higher** than the maximum pt $(3,-3)$.
		Total marks : 8

5

Qn	Suggested Solution	Comments / Common Mistakes
8(i)	$\int \frac{1}{x^2}\ln x \, dx$ $= -\frac{1}{x}\ln x - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx$ $= -\frac{1}{x}\ln x - \frac{1}{x} + B$ (shown) $u = \ln x$, $\frac{dv}{dx} = \frac{1}{x^2}$ $\frac{du}{dx} = \frac{1}{x}$, $v = -\frac{1}{x}$	• It's better to quote the "integration by parts" first, then simplify.
(ii)	The curve C and line $y = -2x$ intersect at (1, -2). Area enclosed $= \int_1^4 \frac{\sqrt{\ln x} - 2x}{x} - (-2x) dx$ $= 10.088$ $= 10.1$ (3 s.f.)	• Note that $\int_1^4 \frac{\sqrt{\ln x} - 2x}{x} dx$ gives a negative value and does not give the required correct shaded region as shown. • Required area $= \int_1^4 (y_{higher} - y_{lower}) dx$ • GC can be used to evaluate the answer
(iii)	$y = \frac{\sqrt{\ln x} - 2x}{x} = \frac{\sqrt{\ln x}}{x} - 2$. Translate 2 units in the positive y-direction. Hence equation of curve becomes $y = \frac{\sqrt{\ln x}}{x}$. Required volume $= \pi \int_1^4 \left(\frac{\sqrt{\ln x}}{x}\right)^2 dx$ $= \pi \int_1^4 \frac{1}{x^2} \ln x \, dx$ $= \pi \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_1^4$ $= \frac{\pi}{4}(-2\ln 2 + 3)$ $p = -2, q = 3$ 	• The expression should be simplified first, before applying part (i)'s result.
		Total marks: 9

6

Qn	Suggested Solution	Comments / Common Mistakes
9(i)		Concept: *one value of* $t \Leftrightarrow (x,y)$ *one pt on curve*. Take note that $\frac{\pi}{2} \le t < 2\pi$, therefore the curve is only a portion of it. The end pts must be indicated by coordinates. The end pts must be indicated by a solid circle (pt is included) / hollow circle (pt is not included).
(ii)	$\frac{dx}{dt} = a(1-\cos t)$, $\frac{dy}{dt} = a\sin t$ $\frac{dy}{dx} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t}$ $\left.\frac{dy}{dx}\right\|_{t=p} = \frac{\sin p}{1-\cos p}$ Gradient of normal at $P = -\frac{1-\cos p}{\sin p} = \frac{\cos p - 1}{\sin p}$ Since the normal at P passes through points $\left(\frac{3}{2}a\pi, 0\right)$ and $\left(a(p-\sin p), a(1-\cos p)\right)$, the gradient between these points must be the same as above. $\frac{0-a(1-\cos p)}{\left(\frac{3}{2}a\pi - a(p-\sin p)\right)} = \frac{\cos p - 1}{\sin p}$ $a(\cos p - 1)\sin p = (\cos p - 1)\left(\frac{3}{2}a\pi - a(p-\sin p)\right)$ Since $(\cos p - 1) \ne 0$, $a\sin p = \left(\frac{3}{2}a\pi - a(p-\sin p)\right)$ $a\sin p = \frac{3}{2}a\pi - ap + a\sin p$ $\frac{3}{2}a\pi - ap = 0$ $\frac{3}{2}\pi - p = 0$ $p = \frac{3}{2}\pi$ $x = a\left(\frac{3}{2}\pi - \sin\left(\frac{3}{2}\pi\right)\right) = \left(\frac{3}{2}\pi + 1\right)a$ $y = a\left(1-\cos\left(\frac{3}{2}\pi\right)\right) = a$ $\therefore$ Coordinates of P is $\left(\left(\frac{3}{2}\pi+1\right)a, a\right)$	Take note that p is a constant, therefore it does not make sense when you write $\frac{dx}{dp}$ and $\frac{dy}{dp}$ Concepts used: 1. Chain rule: $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)$ 2. [grad (normal)][grad (tangent)] = -1 3. Finding gradient between 2 pts. $\cos p = 1$ $p = 0$ or 2π Which is not in the domain. $\left(\frac{3}{2}\pi + 1\right) > \pi$ Pt P is on the "right" of max pt.
(iii)	Area enclosed $= \frac{1}{2}(a)(a) + \int_{\left(\frac{3}{2}\pi+1\right)a}^{2\pi a} y \, dx$ $= \frac{1}{2}a^2 + \int_{\frac{3}{2}\pi}^{2\pi} a(1-\cos t)\cdot a(1-\cos t)\, dt$ $= \frac{1}{2}a^2 + a^2\int_{\frac{3}{2}\pi}^{2\pi} (1-\cos t)^2\, dt$ $= \frac{1}{2}a^2 + a^2\int_{\frac{3}{2}\pi}^{2\pi} 1 + \cos^2 t - 2\cos t\, dt$ $= \frac{1}{2}a^2 + a^2\int_{\frac{3}{2}\pi}^{2\pi} 1 + \frac{1+\cos 2t}{2} - 2\cos t\, dt$ $= \frac{1}{2}a^2 + a^2\left[\frac{3}{2}t + \frac{\sin 2t}{4} - 2\sin t\right]_{\frac{3}{2}\pi}^{2\pi}$ $= \frac{1}{2}a^2 + a^2\left[3\pi - \left(\frac{9}{4}\pi + 2\right)\right]$ $= a^2\left(\frac{3}{4}\pi - \frac{3}{2}\right)$	 Use a diagram to help you to determine the approach of finding the area. Note that you can find the area of triangle (blue) + area under the curve (red). $\int \cos^2 t\, dt \ne \frac{\cos^3 t}{3} + c$ If you differentiate RHS, you will not get the integrand on the LHS.

7

8

		Concepts used: 1. Area = $\int y\,dx$ 2. Integration by substitution method from parametric form. 3. Using double angle formula to integrate $\int \cos^2 t\,dt$
		Total marks : 10

Qn	Suggested Solution	Comments / Common Mistakes
10(i)	$\sum_{r=1}^{n}\left(\frac{1}{2}\right)^r = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^n$ $= \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $= 1-\left(\frac{1}{2}\right)^n$	Concepts used: 1. Sum of GP = $\frac{a(1-r^n)}{1-r}$
(ii)	Rewriting $u_r - u_{r-1} = \left(\frac{1}{2}\right)^r$ When $r=1$, $u_1 - u_0 = \left(\frac{1}{2}\right)^1$ When $r=2$, $u_2 - u_1 = \left(\frac{1}{2}\right)^2$ When $r=3$, $u_3 - u_2 = \left(\frac{1}{2}\right)^3$ … When $r=n-1$, $u_{n-1} - u_{n-2} = \left(\frac{1}{2}\right)^{n-1}$ When $r=n$, $u_n - u_{n-1} = \left(\frac{1}{2}\right)^n$ Sum up these equations, $u_n - u_0 = \sum_{r=1}^{n}\left(\frac{1}{2}\right)^r$ $\therefore u_n = 3+1-\left(\frac{1}{2}\right)^n = 4-\left(\frac{1}{2}\right)^n$	Concepts used: 1. Observe pattern to perform method of differences.

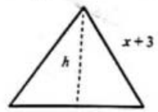
9

(iii)	$S = \sum_{r=0}^{n-1} u_r = \sum_{r=0}^{n-1}\left(4-\left(\frac{1}{2}\right)^r\right)$ $= 4n - \sum_{r=0}^{n-1}\left(\frac{1}{2}\right)^r$ $= 4n - \frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\left(\frac{1}{2}\right)}$ $= 4n-2+2\left(\frac{1}{2}\right)^n$ (shown) $\sum_{r=0}^{n-1} u_r \neq \sum_{r=0}^{n-1} u_n$ $u_0 + u_1 + \ldots + u_{n-1} \neq u_n + u_n + \ldots + u_n$	Concepts used: 1. Sum of GP = $\frac{a(1-r^n)}{1-r}$ 2. Understand that the first n terms of this sequence starts from u_0 to u_{n-1}.
(iv)	When $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$. (a) $u_n \to 4$ (constant), u_n converges. (b) $S \to \infty$, S does not converge.	Some students used: When $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$ From $u_n - u_{n-1} = \left(\frac{1}{2}\right)^n$, $u_n \to u_{n-1}$ This reasoning is **insufficient** to say that u_n will converge. Some counter examples that has this behaviour but u_n will not converge. If $u_n = \sqrt{n}$, $u_n - u_{n-1} = \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n}+\sqrt{n-1}}$ As $n \to \infty$, $u_n \to u_{n-1}$ but u_n is not a convergent sequence. **In summary, you need to obtain $u_n = f(n)$ and use $n \to \infty$ to check for convergence of the sequence.**
		Total marks : 10

10

Qn	Suggested Solution	Comments / Common Mistakes
11(i)	$10-2x>0 \Rightarrow x<5$ .... (1) $(x+3)+(x+3)\geq 10-2x \Rightarrow x\geq 1$ But $x=1$ $\Rightarrow$ the vertices of the 'triangle' are collinear $\Rightarrow$ Area of triangle = 0 $\therefore x>1$ .... (2) From (1) & (2): $1<x<5$ (shown)	• The triangle inequality states that $a+b\geq c$ for a triangle with sides a, b and c, which includes a triangle with zero area. • In this context, the triangle is a plot of land. Hence all sides and the area must be positive.

(ii) Let the height of the isosceles triangle be h m.



$h^2=(x+3)^2-(5-x)^2=16x-16$

$h=4\sqrt{x-1}$

$A=\frac{1}{2}(10-2x)(4\sqrt{x-1})=4(5-x)\sqrt{x-1}$ (shown)

Comments:
- Observing that it is an isosceles triangle in this case, using Area $=\frac{1}{2}\times base \times height$ is an efficient method to show the required expression.

(iii) $\frac{dA}{dx}=4[(5-x)(0.5)(x-1)^{-0.5}+(x-1)^{0.5}(-1)]$

For maximum A, $\frac{dA}{dx}=0$

$\therefore \frac{(5-x)}{2(x-1)^{0.5}}=(x-1)^{0.5}$

$(5-x)=2(x-1)$

$x=\frac{7}{3}$

x	$\left(\frac{7}{3}\right)^-$	$\frac{7}{3}$	$\left(\frac{7}{3}\right)^+$
$\frac{dA}{dx}$	+	0	–

Hence $x=\frac{7}{3}$ gives a maximum A.

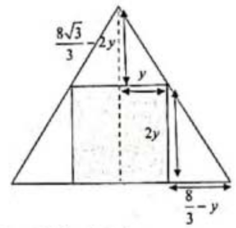
Maximum $A=4\left(5-\frac{7}{3}\right)\sqrt{\frac{7}{3}-1}$

$=4\left(\frac{8}{3}\right)\sqrt{\frac{4}{3}}=\frac{64\sqrt{3}}{9}$

Comments:
- To find the value of x for maximum A, equate $\frac{dA}{dx}$ to zero.
- To prove that this value of x gives a maximum A, the sign test (using $\frac{dA}{dx}$) can be used. Alternatively, $\frac{d^2A}{dx^2}=\frac{-3x-1}{(x-1)^{1.5}}=-5.20$ when $x=\frac{7}{3}$ can also be used.
- Do not write "it gives a max A as the marker would not know what it is".
- The exact maximum value of A needs to be given in the simplest form.

(iv) With $x=\frac{7}{3}$, the triangle is equilateral with sides $\frac{16}{3}$ m.

Let the length of the inscribed square be $(2y)$ m as shown below:



From similar triangles,

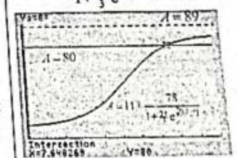
$\frac{y}{\frac{8\sqrt{3}}{3}-2y}=\frac{\frac{8}{3}-y}{2y}$

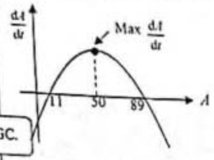
Using GC, $y=1.2376$

Area of the square $=(2y)^2=6.13\ m^2$ (correct to 3 s.f.)

Comments:
- The square is inscribed in this equilateral triangle. Note that the top corners of the square (in the diagram) are **not** the mid-points of the slant sides of the triangle.
- The length of the required square is expressed as $2y$ for ease of calculation.
- Besides using similar triangles, an efficient method is to use $\tan 60° = \frac{2y}{\frac{8}{3}-y}$ to calculate the value of y.
- Other possible methods include the consideration of the total area or the length of the side of the triangle.

Total marks: 12

Qn	Suggested Solution	Comments / Common Mistakes
12(i)	A: the area (in m^2) of algae present at time t days. For $0 \le t < 3$, $\frac{dA}{dt} = kA(100-A)$ $t=2,\ \frac{dA}{dt}=11,\ A=11:\ 11 = k(11)(100-11)$ $k = \frac{1}{89}$ For $t \ge 3$, $\frac{dA}{dt} = \frac{1}{89}A(100-A) - 11$ $= \frac{1}{89}(100A - A^2 - 979)$ $= -\frac{1}{89}(A^2 - 100A + 979)$ $= -\frac{1}{89}(A-11)(A-89)$ (shown) [Working needed]	• Algae growing at the rate of $11\ m^2/\text{day} \Rightarrow \frac{dA}{dt} = 11$ • Note that $\frac{dA}{dt}$ is a piecewise function in A.
(ii)	$\frac{dA}{dt} = -\frac{1}{89}(A-11)(A-89)$ $\int \frac{1}{(A-11)(A-89)}\,dA = -\int \frac{1}{89}\,dt$ $\frac{1}{78}\int -\frac{1}{A-11} + \frac{1}{A-89}\,dA = -\int \frac{1}{89}\,dt$ $\frac{1}{78}\ln\left\lvert\frac{A-89}{A-11}\right\rvert = -\frac{1}{89}t + c$ $\ln\left\lvert\frac{A-89}{A-11}\right\rvert = -\frac{78}{89}t + 78c$ $\frac{A-89}{A-11} = \pm e^{78c} e^{-\frac{78}{89}t} = Be^{-\frac{78}{89}t}$ where $B(=\pm e^{78c})$ is an arbitrary constant. $A=20,\ t=3:\ B = -\frac{23}{3}e^{3(\frac{78}{89})}$ $\frac{A-89}{A-11} = -\frac{23}{3}e^{3(\frac{78}{89})}e^{-\frac{78}{89}t}$ $\frac{A-89}{A-11} = -\frac{23}{3}e^{\frac{78}{89}(3-t)}$, where $q = \frac{78}{89}$	Modulus needed since the sign of $\frac{A-89}{A-11}$ is not stated in the question. Usual rules of calculation apply once c is introduced. $t=3, A=20$ is the <u>only</u> boundary condition where this model is valid ($t \ge 3$). Without the modulus for $\ln\left\lvert\frac{A-89}{A-11}\right\rvert$, you get $e^{78c} = -\frac{23}{3}e^{3(\frac{78}{89})}$ but $e^{78c} > 0$!!
	Alternative : Solve c before simplifying $\frac{1}{78}\ln\left\lvert\frac{A-89}{A-11}\right\rvert = -\frac{1}{89}t + c$ Sub $A=20,\ t=3:\ c = \frac{1}{78}\ln\frac{23}{3} + \frac{3}{89}$ $\frac{1}{78}\ln\left\lvert\frac{A-89}{A-11}\right\rvert = -\frac{1}{89}t + \frac{1}{78}\ln\frac{23}{3} + \frac{3}{89}$ $\left\lvert\frac{A-89}{A-11}\right\rvert = e^{-\frac{78}{89}t + \ln\frac{23}{3} + \frac{3(78)}{89}}$ Now $A=20$ when $t=3 \Rightarrow A-89<0$ and $A-11>0$, $-\frac{A-89}{A-11} = e^{-\frac{78}{89}t + \ln\frac{23}{3} + \frac{3(78)}{89}} = e^{\ln\frac{23}{3}} e^{3(\frac{78}{89}) - \frac{78}{89}t}$ $\frac{A-89}{A-11} = -\frac{23}{3}e^{\frac{78}{89}(3-t)}$, where $q = \frac{78}{89}$	Note that without the modulus for $\ln\left\lvert\frac{A-89}{A-11}\right\rvert$, you get $\ln(-\frac{23}{3})$ which is not defined when you subst $A=20$. Need to justify why $\left\lvert\frac{A-89}{A-11}\right\rvert = -\frac{A-89}{A-11}$
(iii)	As $t \to \infty$, $e^{\frac{78}{89}(3-t)} \to 0$, meaning that $\frac{A-89}{A-11} \to 0$ and thus $A \to 89$. The area of algae eventually tends to $89\ m^2$, exceeding 80% of the pond area ($80\ m^2$) at some point. Thus the owner's wish cannot be attained. **Alternative : Use a counter example** Substitute $A=80$ into $\frac{A-89}{A-11} = -\frac{23}{3}e^{\frac{78}{89}(3-t)}$. $\frac{-3}{23} = -\frac{23}{3}e^{\frac{78}{89}(3-t)}$ $e^{\frac{78}{89}(3-t)} = \left(\frac{3}{23}\right)^2$ $3-t = 2\left(\frac{89}{78}\right)\ln\left(\frac{3}{23}\right)$ $t = 7.65$ days Since $A=80$ when $t=7.65$, the owner's wish cannot be attained. **Alternative : Consider sign in** $\frac{A-89}{A-11} = -\frac{23}{3}e^{\frac{78}{89}(3-t)}$ Since $-\frac{23}{3}e^{\frac{78}{89}(3-t)} < 0$ for all $t \ge 3$, $\frac{A-89}{A-11} < 0 \Rightarrow 11 < A < 89$	Must compare with 80% of the pond area. You may just use a GC to solve. By plotting A against t, we can have an overview of how the area varies with time. Making A the subject, we get $A = 11 + \frac{78}{1+\frac{23}{3}e^{\frac{78}{89}(3-t)}}$.  This graph is valid for $t \ge 3$. From the graph, we can easily infer that • $A \ge 80$ when $t \ge 7.65$ • $A \to 89$ as $t \to \infty$

	With $\frac{dA}{dt} > 0$ for $11 < A < 89$, A increases from $20\,\text{m}^2$ and tend to $89\,\text{m}^2$, thus $A \geq 80$ at some point in time. The owner's wish cannot be attained.	
(iv)	$\frac{dA}{dt} = -\frac{1}{89}(A-11)(A-89)$ is a quadratic function in A. Maximum $\frac{dA}{dt}$ occurs when $A = \frac{11+89}{2} = 50$. This happens when $t = 5.32$. [Again just solve using GC.] **Alternative** At max $\frac{dA}{dt}$, $\frac{d^2A}{dt^2} = 0$. [or consider $\frac{d}{dA}\left(\frac{dA}{dt}\right) = 0$] $\frac{d^2A}{dt^2} = -\frac{1}{89}(2A-100)\frac{dA}{dt} = 0$ when $A = 50$. [Note max $\frac{dA}{dt}$ is positive] Solving $\frac{50-89}{50-11} = -\frac{23}{3}e^{\frac{78}{89}(3-t)}$, $t = 5.32$.	 • $\frac{d}{dA}\left(\frac{dA}{dt}\right) = \frac{d}{dA}\left(-\frac{1}{89}(A-11)(A-89)\right) = -\frac{1}{89}(2A-100)$ • $\frac{d^2A}{dt^2} = \frac{d}{dt}\left(-\frac{1}{89}(A-11)(A-89)\right) = \frac{d}{dA}\left(-\frac{1}{89}(A-11)(A-89)\right)\frac{dA}{dt} = -\frac{1}{89}(2A-100)\frac{dA}{dt}$ [Remember Chain Rule!]
(v)	✓ There could be sudden droughts / heavy rain / changes in intensity of sunlight etc that will impact the growth of algae. [Changes in environmental/weather factors] ✓ There could be sudden heavy rain / droughts such that the area of the pond is no longer $100\ \text{m}^2$. [Pond area not $100\ \text{m}^2$] ✓ There could be new species of aquatic animals that feeds primarily on algae / sudden chemical spills into the pond depleting nutrients etc that will impact the growth of algae. [Changes in the ecosystem of the fish pond] ✓ Farmer is sick on certain days and thus is not able to clear exactly $11\ \text{m}^2$ of algae on those days. [Changes in rate of clearing algae]	• The DE $\frac{dA}{dt} = \frac{1}{89}A(100-A) - 11$, for $t \geq 3$ models the growth (and decay) of algae in a fish pond, whereby factors such as birth (and death) of algae, and fish in pond eating the algae are already considered. • A correct answer targets possible <u>changes</u> to the original conditions (in weather/pond size/nutrients in pond/rate of clearing algae etc) that will impact the area of algae.
		Total marks: 13

2021 EJC H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**
Attempt all questions.
[You may skip Q11 for now, as it is on Integration. Remaining total marks is 86, Duratin is 2 hr 35 mins.]

1 A curve C has equation $5y^2 - 20xy + 25x^2 - 5y - 6 = 0$.
Find the exact x-coordinates of the stationary points of C. [5]

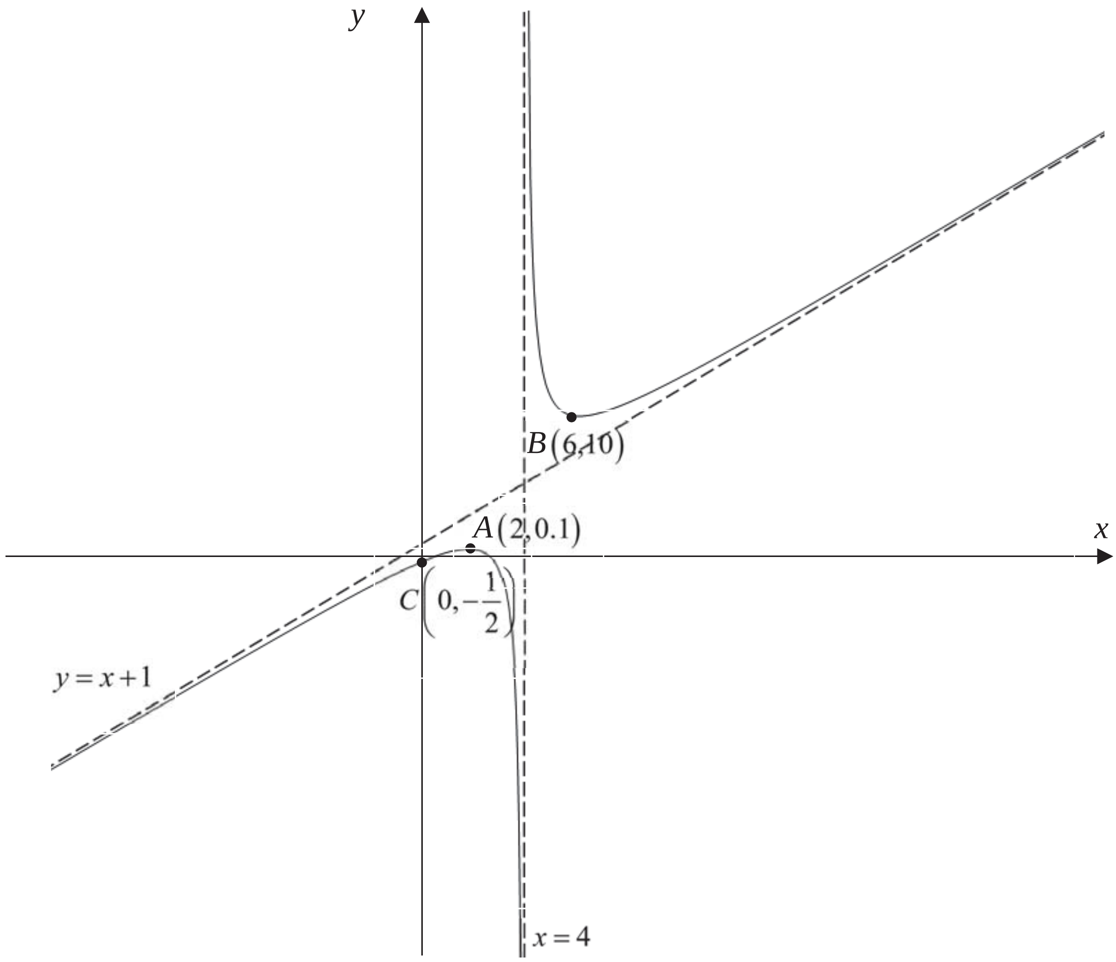
2 A curve C has parametric equations $x = t^4 + t,\ y = t^2 - 2t$. Write down $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [1]

Hence or otherwise, find the equation of the normal to C at point $P\left(\frac{9}{16}, -\frac{3}{4}\right)$. [4]

3 **(i)** Sketch the curve with equation $y = \frac{2|x-3|}{4-x}$, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [3]

(ii) Hence, solve the inequality $-4 < \frac{2|x-3|}{4-x} \le 1$. [3]

4 **(a)** The graph below shows the sketch of the curve $y = \mathrm{f}(x)$.



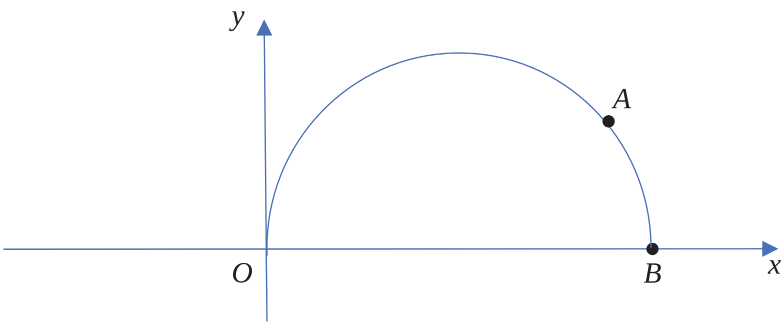
The curve has a maximum point at $A\ (2, 0.1)$, a minimum point at $B(6,10)$, and cuts the y-axis at $C\left(0, -\frac{1}{2}\right)$. The equation of the asymptotes are $y = x + 1$ and $x = 4$. Sketch the graph of $y = \mathrm{f}'(x)$, stating the equation(s) of any asymptote(s), coordinates of any point(s) of intersection with the axes and stationary point(s) whenever possible. [3]

(b) The curve $y = \mathrm{g}(x)$ cuts the axes at $(c, 0)$ and $(0, d)$ such that $c, d > 0$. In the table below, state the coordinates of the points where the following curves cut the axes and indicate if it is not possible to do so.

Equation of curve	x-intercept(s)	y-intercept(s)
$y = \mathrm{g}(cx - 2) - d$		
$y = \mathrm{g}(\lvert x \rvert)$		

[4]

5 The diagram shows a semi-circle with diameter OB. Point A is on the circumference of the semi-circle and point C lies on chord AB such that $AC = 3CB$.



Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ where **a** and **b** are non-zero and non-parallel vectors.

(i) Show that $\mathbf{a}.\mathbf{b} = \lvert \mathbf{a} \rvert^2$. [2]

(ii) Find the position vector of C in terms of **a** and **b**. [1]

(iii) Hence find the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$ if $\lvert \mathbf{a} \rvert = \sqrt{3}$ and $\lvert \mathbf{b} \rvert = 2$. [4]

6 The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = kn^2 + (k+1)n$, where k is a non-zero real constant.

(i) Show that $u_n = 2kn + 1$. [2]

(ii) Show that the sequence is an arithmetic progression. [2]

It is also given that u_{31}, u_6, u_1 are the first three terms of a geometric sequence.

(iii) Find the value of k. [3]

(iv) Give a reason why the geometric series converges and find the value of the sum to infinity. [2]

7 **(i)** The first three terms of a sequence are given by $u_1 = 2007$, $u_2 = 2020$ and $u_3 = 2036$. Given that

$$u_r = \frac{A}{r(r+1)} + Br + C \text{, for all positive integers } r,$$

find the values of the constants A, B and C. [4]

(ii) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{r(r+1)}$. [3]

(iii) Hence, find $\sum_{r=1}^{n} u_r$. Give your answer in the form $an^2 + bn + c - \frac{c}{n+1}$, where a, b and c are constants to be determined. [3]

8 The function f is defined by $\mathrm{f}: x \mapsto \mathrm{e}^{|2-x|},\ x \in \mathbb{R},\ x > a$, where a is a constant.

It is given that a is the least possible value such that f^{-1} exists.

(i) State the value of a. [1]

(ii) Hence find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$, giving the coordinates of all end points. [3]

It is now given that a is replaced by a constant such that the functions ff^{-1} and $\mathrm{f}^{-1}\mathrm{f}$ are equal.

(iv) Explain why the new value of a satisfies the equation $a = \mathrm{f}(a)$, and find a. [2]

(v) The function g is defined by $\mathrm{g}: x \mapsto \mathrm{e}^{|2-x|},\ x \in \mathbb{Z}$. Explain why the composite function g^2 does not exist. [1]

9 **(a)** Fig 1 shows a glass bauble consisting of a square prism inscribed in a spherical glass shell with fixed radius a cm. The corners of the square prism are in contact with the spherical shell. The centre, O, of the prism coincides with that of the sphere. The corners of the square base are labelled A, B, C and D, and θ is the angle that OA makes with the base of the prism.

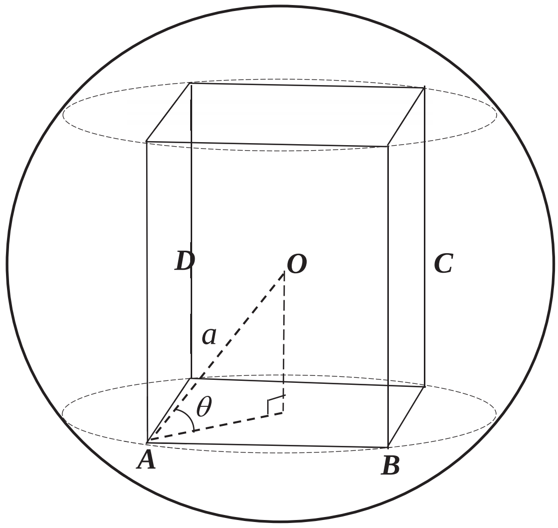


Fig. 1

(i) Show that the volume V of the square prism is given by $V = 4a^3\cos^2\theta\sin\theta$. [3]

(ii) Use differentiation to find, in terms of a, the dimensions of the square prism such that the value of V is maximised. You do not need to show that this value is a maximum. [5]

(iii) State the geometric shape of the square prism in part (ii). [1]

(b) Fig. 2 shows a 2.6 m long pole sliding down along a vertical wall such that the top of the pole is slipping at the constant rate of 0.3 m/s. Determine the speed at which the foot of the pole is moving along the ground when the foot is 1.0 m from the base of the wall. [4]

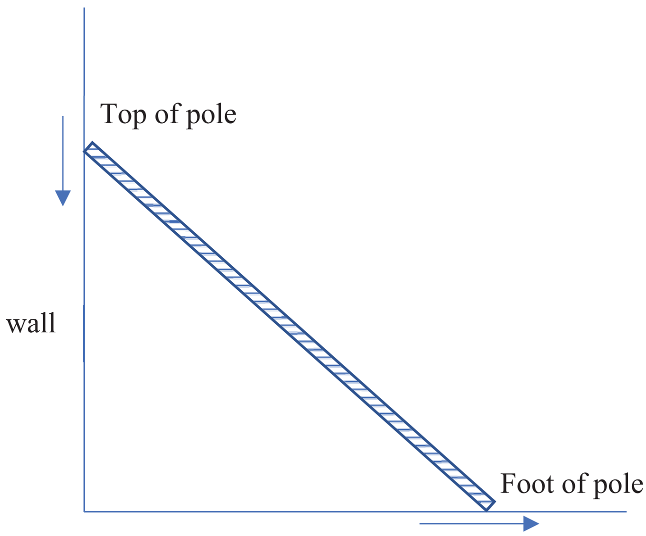
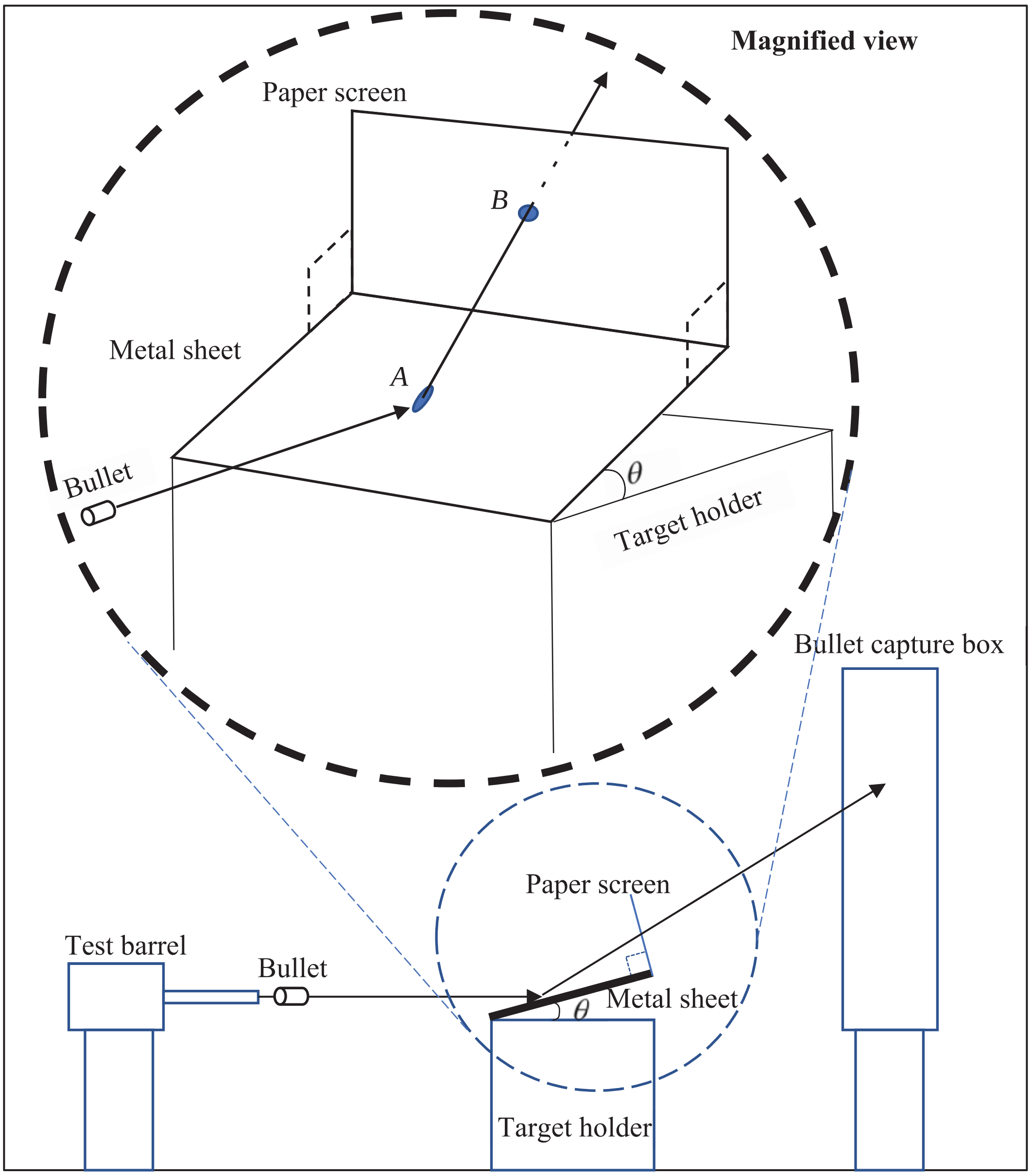


Fig. 2

10 A forensic study on the ricochet behaviour of bullets on a metal sheet was carried out.
A test barrel was used to fire bullets at a metal sheet held at an angle by a target holder. The angle between the metal sheet and the top surface of the target holder is θ (see diagram). It can be assumed that bullets travel in a straight line.
When the test barrel is fired, the bullet emerges, passing through coordinates (0, -2, 1.5) and travels in the direction parallel to **j**. The bullet hits the metal sheet at point *A*, passes through the paper screen at point *B* and is stopped by a box lined with Kevlar fabric.

It is given that the metal sheet is a part of the plane $\pi_1 : \mathbf{r} \cdot \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} = -1$ and the paper screen is part of the plane $\pi_2 : \mathbf{r} \cdot \begin{pmatrix} p \\ 3 \\ q \end{pmatrix} = 15$.



(i) Show, with clear working, that the coordinates of *A* are $(0, 0.875, 1.5)$. [4]

(ii) Given that the paper screen is perpendicular to the metal sheet, show that $q = 4$. [2]

(iii) Given that the coordinates of B are $(0.9, 0.8, 3.15)$, show that $p = 0$. [2]

(iv) Given that the top surface of the target holder is part of the plane with equation $z = 1$, find the acute angle θ. [3]

(v) Find a vector equation of the line of intersection between the metal sheet and the paper screen. [2]

11 **(a)** Find $\int \sec^{n+1} 3x \tan 3x \, \mathrm{d}x$, where n is a positive integer. [2]

(b) Find the exact value, in terms of π, of $\int_0^1 \frac{1}{(2x+1)^2+3} \, \mathrm{d}x$. [3]

(c) Find the exact value, in terms of e, of $\int_0^1 x^2 \mathrm{e}^{2x} \, \mathrm{d}x$. [4]

(d) Use the substitution $x = 2\sin u$ to find $\int \frac{\sqrt{4-x^2}}{x^2} \, \mathrm{d}x$. [5]

2021 EJC H2 Maths Promo Paper Solution

1

$5y^2 - 20xy + 25x^2 - 5y - 6 = 0$

Diff wrt x, $10y\frac{dy}{dx} - 20\left\{x\frac{dy}{dx} + y\right\} + 50x - 5\frac{dy}{dx} = 0$

For stationary points, $\frac{dy}{dx} = 0$

$\Rightarrow -20y + 50x = 0$

$\Rightarrow y = \frac{5}{2}x$

Sub into *C*,

$$5\left(\frac{5}{2}x\right)^2 - 20x\left(\frac{5}{2}x\right) + 25x^2 - 5\left(\frac{5}{2}x\right) - 6 = 0$$

$$\Rightarrow \frac{125}{4}x^2 - 50x^2 + 25x^2 - \frac{25}{2}x - 6 = 0$$

$$\Rightarrow \frac{25}{4}x^2 - \frac{25}{2}x - 6 = 0 \left(\text{or } 25x^2 - 50x - 24 = 0\right)$$

$$\Rightarrow x = -\frac{2}{5} \quad \text{or} \quad \frac{12}{5}$$

2

Since P is $\left(\frac{9}{16}, -\frac{3}{4}\right)$, $t^2 - 2t = -\frac{3}{4} \Rightarrow t^2 - 2t + \frac{3}{4} = 0 \Rightarrow t = \frac{1}{2}$ or $\frac{3}{2}$

$t^4 + t = \frac{9}{16} \Rightarrow t^4 + t - \frac{9}{16} = 0 \Rightarrow t = \frac{1}{2}$ or -1.14

Hence $\Rightarrow t = \frac{1}{2}$

$\frac{dx}{dt} = 4t^3 + 1$ and $\frac{dy}{dt} = 2t - 2$

$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{2t-2}{4t^3+1}$

At P$\left(\frac{9}{16}, -\frac{3}{4}\right)$, $\frac{dy}{dx} = \frac{2\left(\frac{1}{2}\right) - 2}{4\left(\frac{1}{2}\right)^3 + 1} = -\frac{2}{3} \Rightarrow$ gradient of normal $= \frac{3}{2}$

Hence equation of normal at ***P***$\left(\frac{9}{16}, -\frac{3}{4}\right)$ **is** $y - \left(-\frac{3}{4}\right) = \frac{3}{2}\left(x - \frac{9}{16}\right) \Rightarrow y = \frac{3}{2}x - \frac{51}{32}$ **.**

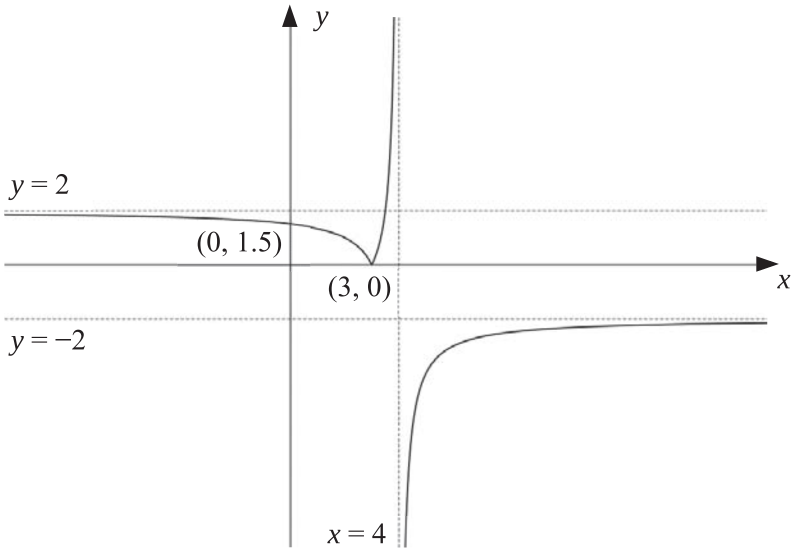
3

(i) Axial intercepts: $(0,1.5),(3,0)$

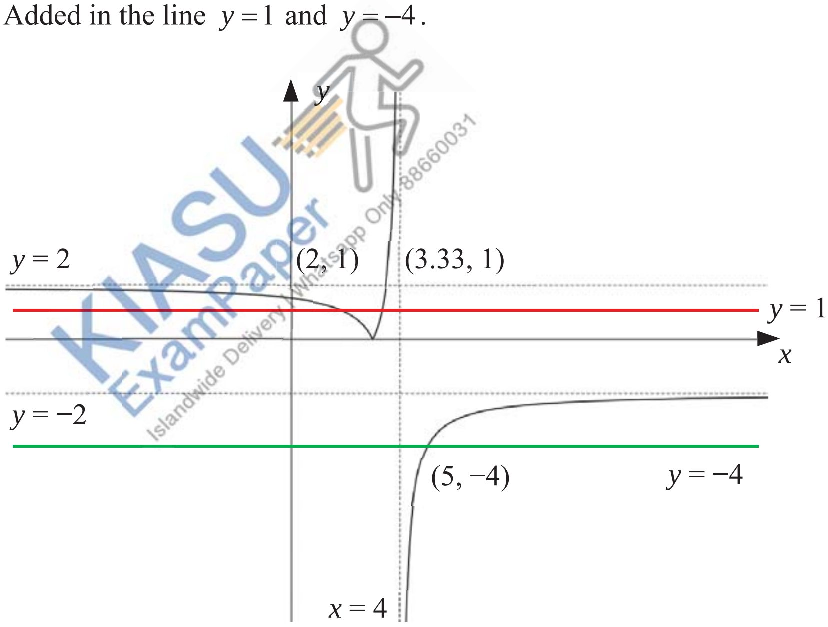
For $x<3$, $y=\dfrac{2|x-3|}{4-x}=\dfrac{2(3-x)}{4-x}=2-\dfrac{2}{4-x}$ $\Rightarrow$ as $x\to-\infty$, $y\to 2$

For $x>3$, $y=\dfrac{2|x-3|}{4-x}=\dfrac{2(x-3)}{4-x}=-2+\dfrac{2}{4-x}$ $\Rightarrow$ as $x\to\infty$, $y\to -2$

Hence, the asymptotes are: $y=2$, $y=-2$, $x=4$,



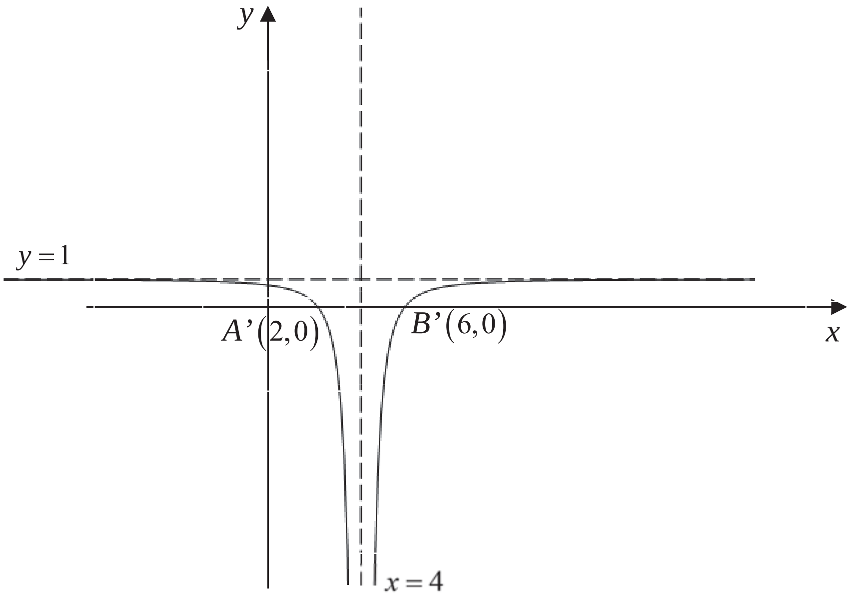
(ii) Added in the line $y=1$ and $y=-4$.



Hence, $2\le x\le 3.33$ or $x>5$.

4

(a)



(b)

Equation	x-intercept	y-intercept
$y=\mathrm{g}(cx-2)-d$	$\left(\frac{2}{c},0\right)$	cannot be determined
$y=\mathrm{g}(\lvert x\rvert)$	$(c,0)$, $(-c,0)$	$(0,d)$

5

(i)

Method 1:

$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$

Since $\angle OAB=90^\circ$ (angle in a semi circle), $\overrightarrow{AB}\perp\overrightarrow{OA}\Rightarrow\overrightarrow{AB}\cdot\overrightarrow{OA}=0$

$$\Rightarrow(\mathbf{b}-\mathbf{a})\cdot\mathbf{a}=0$$

$$\Rightarrow\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{a}=0$$

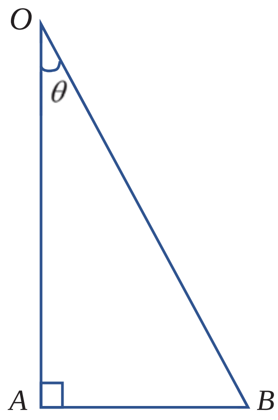
$$\Rightarrow\mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$$

$$\Rightarrow\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|^2\ (\because\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a})\text{ [shown]}$$

Method 2:

Let θ be the angle between **a** and **b.**

Note that triangle OAB is a right angled triangle.



Hence $OA = OB\cos\theta = |\mathbf{b}|\cos\theta$

Since $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{a}|) = |\mathbf{a}|^2$ [shown]

Method 3:

Since angle OAB is a right angle in a semi circle,

a is the projection vector of **b** onto **a**.

$$\frac{|\mathbf{b}\cdot\mathbf{a}|}{|\mathbf{a}|} = |\mathbf{a}|$$

$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}|^2$$

As angle between **a** and **b** is acute,

$$|\mathbf{a}\cdot\mathbf{b}| = \mathbf{a}\cdot\mathbf{b}$$

$$\therefore \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|^2 \text{ (shown)}$$

(ii)

$$AC = 3CB \Rightarrow \frac{AC}{CB} = \frac{3}{1}$$

By ratio theorem, $\overrightarrow{OC} = \frac{\overrightarrow{OA} + 3\overrightarrow{OB}}{4} = \frac{\mathbf{a} + 3\mathbf{b}}{4}$

(iii)

length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB} = \left|\overrightarrow{OC}\cdot\frac{\overrightarrow{OB}}{|\overrightarrow{OB}|}\right| = \left|\left(\frac{\mathbf{a}+3\mathbf{b}}{4}\right)\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right| = \left|\left(\frac{\mathbf{a}\cdot\mathbf{b}+3\mathbf{b}\cdot\mathbf{b}}{4|\mathbf{b}|}\right)\right| = \left|\left(\frac{|\mathbf{a}|^2+3|\mathbf{b}|^2}{4|\mathbf{b}|}\right)\right|$ (from (i))

Since $|\mathbf{a}| = \sqrt{3}$ and $|\mathbf{b}| = 2$, $\left|\overrightarrow{OC}\cdot\frac{\overrightarrow{OB}}{|\overrightarrow{OB}|}\right| = \left|\left(\frac{|\mathbf{a}|^2+3|\mathbf{b}|^2}{4|\mathbf{b}|}\right)\right| = \left|\left(\frac{3+3(4)}{4(2)}\right)\right| = \frac{15}{8}$

6

(i)

$S_n = kn^2 + (k+1)n$

$S_{n-1} = k(n-1)^2 + (k+1)(n-1)$

For $n \geq 2$, $u_n = kn^2 + (k+1)n - \left[k(n-1)^2 + (k+1)(n-1)\right]$

$$= kn^2 + (k+1)n - k(n-1)^2 - (k+1)(n-1)$$
$$= kn^2 + (k+1)n - kn^2 + 2kn - k - (k+1)n + k + 1$$
$$= 2kn + 1$$

$u_1 = S_1 = k + (k+1) = 2k + 1 = 2k(1) + 1$ which follows the form of $u_n = 2nk + 1$ when $n = 1$.

Thus, $u_n = 2nk + 1$

(ii)

$u_n = 2nk + 1$

$u_{n-1} = 2k(n-1) + 1$

$u_n - u_{n-1}$
$= 2nk + 1 - \left[2k(n-1) + 1\right]$
$= 2nk + 1 - 2nk + 2k - 1$
$= 2k$ is a constant independent of n

$\therefore$ the sequence is an arithmetic sequence

(iii) $u_{31} = 2(31)k + 1 = 62k + 1$; $u_6 = 2(6)k + 1 = 12k + 1$; $u_1 = 2(1)k + 1 = 2k + 1$

$$\frac{12k+1}{62k+1} = \frac{2k+1}{12k+1}$$
$$(12k+1)^2 = (2k+1)(62k+1)$$
$$144k^2 + 24k + 1 = 124k^2 + 64k + 1$$
$$20k^2 - 40k = 0$$
$$20k(k-2) = 0$$
$$k = 0 \left(rej \because k \neq 0\right) \quad or \quad k = 2$$

(iv) $r = \frac{12k+1}{62k+1} = \frac{25}{125} = \frac{1}{5}$

Since $|r| = \frac{1}{5} < 1$, the geometric series is convergent

$$\therefore S_\infty = \frac{u_{31}}{1-r} = \frac{125}{1-\frac{1}{5}} = 156.25$$

7

(i)

$$\frac{1}{2}A + B + C = 2007$$

$$\frac{1}{6}A + 2B + C = 2020$$

$$\frac{1}{12}A + 3B + C = 2036$$

From GC, $A = 12, B = 17, C = 1984$

(ii)

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1} \right)$$

$$= \frac{1}{1} - \frac{1}{2}$$

$$+ \frac{1}{2} - \frac{1}{3}$$

$$+ \frac{1}{3} - \frac{1}{4}$$

$$\vdots$$

$$+ \frac{1}{n-1} - \frac{1}{n}$$

$$+ \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

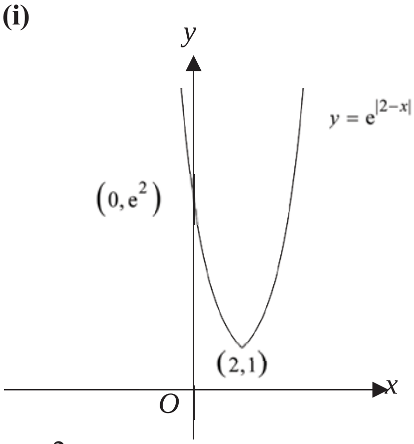
(iii)

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \frac{12}{r(r+1)} + \sum_{r=1}^{n} (17r + 1984)$$

$$= 12 - \frac{12}{n+1} + \frac{n}{2}(2001 + 17n + 1984)$$

$$= 8.5n^2 + 1992.5n + 12 - \frac{12}{n+1}$$

(i)



$a = 2$

(ii) Let $y = e^{|2-x|}$.

Since $x > a \Rightarrow |2-x| = x-2$, $\therefore$ $y = e^{x-2}$

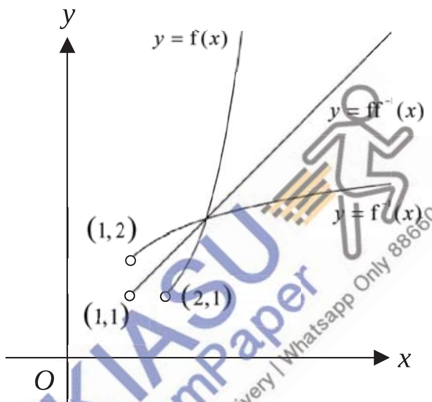
$\ln y = x - 2$

$x = \ln y + 2$

$f^{-1}(x) = \ln x + 2$

$D_{f^{-1}} = R_f = (1, \infty)$

(iii)



(iv) As f^{-1} exists, $a > 2$.

$ff^{-1}(x) = f^{-1}f(x) = x$ (same rule)

For ff^{-1} and $f^{-1}f$ to be equal, we also need $D_{ff^{-1}} = D_{f^{-1}f}$ (same domain).

$\Rightarrow D_{f^{-1}} = D_f$

$\Rightarrow R_f = D_f$

$\Rightarrow (f(a), \infty) = (a, \infty)$

$\therefore$ $a = f(a)$

Using GC, $a = 3.15$ (3 s.f.).

(v) If g^2 exists, $R_g \subseteq D_g$.

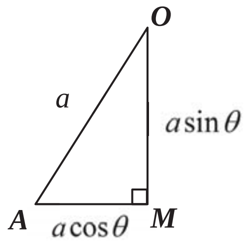
$g(0) = e^2 \in R_g$

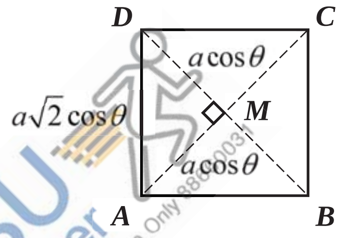
As e^2 is not an integer, $e^2 \notin D_g$.

Hence $R_g \not\subseteq D_g$, and so g^2 does not exist.

9

(a)(i) Let M be the centre of the square base, then





Height of square prism $= 2a\sin\theta$; Length of square $= a\sqrt{2}\cos\theta$

$V = (\text{base area}) \times (\text{height}) = \left(a\sqrt{2}\cos\theta\right)^2 (2a\sin\theta) = 4a^3\cos^2\theta\sin\theta$ (shown)

Alternative: Base area $= 4 \times \frac{1}{2}(a\cos\theta)^2$, using triangle AMD

(ii) $\frac{dV}{d\theta} = -4a^3(2\cos\theta\sin\theta)\sin\theta + 4a^3\cos^2\theta\cos\theta$

$= -8a^3\cos\theta\sin^2\theta + 4a^3\cos^3\theta = 4a^3\cos\theta\left(-2\sin^2\theta + \cos^2\theta\right) = 4a^3\cos\theta\left(1 - 3\sin^2\theta\right)$

At max V, $\frac{dV}{d\theta} = 0$

$4a^3\cos\theta\left(1 - 3\sin^2\theta\right) = 0$

$\cos\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2} \quad \left(\text{rejected since } 0 < \theta < \frac{\pi}{2}\right)$

OR

$\left(1 - 3\sin^2\theta\right) = 0 \quad \Rightarrow \quad \sin\theta = \sqrt{\frac{1}{3}} \quad \text{or} \quad \sin\theta = -\sqrt{\frac{1}{3}}$

$\left(\text{rejected since } 0 < \theta < \frac{\pi}{2}\right)$

Hence, $\sin\theta = \sqrt{\frac{1}{3}}$ and $\theta = \sin^{-1}\sqrt{\frac{1}{3}}$

Height of square prism $= 2a\sin\theta = 2a\sqrt{\frac{1}{3}} = \frac{2}{\sqrt{3}}a$

Length of square base $= a\sqrt{2}\cos\theta = a\sqrt{2}\sqrt{1 - \frac{1}{3}} = \frac{2}{\sqrt{3}}a$

(iii) It is a cube.

(b) Let y m be the vertical distance between the top of the pole and the ground.
Let x m be the horizontal distance between the foot of the pole and the base of the wall.

$x^2 + y^2 = 2.6^2$ (Pythagoras' Theorem) and given $\frac{dy}{dt} = -0.3$.

When $x = 1.0$, $y = 2.4$.

By implicit differentiation, $2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$

$2(1.0)\frac{dx}{dt} + 2(2.4)(-0.3) = 0 \quad \Rightarrow \quad \frac{dx}{dt} = 0.72$

The foot of the pole is moving at a speed of 0.72 m/s.

10

(i)	**Method 1** Let the path of bullet (equation of line) be l. $$l:\mathbf{r}=\begin{pmatrix}0\\-2\\1.5\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \lambda\in\mathbb{R};\qquad p_1:\mathbf{r}\bullet\begin{pmatrix}0\\4\\-3\end{pmatrix}=-1$$ A is the point of intersection between l and p_1. A lies on l so $\overrightarrow{OA}=\begin{pmatrix}0\\-2\\1.5\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\-2+\lambda\\1.5\end{pmatrix}$, for some $\lambda\in\mathbb{R}$ A also lies on π, so $\overrightarrow{OA}\bullet\begin{pmatrix}0\\4\\-3\end{pmatrix}=-1 \Rightarrow \begin{pmatrix}0\\-2+\lambda\\1.5\end{pmatrix}\bullet\begin{pmatrix}0\\4\\-3\end{pmatrix}=-1\Rightarrow -8+4\lambda-4.5=-1\Rightarrow \lambda=\frac{11.5}{4}=2.875.$ Therefore $\overrightarrow{OA}=\begin{pmatrix}0\\-2+2.875\\1.5\end{pmatrix}=\begin{pmatrix}0\\0.875\\1.5\end{pmatrix}$ So the coordinates of A is $(0,0.875,1.5)$ [shown] **Method 2** Since the bullet travels in the direction parallel to **j** only, the x and z-coordinates remain the same. Let $\overrightarrow{OA}=\begin{pmatrix}0\\y\\1.5\end{pmatrix}$, where y is a value to be found. A also lies on π, so $\overrightarrow{OA}\bullet\begin{pmatrix}0\\4\\-3\end{pmatrix}=-1\Rightarrow\begin{pmatrix}0\\y\\1.5\end{pmatrix}\bullet\begin{pmatrix}0\\4\\-3\end{pmatrix}=-1$ Solving gives us $y=\frac{3.5}{4}=0.875$ Hence, the coordinates of A is $(0,0.875,1.5)$ [shown]
(ii)	Since paper screen is perpendicular to the metal sheet and we know: $$p_1:\mathbf{r}\bullet\begin{pmatrix}0\\4\\-3\end{pmatrix}=-1 \text{ and } p_2:\mathbf{r}\bullet\begin{pmatrix}p\\3\\q\end{pmatrix}=15$$ Then $\begin{pmatrix}0\\4\\-3\end{pmatrix}\bullet\begin{pmatrix}p\\3\\q\end{pmatrix}=0\Rightarrow 0+12-3q=0\Rightarrow q=4$ [shown]
(iii)	Since B $(0.9,0.8,3.15)$ is on p_2. $$p_2:\mathbf{r}\bullet\begin{pmatrix}p\\3\\q\end{pmatrix}=15\quad\Rightarrow\begin{pmatrix}0.9\\0.8\\3.15\end{pmatrix}\bullet\begin{pmatrix}p\\3\\4\end{pmatrix}=15\Rightarrow 0.9p+2.4+12.6=15\Rightarrow p=\frac{15-15}{0.9}=0 \text{ [shown]}$$
(iv)	Since paper screen is perpendicular to the metal sheet and we know:Let equation of top surface be p_3.

So $z = 1$ is actually $p_3 : \mathbf{r} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$.

The acute angle between p_1 and p_3 is given as θ

So $\cos\theta = \dfrac{\left| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} \right|}{\sqrt{1}\sqrt{25}} = \dfrac{3}{5}$

Therefore $\theta = \cos^{-1}\dfrac{3}{5} = 53.1^\circ$

(v) .

Line CD is the line of intersection between

$p_1 : \mathbf{r} \bullet \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} = -1$ and $p_2 : \mathbf{r} \bullet \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = 15$

$p_1 : 4y - 3z = -1$ and $p_2 : 3y + 4z = 15$

Solving SLE via GC gives: $x = x$, $y = \dfrac{41}{25}$, **and** $z = \dfrac{63}{25}$

So an equation of the line of intersection is $l_{CD} : \mathbf{r} = \begin{pmatrix} 0 \\ 41/25 \\ 63/25 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$

11

(a) $\int \sec^{n+1} 3x \tan 3x \, dx = \dfrac{1}{3}\int \sec^n 3x \,(3\sec 3x \tan 3x)\, dx = \dfrac{1}{3}\left(\dfrac{\sec^{n+1} 3x}{n+1}\right) + C = \dfrac{1}{3(n+1)}\sec^{n+1} 3x + C$

(b) $\int_0^1 \dfrac{1}{(2x+1)^2 + 3}\, dx = \left[\dfrac{1}{2\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)\right]_0^1 = \dfrac{1}{2\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\dfrac{1}{\sqrt{3}}\right] = \dfrac{1}{2\sqrt{3}}\left(\dfrac{\pi}{3} - \dfrac{\pi}{6}\right) = \dfrac{\pi}{12\sqrt{3}}$

(c) $\int_0^1 x^2 e^{2x}\, dx = \left[\dfrac{x^2 e^{2x}}{2}\right]_0^1 - \int_0^1 x e^{2x}\, dx = \dfrac{1}{2}e^2 - \left(\left[\dfrac{xe^{2x}}{2}\right]_0^1 - \int_0^1 \dfrac{e^{2x}}{2}\, dx\right) = \dfrac{1}{2}e^2 - \dfrac{1}{2}e^2 + \left[\dfrac{e^{2x}}{4}\right]_0^1 = \dfrac{e^2 - 1}{4}$

(d)

$\int \dfrac{\sqrt{4 - x^2}}{x^2}\, dx$ $\qquad \dfrac{dx}{du} = 2\cos u$

$= \int \dfrac{\sqrt{4 - 4\sin^2 u}}{4\sin^2 u}\, 2\cos u\, du = \int \dfrac{\sqrt{4\cos^2 u}}{4\sin^2 u}\, 2\cos u\, du$

$= \int \dfrac{2\cos u}{4\sin^2 u}\, 2\cos u\, du$

$= \int \cot^2 u\, du = \int \operatorname{cosec}^2 u - 1\, du$

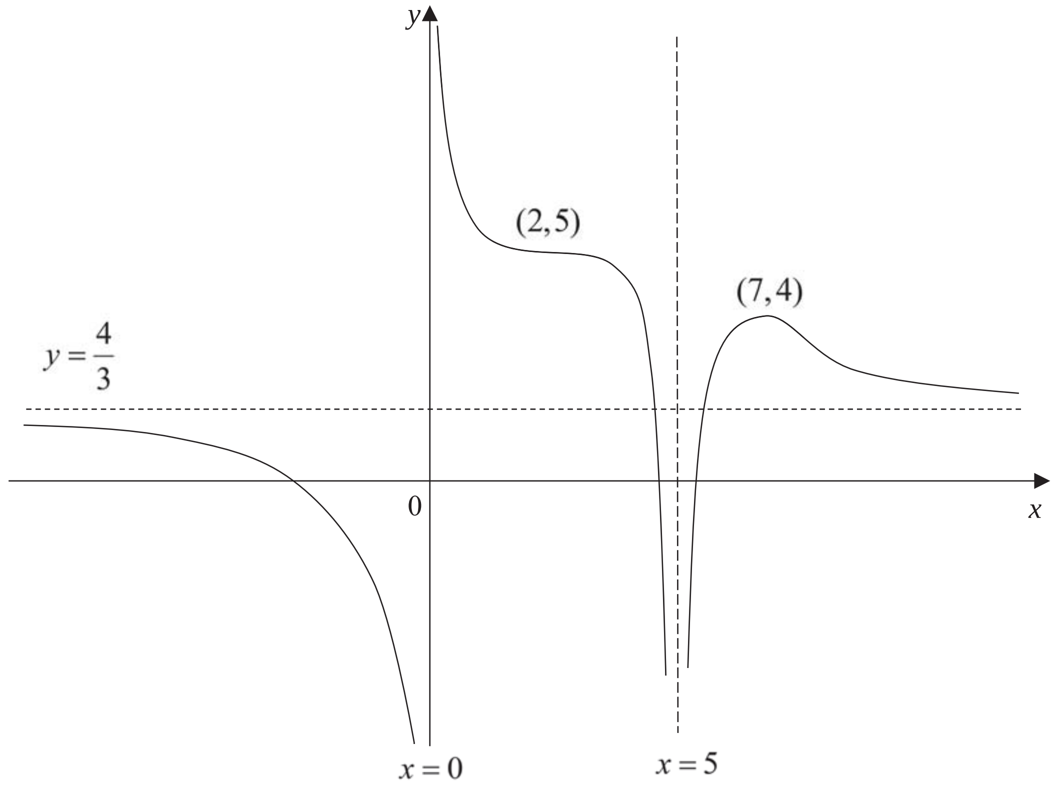
$= -\cot u - u + C = -\dfrac{\sqrt{4 - x^2}}{x} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$

2021 HCI H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**

Attempt all questions.

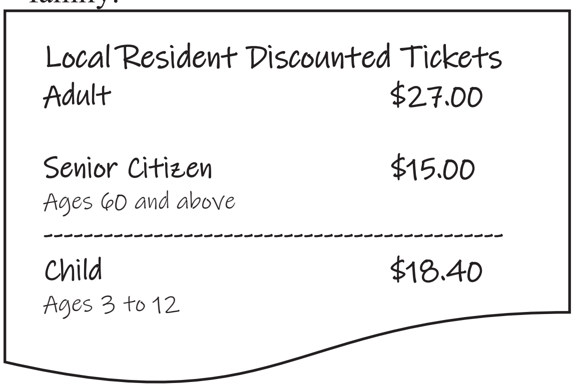
[You may skip Q5, 7(iv) & 9(v) for now, as they are on Integration. Remaining total marks is 85, Duratin is 2 hr 33 mins.]

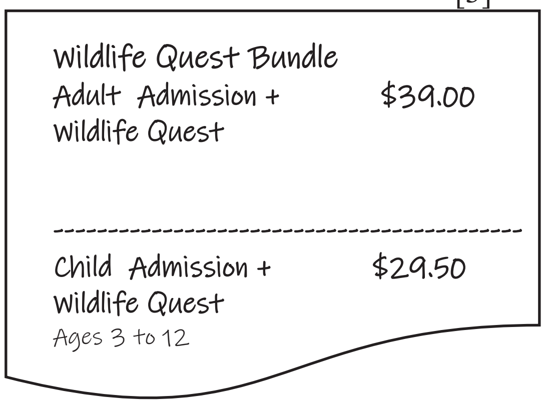
1 The diagram below shows the curve $y = \mathrm{f}(x)$. The curve has a stationary point of inflexion at $(2,5)$ and a maximum point at $(7,4)$. The curve also has vertical asymptotes $x = 0$, $x = 5$ and a horizontal asymptote $y = \frac{4}{3}$.



Sketch the curve $y = \mathrm{f}'(x)$, labelling clearly the coordinates of any axial intercepts, turning points and equations of asymptotes where applicable. [3]

2 A local family consisting of adults, children and senior citizens, is planning for a trip to the Jewel Bird Park. The ticket prices are listed as shown below. If all members in the family purchase the Local Resident Discounted tickets, they will need to pay a total price of \$301.20. If all members in the family purchase the Wildlife Quest Bundle, they will need to pay a total price of \$478.50. Given that the number of adults is four times the number of senior citizens, find the number of adults, children and senior citizens in the family. [3]





3 A curve C has equation $kxe^{y}+ke^{x}=y^{2}+k^{2}$, where k is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of x, y and k. [3]

(ii) Explain why there is no point on C where the tangent is parallel to the x-axis. [2]

4 With respect to origin O, the distinct points P, Q, R and S have position vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ respectively. It is known that $\mathbf{q}$ is a unit vector.

(i) Given that $(\mathbf{r}-\mathbf{p})\times(\mathbf{p}-\mathbf{q})=\mathbf{0}$, state, with justification, the relationship between the points P, Q and R. [2]

(ii) Give a geometrical interpretation of $|\mathbf{q}\cdot(\mathbf{r}-\mathbf{s})|$. [1]

(iii) Given that $\mathbf{p}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$ and that $\mathbf{q}$ is parallel to and in the opposite direction of $\mathbf{p}$, find $\mathbf{q}$. [2]

5 A curve has equation $y=xe^{x^{2}}$.

(i) The region R is bounded by the curve, the lines $y=-e$, $y=e$ and the y-axis. Find the exact area of R. [4]

(ii) Find the volume generated when R is rotated about the x-axis through $360°$, giving your answer correct to 3 decimal places. [3]

6 The sum of the first n terms of a series is given by the expression $e^{2}-(-2)^{n}\left(e^{2-n}\right)$.

(i) State the first term of the series in terms of e. [1]

(ii) By finding the n^{th} term of the series, show that this is a geometric series. [3]

(iii) Explain why the sum to infinity, S, of the series exists, and determine the exact value of S. [3]

7 A curve C has parametric equations $x=\theta^{2}+1$, $y=2\sin\theta+1$ where $-\pi\le\theta\le\pi$.

(i) Find the exact values of θ at which C crosses the x-axis. [1]

(ii) Sketch C, labelling the coordinates of the points at which C crosses the x-axis. [2]

(iii) Find the equation of the tangent to C at the point P with parameter p, where $-\pi\le p\le\pi$. [3]

(iv) Q is a point on C such that the tangent at Q is parallel to the y-axis. Find the area bounded by C, the tangent at Q and the x-axis. [3]

8 A function f is said to be self-inverse if $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of f.

It is given that g is a self-inverse function and is defined by $\mathrm{g}: x \mapsto \dfrac{x+a}{3x+b}$, for $x \in \mathbb{R},\ x > -\dfrac{b}{3}$, where a and b are constants and $\mathrm{g}(1) = 5$.

(i) Find the value of b and show that $a = 9$. [3]

(ii) Find $\mathrm{g}^{2021}(1)$. [1]

The function h is defined by $\mathrm{h}: x \mapsto |1-x|(x+5)$, for $x \in \mathbb{R}, x \geq 2$.

(iii) Given that h^{-1} exists, find h^{-1} in similar form. [4]

(iv) Show that gh exists and find the exact range of gh. [3]

9 The curve C has equation $y = \dfrac{(1-x)(x-2)}{x^2-2x-8}$.

(i) Find the value of x when $y = -1$. [1]

(ii) Sketch C, showing clearly the equations of any asymptotes and coordinates of any turning points and axial intercepts. [3]

(iii) Solve the inequality [1]

$$\frac{-x^2+3x-2}{x^2-2x-8} \geq -1.$$

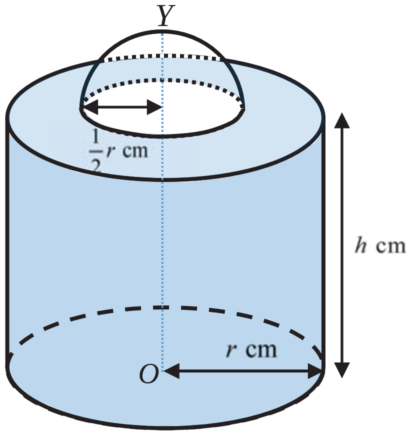
(iv) Using the result in part **(iii)**, solve the inequality [2]

$$\frac{-1+3x-2x^2}{1-2x-8x^2} \geq -1.$$

(v) Find the area bounded by the curve C and the x-axis, leaving your answer in the form $a + \ln\dfrac{b}{c}$, where a, b and c are integers. [5]

[It is given that a sphere of radius R has surface area $4\pi R^2$ and volume $\dfrac{4}{3}\pi R^3$.]

10 A perfume maker designs a prototype of a perfume bottle of fixed volume V cm^3 for a new fragrance as shown in the diagram below.



The prototype comprises 2 different segments where the vertical axis of the prototype, OY, is $\left(h+\frac{1}{2}r\right)$ cm. The bottom segment is a glass cylinder of radius r cm and height h cm. The top segment is a chrome-plated plastic hemisphere of radius $\frac{1}{2}r$ cm. It is assumed that the prototype is of negligible thickness and there is no gap between the 2 segments.

(i) Find h in terms of V, r and π. [2]

(ii) Given that the cost of manufacturing the glass cylinder is \$8 per cm^2 and the chrome-plated plastic is \$3 per cm^2, show that the total cost of manufacturing the prototype is $\$\left(\frac{85}{6}\pi r^2+\frac{16V}{r}\right)$. Hence, using differentiation, find the exact value of $\frac{h}{r}$ such that the cost of manufacturing the prototype is minimum. [7]

It is now given that the diameter of the chrome-plated hemisphere is 3 cm and the height of the glass cylinder is 5 cm.

(iii) A crack at the bottom of the prototype causes the perfume to leak out of the prototype at a constant rate of 1.5 cm^3 / s. Given that perfume is initially filled to the brim of the top segment of the prototype, find the exact rate of decrease of the height of the perfume in the prototype 5 seconds after the prototype has cracked. [3]

11 The points A, B and C have coordinates $(-1,-12,4)$, $(5,0,7)$ and $(6,1,4)$ respectively. The line l_1 has equations $\frac{x-1}{2}=\frac{2-y}{3}$, $z=4$ and the line l_2 passes through A and B.

(i) Find the coordinates of the foot of perpendicular from C to l_1. [4]

(ii) Find the acute angle between l_1 and l_2. [3]

(iii) The point D is on l_2 such that the distance from D to A is twice the distance from D to B. Find the possible point(s) D. [4]

(iv) The line l_3 passes through point A and is perpendicular to both l_1 and l_2. Find the equation of l_3. [2]

12 A couple takes up a housing loan of $\$L$ and the interest is charged before each monthly repayments at a fixed rate of p% per annum. Their monthly repayment commences on 1 September 2021. Monthly repayments of $\$x$ are due and payable on the first day of subsequent months until their housing loan is fully repaid.

(i) State an expression in terms of L and p for the interest charged before their first repayment on 1 September 2021. [1]

(ii) Show that the outstanding loan at the start of the n^{th} month after their monthly repayment is given by

$$\left(1+\frac{p}{1200}\right)^{n} L-\frac{1200x}{p}\left[\left(1+\frac{p}{1200}\right)^{n}-1\right].$$

[3]

The couple is taking a housing loan of \$504,000 at a fixed interest rate of 2.6% per annum.

(iii) Calculate the monthly repayment if the couple plans to repay the loan in 30 years. [2]

(iv) Given that the couple decides to pay monthly repayments of \$4000, find the date at which the couple will be able to fully repay their housing loan and the amount that the couple pays for their final monthly repayment. [3]

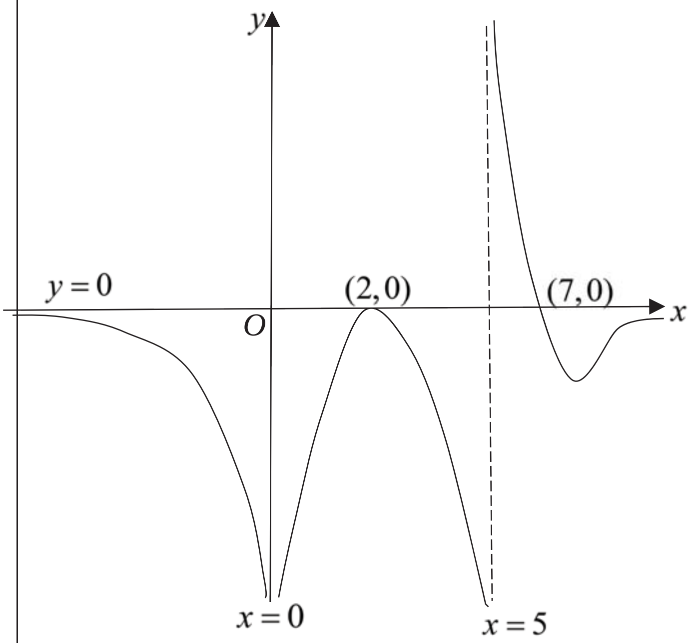
The couple decides to start adopting a savings plan on 1 September 2021. The couple decides to deposit $\$k$ on 1 September 2021 to the savings plan and for each subsequent month, they will deposit $\$a$ more than the previous month. Each month, the savings plan gives a fixed interest of 0.1% for the amount deposited for that month. The total amount that the couple will have in the savings plan after n months is given by

$$\sum_{r=1}^{n}\left(450.45+50.05r\right).$$

(v) Find the values of k and a. [2]

(vi) Assuming that the couple intends to pay monthly repayments of \$4000 for their housing loan, find the least number of months that is needed so that the couple can use the amount in their savings plan to make a one-time repayment to fully repay their outstanding housing loan. [2]

2021 HCI H2 Maths Promo Paper Solution

<table>
<tr><th>No.</th><th>Suggested Solutions</th></tr>
<tr><td>1</td><td>
</td></tr>
<tr><td>2</td><td>Let A, C and S be the number of adults, children and senior citizens in the local family.

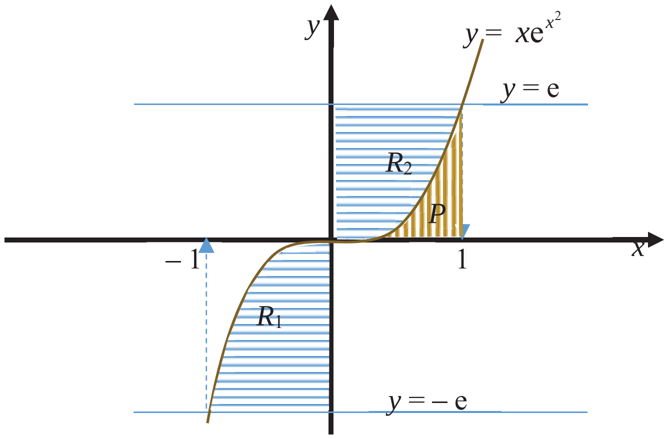
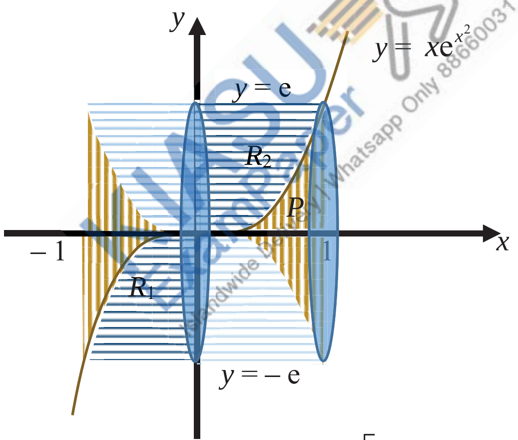
$27A+18.4C+15S=301.2$ ------------- (1)
$39A+29.5C+39S=478.50$ ------------ (2)
$A-4S=0$ -------------------------------- (3)

By GC,
$A=8$
$C=3$
$S=2$

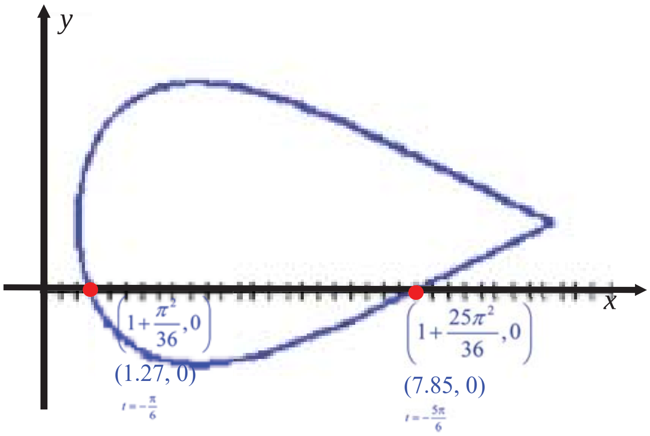
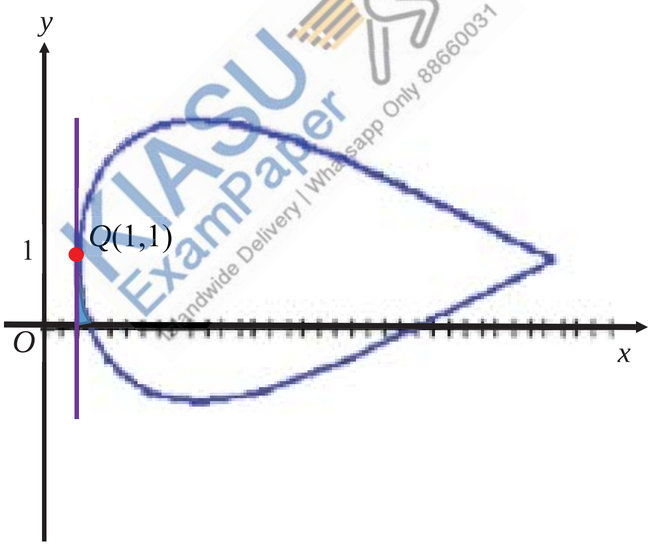
$\therefore A=8,\ C=3,\ S=2$
The family consists of 8 adults, 3 children and 2 senior citizens.</td></tr>
<tr><td>3(i)</td><td>$kxe^{y}+ke^{x}=y^{2}+k^{2}$

Differentiating with respect to x,</td></tr>
</table>

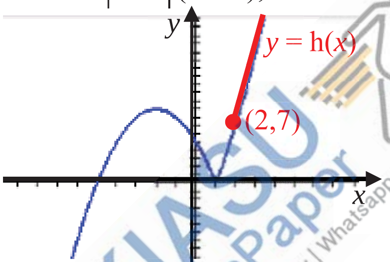
No.	Suggested Solutions
	$k\mathrm{e}^{y}+kx\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x}+k\mathrm{e}^{x}=2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}\left(2y-kx\mathrm{e}^{y}\right)=k\mathrm{e}^{x}+k\mathrm{e}^{y}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{k\mathrm{e}^{x}+k\mathrm{e}^{y}}{2y-kx\mathrm{e}^{y}}$
3 (ii)	For tangent to the curve to be parallel to the *x*-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\therefore k\mathrm{e}^{x}+k\mathrm{e}^{y}=0$ For all x, y, $\mathrm{e}^{x}>0$, $\mathrm{e}^{y}>0$ Hence, $k\mathrm{e}^{x}+k\mathrm{e}^{y}>0$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}\neq 0$ for all $x, y \in \mathbb{R}$ Therefore, there is no point where the tangent is parallel to the *x*-axis.
4(i)	$\left(\underset{\sim}{r}-\underset{\sim}{p}\right)\times\left(\underset{\sim}{p}-\underset{\sim}{q}\right)=\underset{\sim}{0}$ $\Rightarrow\left(\overrightarrow{PR}\right)\times\left(\overrightarrow{QP}\right)=\underset{\sim}{0}$ $\Rightarrow \overrightarrow{PR}\,//\,\overrightarrow{QP}$ Common point P, $\therefore P, Q$ and R are collinear **Alternative method** $\left(\underset{\sim}{r}-\underset{\sim}{p}\right)=\lambda\left(\underset{\sim}{p}-\underset{\sim}{q}\right),\quad \lambda\in\mathbb{R}$ $\underset{\sim}{r}=\underset{\sim}{p}+\lambda\left(\underset{\sim}{p}-\underset{\sim}{q}\right),\quad \lambda\in\mathbb{R}$ $\therefore R$ is a point on line passing through P and Q
4(ii)	Since $\left\|\underset{\sim}{q}\right\|=1$ $\left\|\underset{\sim}{q}\cdot\left(\underset{\sim}{r}-\underset{\sim}{s}\right)\right\|=\left\|\overrightarrow{SR}\cdot\underset{\sim}{q}\right\|$ $\therefore \left\|\underset{\sim}{q}\cdot\left(\underset{\sim}{r}-\underset{\sim}{s}\right)\right\|$ is the length of projection of $\overrightarrow{SR}$ onto $\underset{\sim}{q}$
4(iii)	$\underset{\sim}{p}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$ $\underset{\sim}{q}=-\frac{1}{\sqrt{(-1)^{2}+2^{2}+4^{2}}}\begin{pmatrix}-1\\2\\4\end{pmatrix}=-\frac{1}{\sqrt{21}}\begin{pmatrix}-1\\2\\4\end{pmatrix}$

No.	Suggested Solutions
5(i)	 Since $f(x) = xe^{x^2}$ is an odd function, i.e. symmetrical about the origin, $R_1 = R_2 = \frac{1}{2}R$ Area of R $= 2\left[e - \int_0^1 xe^{x^2}\,dx\right]$ $= 2\left[e - \frac{1}{2}\int_0^1 2xe^{x^2}\,dx\right]$ $= 2e - \left[e^{x^2}\right]_0^1 = 2e - [e-1] = e+1$ units^2
5 (ii)	 Volume generated by $R = 2\left[\pi(e^2)(1) - \pi\int_0^1 \left(xe^{x^2}\right)^2 dx\right] = 38.534$ units^3 (to 3 d.p.) **<u>Method 2</u>** Volume generated by $R = \pi(e^2)(2) - \pi\int_{-1}^1 \left(xe^{x^2}\right)^2 dx = 38.534$ units^3 (to 3 d.p.)

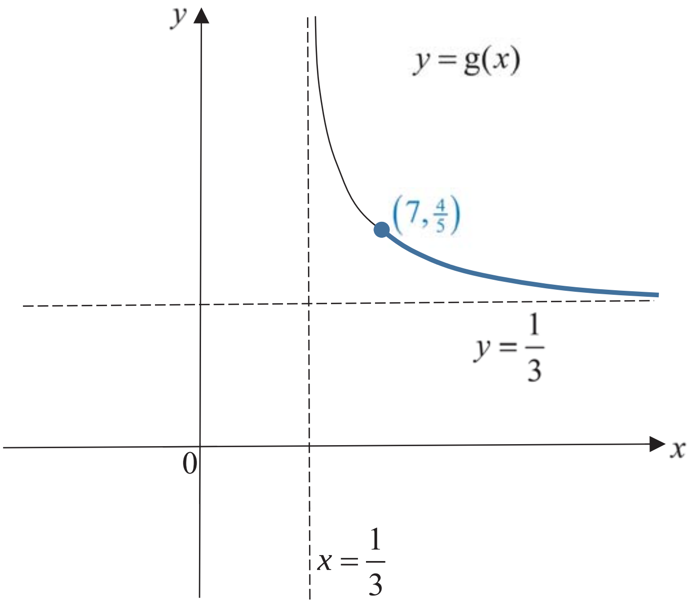
No.	Suggested Solutions
6(i)	$S_n = e^2 - (-2)^n (e^{2-n})$ $T_1 = S_1 = e^2 - (-2)(e^{2-1}) = e^2 + 2e$
6(ii)	$T_n = S_n - S_{n-1}$ $= \left[e^2 - (-2)^n (e^{2-n})\right] - \left[e^2 - (-2)^{n-1}(e^{2-n+1})\right]$ $= (-2)^{n-1}(e^{2-n+1}) - (-2)^n (e^{2-n})$ $= (-2)^{n-1}(e^{2-n})[e - (-2)]$ $= (-2)^{n-1}(e^{2-n})[e+2]$ Since $\frac{T_{n+1}}{T_n} = \frac{(-2)^n (e^{1-n})[e+2]}{(-2)^{n-1}(e^{2-n})[e+2]} = -\frac{2}{e}$ is a constant, this is a Geometric Series with common ratio $-\frac{2}{e}$.
6(iii)	Since $\lvert r \rvert = \left\lvert -\frac{2}{e} \right\rvert = 0.736\ (3\text{ s.f.}) < 1$ Sum to infinity exits. $S = \frac{a}{1-r}$ $= \frac{e(2+e)}{1 - \left(-\frac{2}{e}\right)}$ $= \frac{e(2+e)}{\left(\frac{e+2}{e}\right)}$ $= e^2$
7(i)	When C crosses the x-axis , $y = 0$ $y = 2\sin\theta + 1 = 0$ $\sin\theta = -\frac{1}{2}$ $\theta = -\frac{\pi}{6}, -\frac{5\pi}{6}$ since $-\pi \le \theta \le \pi$

7 **(ii)**	
7 **(iii)**	$\frac{\mathrm{d}x}{\mathrm{d}\theta}=2\theta$, $\frac{\mathrm{d}y}{\mathrm{d}\theta}=2\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}\theta}\div\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{\cos\theta}{\theta}$ $\Rightarrow$ At P, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos p}{p}$ Equation of tangent at P : $y-(2\sin p+1)=\frac{\cos p}{p}\left(x-p^2-1\right)$ $y=\frac{x\cos p}{p}-\frac{\left(p^2+1\right)\cos p}{p}+2\sin p+1$
7 **(iv)**	 Tangent parallel to the y-axis: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos\theta}{\theta}$ is undefined at $\theta=0$, $Q\ (1,1)$

	Method 1: Using parametric equations Required Area $=\int_{1}^{1+\frac{\pi^2}{36}} y\,\mathrm{d}x$ $=\int_{0}^{-\frac{\pi}{6}} (1+2\sin\theta)(2\theta)\,\mathrm{d}\theta$ $= 0.087955042$ $= 0.0880$ units^2 (3 s.f.)
	Method 2: Using Cartesian equation $x=\theta^2+1$ $\theta=\pm\sqrt{x-1}$, $y=2\sin\theta+1$ $y=2\sin\left(-\sqrt{x-1}\right)+1$ Required Area $=\int_{1}^{1+\frac{\pi^2}{36}} y\,\mathrm{d}x$ $=\int_{1}^{1+\frac{\pi^2}{36}} 2\sin\left(-\sqrt{x-1}\right)+1\,\mathrm{d}x$ $= 0.087954559$ $= 0.0880$ units^2 (3 s.f.)
8(i)	**<u>Method 1</u>** $\mathrm{g}(1)=5 \Rightarrow \frac{1+a}{3+b}=5 \Rightarrow a-5b=14$ Since g is self-inverse, $\mathrm{g}(5)=1 \Rightarrow \frac{5+a}{15+b}=1 \Rightarrow a-b=10$ Using GC, $a=9,\ b=-1$.
	<u>Method 2</u> Let $y=\frac{x+a}{3x+b} \Rightarrow 3xy+by=x+a \Rightarrow x=\frac{a-by}{3y-1}$ $\mathrm{g}^{-1}(x)=\frac{a-bx}{3x-1}$ Since g is self-inverse, $\frac{a-bx}{3x-1}=\frac{x+a}{3x+b}$

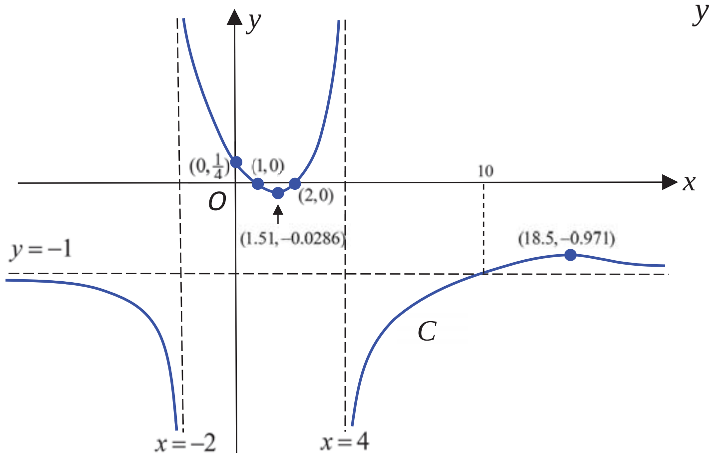
	$b=-1$ $g(1)=5$ $\frac{1+a}{3+b}=5 \Rightarrow 1+a=15+5b \Rightarrow a-5b=14$ $a=14-5=9$
	Method 3 $g(x)=\frac{x+a}{3x+b}=\frac{\frac{1}{3}(3x+b)-\frac{1}{3}b+a}{3x+b}=\frac{1}{3}-\frac{b-3a}{3(3x+b)}$ Since $y=\frac{1}{3}$ is the horizontal asymptote, by symmetry $x=\frac{1}{3}$ is the vertical asymptote. $\therefore b=-1$ $g(1)=5$ $\frac{1+a}{3+b}=5 \Rightarrow 1+a=15+5b \Rightarrow a-5b=14$ $a=14-5=9$
8 (ii)	Since $g(x)= g^{-1}(x)$, $g^2(x)=x$ $g^3(x)=g\left[g^2(x)\right]=g(x)$ $\therefore g^{2021}(1)=g\left[g^{2020}(1)\right]=g(1)=5$
8 (iii)	$h: x \mapsto \lvert 1-x \rvert (x+5)$, for $x \in \mathbb{R}, x \geq 2$.  Since $x \geq 2$, $\lvert 1-x \rvert = x-1$. $\therefore h(x)=(x-1)(x+5)$ Let $y=x^2+4x-5$ $x^2+4x-5-y=0$ $x=\frac{-4\pm\sqrt{16-4(-5-y)}}{2}$ $=\frac{-4\pm\sqrt{36+4y}}{2}$ $=-2\pm\sqrt{9+y}$ Since $x \geq 2$, $x=-2+\sqrt{9+y}$ $h^{-1}: x \mapsto \sqrt{9+x}-2, \quad x \in \mathbb{R}, x \geq 7.$

<table>
<tr><td>8
(iv)</td><td>



$R_h = [7,\infty)$ and $D_g = \left(\frac{1}{3},\infty\right)$
Since $R_h \subseteq D_g$, $\therefore$ gh exits
$D_h = [2,\infty) \xrightarrow{h} R_h = [7,\infty) \xrightarrow{g} R_{gh} = \left(\frac{1}{3},\frac{4}{5}\right]$
$R_{gh} = \left(\frac{1}{3},\frac{4}{5}\right]$
</td></tr>
<tr><td>9(i)</td><td>
<u>Method 1</u>: (solve algebraically)
When $y=-1$, $-1 = \frac{(1-x)(x-2)}{x^2-2x-8}$
$-x^2+2x+8 = -x^2+3x-2$
$\therefore x = 10$
<u>Method 2</u>: (solve graphically using GC)
Using GC, $\therefore x = 10$
</td></tr>
</table>

9(ii)



$$y = \frac{(1-x)(x-2)}{x^2-2x-8}$$
$$= \frac{-x^2+3x-2}{x^2-2x-8}$$
$$= -1 + \frac{x-10}{(x+2)(x-4)}$$

Long division: $x^2-2x-8 \,\big)\, -x^2+3x-2$, quotient -1

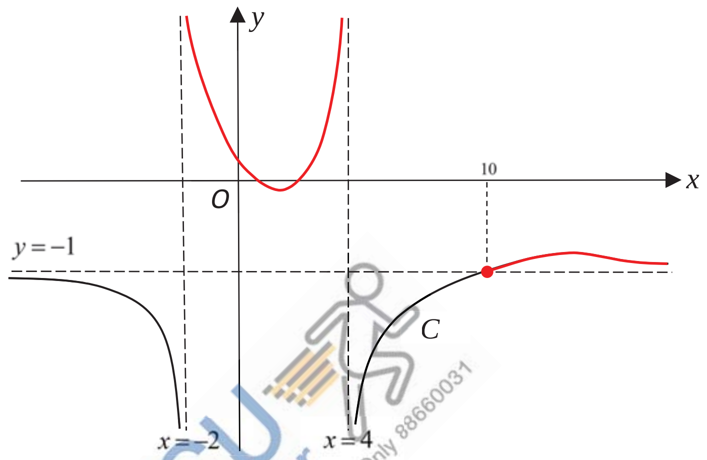
$-x^2+2x+8$

$x-10$

9 (iii)

Method 1: [using graph in part (ii)]

$\frac{-x^2+3x-2}{x^2-2x-8} \geq -1 \Rightarrow$ Curve $C \geq -1$



$\therefore$ from graph in (ii), $-2 < x < 4$ or $x \geq 10$

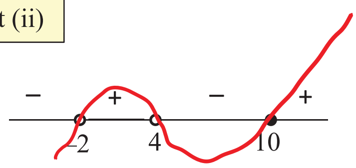
Method 2: (solve inequality directly)

$$\frac{-x^2+3x-2}{x^2-2x-8} \geq -1$$
$$-1 + \frac{x-10}{(x+2)(x-4)} \geq -1$$
$$\frac{x-10}{(x+2)(x-4)} \geq 0$$

Use simplified form after long division from part (ii)

$(x-10)(x+2)(x-4) \geq 0, \; x \neq -2, 4$

$\therefore \; -2 < x < 4$ or $x \geq 10$



9 (iv)

Replace x with $\frac{1}{x}$ in (iii):

$$\frac{-(\frac{1}{x})^2+3(\frac{1}{x})-2}{(\frac{1}{x})^2-2(\frac{1}{x})-8} \geq -1$$

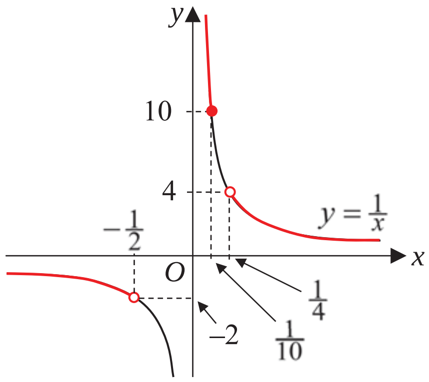
$$\frac{\frac{-1+3x-2x^2}{x^2}}{\frac{1-2x-8x^2}{x^2}} \geq -1$$

$$\frac{-1+3x-2x^2}{1-2x-8x^2} \geq -1$$

Hence using result in (iii):

$-2 < \frac{1}{x} < 4$ or $\frac{1}{x} \geq 10$

<u>Method 1</u>: (using graph of $y = \frac{1}{x}$)



From graph,

$x < -\frac{1}{2}$ or $x > \frac{1}{4}$ $\quad 0 < x \leq \frac{1}{10}$

Note that $x = 0$ is also a solution to $\frac{-1+3x-2x^2}{1-2x-8x^2} \geq -1$

Hence the final solution is $x < -\frac{1}{2}$ or $x > \frac{1}{4}$ or $0 \leq x \leq \frac{1}{10}$

<u>Method 2</u>: (solving algebraically)

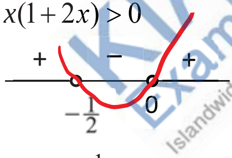
For $-2 < \frac{1}{x} < 4$:

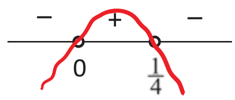
$-2 < \frac{1}{x}$ and $\frac{1}{x} < 4$

$\frac{1}{x} + 2 > 0$ $\quad \frac{1}{x} - 4 < 0$

$\frac{1+2x}{x} > 0$ $\quad \frac{1-4x}{x} < 0$

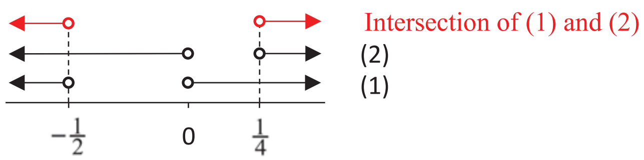
$x(1+2x) > 0$ $\quad x(1-4x) < 0$





$\therefore\ x < -\frac{1}{2}$ or $x > 0$ ...(1) and $\therefore\ x < 0$ or $x > \frac{1}{4}$ ...(2)

Combining (1) and (2), ie. taking intersection:



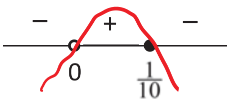
Hence $x < -\frac{1}{2}$ or $x > \frac{1}{4}$ (*)

For $\frac{1}{x} \geq 10$:

$\frac{1}{x}-10\geq 0$

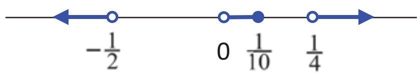
$\frac{1-10x}{x}\geq 0$

$x(1-10x)\geq 0$



$\therefore\ 0<x\leq\frac{1}{10}$ (**)

Hence combining (*) or (**), ie. taking union:

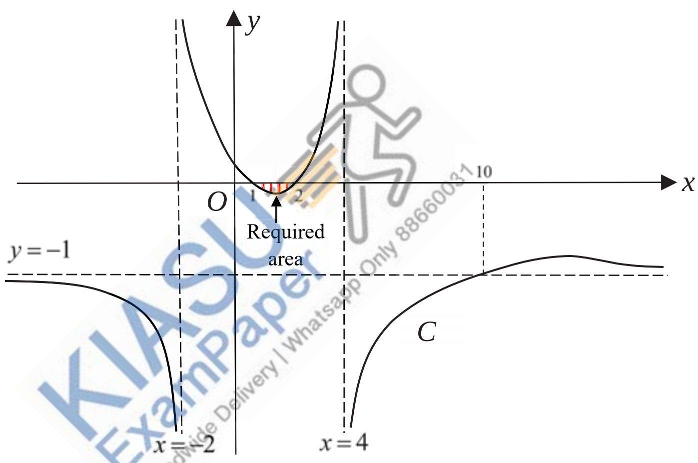


$\therefore\ x<-\frac{1}{2}$ or $0<x\leq\frac{1}{10}$ or $x>\frac{1}{4}$

Note that $x=0$ is also a solution to $\frac{-1+3x-2x^2}{1-2x-8x^2}\geq -1$

Hence the final solution is $x<-\frac{1}{2}$ or $x>\frac{1}{4}$ or $0\leq x\leq\frac{1}{10}$

9 (v) From graph in (iii):



Method 1: (using partial fractions)

$$\text{Area} = -\int_1^2 \frac{-x^2+3x-2}{x^2-2x-8}\,dx$$

$$= -\int_1^2 -1+\frac{x-10}{x^2-2x-8}\,dx$$

Use simplified form after long division from part (ii)

$$= \int_1^2 1-\frac{x-10}{(x+2)(x-4)}\,dx$$

Let $\frac{x-10}{(x+2)(x-4)}=\frac{A}{x+2}+\frac{B}{x-4}$

$x-10=A(x-4)+B(x+2)$

When $x=4$, $B=-1$

	When $x=-2$, $A=2$ $\therefore$ Area $=\int_1^2 1-\left[\frac{2}{x+2}-\frac{1}{x-4}\right]\mathrm{d}x$ $=\int_1^2 1-\frac{2}{x+2}+\frac{1}{x-4}\,\mathrm{d}x$ $=\left[x-2\ln\lvert x+2\rvert+\ln\lvert x-4\rvert\right]_1^2$ $=2-2\ln 4+\ln 2-1+2\ln 3-\ln 3$ $=1-3\ln 2+\ln 3$ $=1-\ln 8+\ln 3$ $=1+\ln\frac{3}{8}$ unit^2 where $a=1$, $b=3$, $c=8$ <u>Method 2</u>: (using $\int\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x$ and MF26) Area $=-\int_1^2 \frac{-x^2+3x-2}{x^2-2x-8}\mathrm{d}x$ $=\int_1^2 1-\frac{x-10}{x^2-2x-8}\mathrm{d}x$ ← Use simplified form after long division from part (ii) $=\int_1^2 1-\left[\frac{\frac{1}{2}(2x-2)-9}{x^2-2x-8}\right]\mathrm{d}x$ $=\int_1^2 1\,\mathrm{d}x-\frac{1}{2}\int_1^2\frac{2x-2}{x^2-2x-8}\mathrm{d}x+9\int_1^2\frac{1}{(x-1)^2-9}\mathrm{d}x$ ← Can also use partial fractions: $9\int_1^2\frac{1}{x^2-2x-8}\mathrm{d}x=9\int_1^2\frac{1}{(x+2)(x-4)}\mathrm{d}x=\frac{3}{2}\int_1^2-\frac{1}{x+2}+\frac{1}{x-4}\mathrm{d}x$ $=[x]_1^2-\frac{1}{2}\left[\ln\lvert x^2-2x-8\rvert\right]_1^2+9\left[\frac{1}{2(3)}\ln\left\lvert\frac{x-1-3}{x-1+3}\right\rvert\right]_1^2$ ← From MF26 $=1-\frac{1}{2}[\ln 8-\ln 9]+\frac{3}{2}\left[\ln\frac{1}{2}-\ln 1\right]$ $=1-\frac{3}{2}\ln 2+\ln 3-\frac{3}{2}\ln 2$ $=1+\ln 3-\ln 8$ $=1+\ln\frac{3}{8}$ unit^2 where $a=1$, $b=3$, $c=8$
10 (i)	$V=\pi r^2h+\frac{1}{2}\left[\frac{4}{3}\pi\left(\frac{r}{2}\right)^3\right]$ $V=\pi r^2h+\frac{2}{3}\pi\left(\frac{r^3}{8}\right)$ $h=\frac{V}{\pi r^2}-\frac{r}{12}$
10 (ii)	Let C be the total cost of manufacturing the prototype. $C=8[\text{Cylinder's: Base area + Curved Surface + Top}]$ $+3[\text{Curved Surface of hemisphere}]$

	$C=8\left[\pi r^2+2\pi rh+\left\{\pi r^2-\pi\left(\frac{r}{2}\right)^2\right\}\right]+3\times\frac{1}{2}\left[4\pi\left(\frac{r}{2}\right)^2\right]$ $C=8\left[\pi\left(2r^2-\frac{r^2}{4}\right)+2\pi rh\right]+3\left[2\pi\left(\frac{r^2}{4}\right)\right]$ $=8\left[\frac{7\pi r^2}{4}+2\pi r\left(\frac{V}{\pi r^2}-\frac{r}{12}\right)\right]+\frac{3\pi r^2}{2}$ $=8\left[\frac{19\pi r^2}{12}+\frac{2V}{r}\right]+\frac{3\pi r^2}{2}$ $=\frac{16V}{r}+\frac{85\pi r^2}{6}$ (shown)
	$\frac{dC}{dr}=\frac{85}{3}\pi r-\frac{16V}{r^2}$ $\frac{dC}{dr}=\frac{1}{r^2}\left(\frac{85}{3}\pi r^3-16V\right)$ For stationary value, $\frac{dC}{dr}=0$ $r^3=\frac{48V}{85\pi}$ $r=\sqrt[3]{\frac{48V}{85\pi}}$
	Using 2nd Derivative Test: $\frac{d^2C}{dr^2}=\frac{85}{3}\pi+\frac{32V}{r^3}=\frac{85}{3}\pi+\frac{32V}{\frac{48V}{85\pi}}=85\pi>0$ Thus, C is a minimum when $r=\sqrt[3]{\frac{48V}{85\pi}}$.
	$h=\frac{V}{\pi r^2}-\frac{r}{12}$

$$\frac{h}{r}=\frac{V}{\pi r^3}-\frac{1}{12}$$

$$\frac{h}{r}=\frac{V}{\pi\left(\frac{48V}{85\pi}\right)}-\frac{1}{12}$$

$$\frac{h}{r}=\frac{85}{48}-\frac{1}{12}=\frac{27}{16}$$

$$\frac{\mathrm{d}C}{\mathrm{d}r}=\frac{85}{3}\pi r-\frac{16V}{r^2}$$

$$\frac{\mathrm{d}C}{\mathrm{d}r}=\frac{1}{r^2}\left(\frac{85}{3}\pi r^3-16V\right)$$

$$\frac{\mathrm{d}C}{\mathrm{d}r}=0$$

Using 1st Derivative Test:

r	$\sqrt[3]{\frac{48V}{85\pi}}^{-}$	$\sqrt[3]{\frac{48V}{85\pi}}$	$\sqrt[3]{\frac{48V}{85\pi}}^{+}$
$\frac{\mathrm{d}C}{\mathrm{d}r}$	$-$	0	$+$
Shape	\	–	/

---

10 (iii)

After 5 seconds, amount of perfume leaked is

$1.5\times 5=7.5\text{ cm}^3$

Total volume of hemisphere is

$$\frac{2}{3}\pi\left(\frac{3}{2}\right)^3=\frac{9\pi}{4}=7.06858\text{ cm}^3<7.5$$

After 5 seconds, perfume is in the bottom segment, i.e. cylinder.

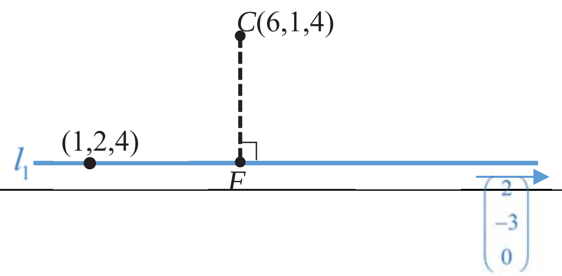
$V=\pi r^2h=9\pi h$

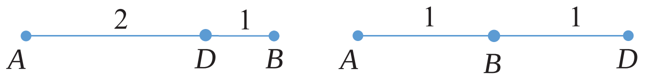
$$\frac{\mathrm{d}V}{\mathrm{d}h}=9\pi$$

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{9\pi}\times(-1.5)=-\frac{1}{6\pi}\text{ cm / s}$$

Height of perfume decreases at $\frac{1}{6\pi}$ cm / s.

11(i)



	Let the foot of perpendicular be F. Since F lies on l_1, $\overrightarrow{OF}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\0\end{pmatrix}$ for some λ $\overrightarrow{CF}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\0\end{pmatrix}-\begin{pmatrix}6\\1\\4\end{pmatrix}=\begin{pmatrix}-5+2\lambda\\1-3\lambda\\0\end{pmatrix}$ Since $\overrightarrow{CF}\perp l_1$, $\begin{pmatrix}-5+2\lambda\\1-3\lambda\\0\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\0\end{pmatrix}=0$ $-10+4\lambda-3+9\lambda=0$ $\lambda=1$ $\overrightarrow{OF}=\begin{pmatrix}1\\2\\4\end{pmatrix}+1\begin{pmatrix}2\\-3\\0\end{pmatrix}=\begin{pmatrix}3\\-1\\4\end{pmatrix}$ $\therefore F(3,-1,4)$
11 (ii)	$\overrightarrow{AB}=\begin{pmatrix}5\\0\\7\end{pmatrix}-\begin{pmatrix}-1\\-12\\4\end{pmatrix}=\begin{pmatrix}6\\12\\3\end{pmatrix}=3\begin{pmatrix}2\\4\\1\end{pmatrix}$ $l_2: \underset{\sim}{r}=\begin{pmatrix}5\\0\\7\end{pmatrix}+\mu\begin{pmatrix}2\\4\\1\end{pmatrix}\quad \mu\in\mathbb{R}$ $\theta=\cos^{-1}\dfrac{\left\|\begin{pmatrix}2\\4\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\0\end{pmatrix}\right\|}{\sqrt{21}\sqrt{13}}=\cos^{-1}\dfrac{8}{\sqrt{21}\sqrt{13}}=61.0^\circ$ (1 d.p)
11 (iii)	

	$\overrightarrow{OD}=\dfrac{\overrightarrow{OA}+2\overrightarrow{OB}}{3}$ or $\overrightarrow{OB}=\dfrac{\overrightarrow{OA}+\overrightarrow{OD}}{2}$ $\overrightarrow{OD}=\dfrac{\begin{pmatrix}-1\\-12\\4\end{pmatrix}+2\begin{pmatrix}5\\0\\7\end{pmatrix}}{3}$ or $\overrightarrow{OD}=2\overrightarrow{OB}-\overrightarrow{OA}$ $\overrightarrow{OD}=\begin{pmatrix}3\\-4\\6\end{pmatrix}$ or $\overrightarrow{OD}=2\begin{pmatrix}5\\0\\7\end{pmatrix}-\begin{pmatrix}-1\\-12\\4\end{pmatrix}=\begin{pmatrix}11\\12\\10\end{pmatrix}$ $\therefore D(3,-4,6)$ or $D(11,12,10)$
	Since D lies on l_2 and $l_2: \underset{\sim}{r}=\begin{pmatrix}5\\0\\7\end{pmatrix}+\mu\begin{pmatrix}2\\4\\1\end{pmatrix}$ $\mu\in\mathbb{R}$, $\overrightarrow{OD}=\begin{pmatrix}5+2\mu\\4\mu\\7+\mu\end{pmatrix}$ for some μ $\overrightarrow{AD}=(3+\mu)\begin{pmatrix}2\\4\\1\end{pmatrix}$ and $\overrightarrow{BD}=\mu\begin{pmatrix}2\\4\\1\end{pmatrix}$ $\overrightarrow{AD}=2\overrightarrow{DB} \Rightarrow 3+\mu=-2\mu \Rightarrow \mu=-1$ $\overrightarrow{AD}=2\overrightarrow{BD} \Rightarrow 3+\mu=2\mu \Rightarrow \mu=3$ $\therefore D(3,-4,6)$ or $D(11,12,10)$
	Since D lies on l_2 and $l_2: \underset{\sim}{r}=\begin{pmatrix}5\\0\\7\end{pmatrix}+\mu\begin{pmatrix}2\\4\\1\end{pmatrix}$ $\mu\in\mathbb{R}$, $\overrightarrow{OD}=\begin{pmatrix}5+2\mu\\4\mu\\7+\mu\end{pmatrix}$ for some μ

	$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\begin{pmatrix}6+2\mu\\12+4\mu\\3+\mu\end{pmatrix}$ $\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=\begin{pmatrix}2\mu\\4\mu\\\mu\end{pmatrix}$ $\lvert\overrightarrow{AD}\rvert=2\lvert\overrightarrow{BD}\rvert$ $\lvert\overrightarrow{AD}\rvert^2=4\lvert\overrightarrow{BD}\rvert^2$ $(6+2\mu)^2+(12+4\mu)^2+(3+\mu)^2=4\left(4\mu^2+16\mu^2+\mu^2\right)$ From GC, $\mu=-1$ or $\mu=3$ $\therefore D(3,-4,6)$ or $D(11,12,10)$
11 (iv)	Direction vector of l_3 $=\begin{pmatrix}2\\-3\\0\end{pmatrix}\times\begin{pmatrix}2\\4\\1\end{pmatrix}$ $=\begin{pmatrix}-3\\-2\\14\end{pmatrix}$ $l_3:\underset{\sim}{r}=\begin{pmatrix}-1\\-12\\4\end{pmatrix}+\alpha\begin{pmatrix}-3\\-2\\14\end{pmatrix}\quad\alpha\in\mathbb{R}$
12 (i)	Interest rate for 12 months = $p\%$ Interest rate per month = $\frac{p}{100}\times\frac{1}{12}$ $\therefore$ Interest before 1 Sept $=\frac{pL}{1200}$
12 (ii)	<table><tr><th>Month</th><th>Outstanding Loan</th></tr><tr><td>0</td><td>L</td></tr><tr><td>1</td><td>$\left(1+\frac{p}{1200}\right)L-x$</td></tr></table>

2	$\left(1+\frac{p}{1200}\right)\left[\left(1+\frac{p}{1200}\right)L-x\right]-x$ $=\left(1+\frac{p}{1200}\right)^2 L-\left(1+\frac{p}{1200}\right)x-x$
3	$\left(1+\frac{p}{1200}\right)\left[\left(1+\frac{p}{1200}\right)^2 L-\left(1+\frac{p}{1200}\right)x-x\right]-x$ $=\left(1+\frac{p}{1200}\right)^3 L-\left(1+\frac{p}{1200}\right)^2 x-\left(1+\frac{p}{1200}\right)x-x$
n	$\left(1+\frac{p}{1200}\right)^n L-\left(1+\frac{p}{1200}\right)^{n-1} x$ $\ldots-\left(1+\frac{p}{1200}\right)^2 x-\left(1+\frac{p}{1200}\right)x-x$

$$L_n=\left(1+\frac{p}{1200}\right)^n L-\left(1+\frac{p}{1200}\right)^{n-1} x-\ldots-\left(1+\frac{p}{1200}\right)^2 x-\left(1+\frac{p}{1200}\right)x-x$$

$$=\left(1+\frac{p}{1200}\right)^n L-x\left[\left(1+\frac{p}{1200}\right)^{n-1}+\left(1+\frac{p}{1200}\right)^{n-2}\ldots+1\right]$$

$$=\left(1+\frac{p}{1200}\right)^n L-x\left[\frac{\left(1+\frac{p}{1200}\right)^n-1}{\left(1+\frac{p}{1200}\right)-1}\right]$$

$$=\left(1+\frac{p}{1200}\right)^n L-\frac{1200x}{p}\left[\left(1+\frac{p}{1200}\right)^n-1\right] \text{ (Shown)}$$

12 (iii)

In 30 years, $n=360$,

$$504000\left(1+\frac{2.6}{1200}\right)^{360}-\frac{1200x}{2.6}\left(\left(1+\frac{2.6}{1200}\right)^{360}-1\right)\le 0$$

$$\frac{1200x}{2.6}\left(\left(1+\frac{2.6}{1200}\right)^{360}-1\right)\ge 504000\left(1+\frac{2.6}{1200}\right)^{360}$$

$$x\ge\frac{1098534.707}{544.4457028}$$

$$\ge 2017.712145$$

OR

Outstanding loan, $Y_1=504000\left(1+\frac{2.6}{1200}\right)^{360}-\frac{1200x}{2.6}\left(\left(1+\frac{2.6}{1200}\right)^{360}-1\right)$

By GC, when $Y_1=0$, $x=2017.712146$

Check: When $x = 2017.71$, $Y_1 = 1.1684474 > 0$

$\therefore$ Their monthly instalment is \$2017.72. (nearest cent)
Or
Their monthly instalment is \$2020. (3 s.f.)

12 (iv)

When $504000\left(1+\frac{2.6}{1200}\right)^n - \frac{1200\times 4000}{2.6}\left[\left(1+\frac{2.6}{1200}\right)^n - 1\right] \le 0$,

by GC table,

X	Y2
140	28983
141	25046
142	21100
143	17146
144	13183
145	9211.4
146	5231.3
147	1242.7
148	-2755
149	-6761
150	-10775

$\therefore n = 148 \Rightarrow$ 12 years 4 months

The couple will fully repay the loan on 1 December 2033.

Final repayment = \$1245.38

12 (v)

$$\sum_{r=1}^{n}(450.45+50.05r)$$

$$=1.001\sum_{r=1}^{n}\left[k+(n-1)a\right]$$

$$=1.001\sum_{r=1}^{n}\left[450+50r\right]$$

$$=1.001\sum_{r=1}^{n}\left[500+50(r-1)\right]$$

Hence, $k = 500$, $a = 50$

Alternative solution

when $n = 1$,

$1.001k = 450.45 + 50.05$

$k = 500$

when $n = 2$,

$1.001k + 1.001(k+a) = 450.45 + 50.05 + 450.45 + 50.45(2)$

$2k + a = 1050$

$a = 50$

Hence, $k = 500$, $a = 50$

12 (vi)

$$\sum_{r=1}^{n}(450.45+50.05r)\geq 504000\left(1+\frac{2.6}{1200}\right)^{n}-\frac{1200\times 4000}{2.6}\left[\left(1+\frac{2.6}{1200}\right)^{n}-1\right]$$

Let $Y_1=\sum_{r=1}^{n}(450.45+50.05r)$

$$Y_2=504000\left(1+\frac{2.6}{1200}\right)^{n}-\frac{1200\times 4000}{2.6}\left[\left(1+\frac{2.6}{1200}\right)^{n}-1\right]$$

Use GC directly:

```
1+2.6/1200
                 1.002166667
Ans→I
                 1.002166667
\Y1=Σ(A=1 to X)(450.45+50.05A)
\Y2=504000(I)^X−(1200*4000*(I^X…)/2.6
\Y3=Y1−Y2
```

X	Y1	Y2	Y3
85	221221	232916	-11695
86	225976	229421	-3445
87	230781	225918	4862.6
88	235635	222407	13228

∴ Least number of months = 87

Or use

$$450.45n+\frac{50.05n(n+1)}{2}\geq 504000\left(\frac{6013}{6000}\right)^{n}-\frac{1200(4000)}{2.6}\left[\left(\frac{6013}{6000}\right)^{n}-1\right]$$

To be keyed into GC to solve

1 The parametric equations of a curve C are

$$x = t + 2\mathrm{e}^{t},\ y = 2t + \mathrm{e}^{-t},\ t > -2.$$

(i) Sketch C. [2]

(ii) Find the equation of the normal to the curve at the point P where $t = 0$. [4]

2 A curve C has equation $y = \dfrac{a}{x+3} + bx + c$, where a, b and c are constants. It is given that C passes through the point with coordinates (1, –3) and has a stationary point (–5, –21).

(i) Find the values of a, b and c. [4]

(ii) Hence find the equations of the asymptotes of C. [2]

3 It is given that $x^2 y - \tan^{-1} y = \dfrac{3}{4}\pi$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

Hence, find the exact value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = 1$, given that $x > 0$. [3]

4 **(i)** Sketch the curve with equation $y = \ln(x+1), x > -1$, stating the equation of asymptote and intercepts with the axes. On the same diagram, sketch the curve with equation $y = \sqrt{9 - x^2}, -3 \le x \le 3$, stating the intercepts with the axes. [4]

(ii) Use your answer in part **(i),** solve the inequality $\ln(x+1) < \sqrt{9 - x^2}$. [2]

(iii) Hence solve the inequality $\ln(x^2 + 1) < \sqrt{9 - x^4}$. [3]

5 The function f is defined by $\mathrm{f} : x \to \dfrac{1}{3}\left(\mathrm{e}^{x-2} - 1\right), x > 2$.

(i) Find f^{-1} and write down the its domain. [3]

(ii) Explain why the solution of $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ satisfies the equation $\mathrm{e}^{x-2} = 3x + 1$ and find the value of this solution. [3]

The function g is defined by $\mathrm{g} : x \to 1 + x^2,\ x \in \mathbb{R}$.

(iii) Show that gf exists . Hence find the composite function gf , stating its domain and the corresponding range. [4]

6 **(a)** Given that $\mathbf{u} \bullet \mathbf{v} = 0$, what can be deduced about the vectors **u** and **v**? [2]

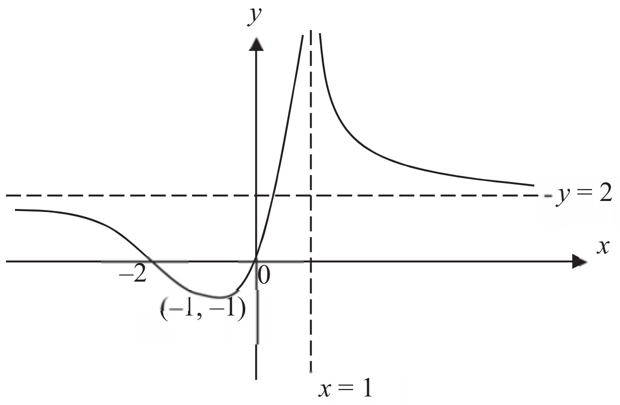
(b) Referred to an origin O, the position vectors of two points A and B are **a** and **b** respectively. A line l has vector equation given by $\mathbf{r} = \frac{1}{3}\mathbf{a} + \lambda(2\mathbf{b} - \mathbf{a})$, where $\lambda \in \mathbb{R}$.

The point N is the foot of perpendicular from A to l. It is given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$ and **a** is perpendicular to **b**.

(i) Find the position vector of N in terms of **a** and **b**. [5]

(ii) Find the exact area of triangle OAN. [3]

7 **(a)** The graph of $y = \mathrm{f}(x)$ has asymptotes $x = 1$, $y = 2$ and a minimum point at $(-1, -1)$ as shown in the diagram. It cuts the x-axis at the origin and at $(-2, 0)$.



Sketch the following graphs on separate diagrams, labeling clearly the asymptotes, turning points and intercepts on the axes where applicable.

(i) $y = \mathrm{f}(2x) + 1$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

(b) The curve whose equation is $y = \dfrac{1}{x+1}$ undergoes, in succession, the following transformations:

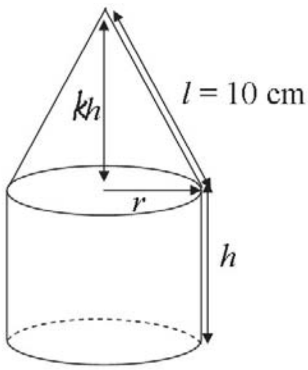
A: A reflection in the y – axis.

B: A translation of 5 units in the negative x – direction.

Give the equation of the resulting curve. [3]

[Turn over

8 **(a)**



A closed container is constructed using a sheet of metal with area 100π cm^2. The container comprises 2 shapes, a cone and a cylinder. The slant height, l, of the cone is 10 cm. Given that the cylinder has height h, and radius r, and the height of the cone is kh, where k is a positive constant.

(i) show that $h = \dfrac{100 - r^2 - 10r}{2r}$, [1]

(ii) use differentiation to show that the exact maximum volume of the container is given that $V = \left(\dfrac{1}{3}k + 1\right)\dfrac{2500\pi}{27}$ cm^3, proving that it is a maximum. [6]

[Volume of Cone $= \dfrac{1}{3}\pi r^2 h$, Curved Surface Area of Cone $= \pi rl$]

(b) The height of an upright cone is twice the radius, r, of its circular base. It is known that the volume of the cone is increasing at the rate of 15 cm^3 min^{-1} when the radius is 3 cm. Find the rate of increase of the base area of the cone at this instant. [4]

9 A plane $\prod_1$ has equation $x + y + z = 3$. A line passes through the points P and Q with position vectors $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ respectively.

(i) Find the exact length of projection of $\overrightarrow{PQ}$ onto $\prod_1$. [3]

(ii) Find the position vector of the point of intersection of line PQ and $\prod_1$. [3]

A plane $\prod_2$ is parallel to the y-z plane and contains the point $(-2, 1, 4)$.

(iii) Find the cartesian equation of $\prod_2$. [2]

(iv) A point $S(a, 7, b)$ lies on both $\prod_1$ and $\prod_2$. Write down the values of a and b. [2]

(v) Hence or otherwise, find a vector equation of the line of intersection of $\prod_1$ and $\prod_2$. [2]

10 **(a)** Find $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\, dx$. [2]

(b) Express $f(x) = \frac{x^2 - 6x + 1}{(3x+1)(x^2+3)}$ in the form

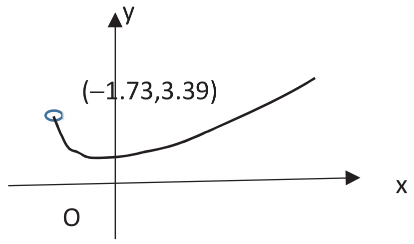
$$\frac{A}{3x+1} + \frac{Bx + C}{x^2 + 3},$$

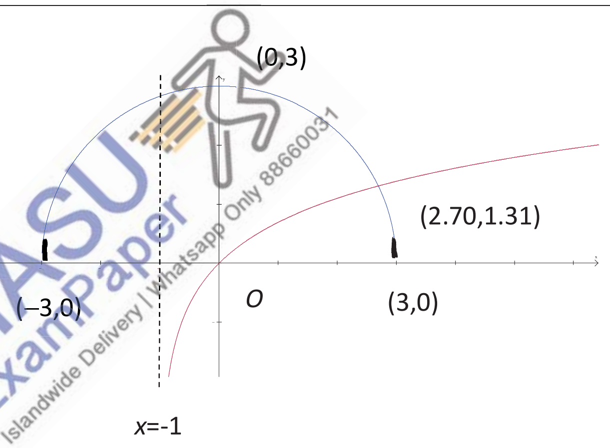
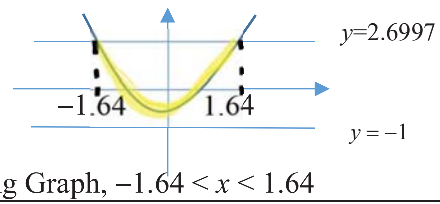
where the values of A, B and C to be determined. [3]

Hence find $\int f(x)\, dx$. [2]

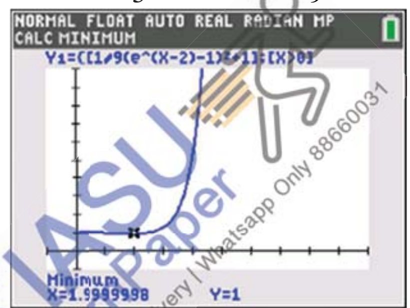
(c) Find the exact value of $\int_{\pi}^{2\pi} \sin^2\left(\frac{x}{8}\right) dx$. [3]

- END OF PAPER -

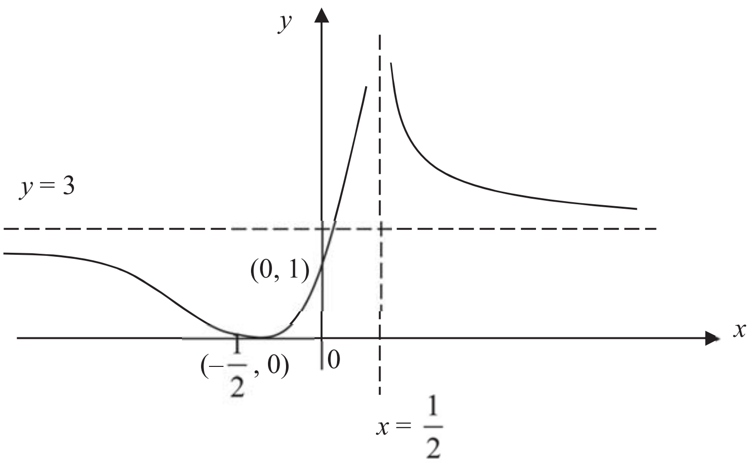
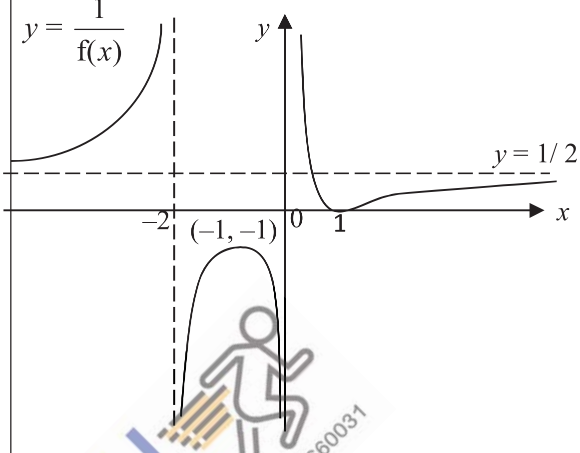
u i	**u i**
1(i)	
1(ii)	$\frac{dx}{dt} = 1 + 2e^{t}, \frac{dy}{dt} = 2 - e^{-t}$ $\frac{dy}{dx} = \frac{2 - e^{-t}}{1 + 2e^{t}}$ At P, $\frac{dy}{dx} = \frac{1}{3}$ Gradient of normal at $P = -3$ At P, $x = 2$, $y = 1$ The equation of normal is $y \quad 1 = -3(x - 2)$ $y = -3x + 7$
2(i)	$(1, -3)$: $\frac{a}{4} + b + c = -3 - - - (1)$ $(-5, -21)$: $\frac{a}{-2} - 5b + c = -21 - - - (2)$ $\frac{dy}{dx} = -\frac{a}{(x+3)^2} + b$ At $x = -5$: $\frac{dy}{dx} = 0 \qquad -\frac{a}{4} + b = 0 - - - (3)$ Using GC: $a = 8$, $b = 2$, $c = -7$ So the equation of C is $y = \frac{8}{x+3} + 2x - 7$
(ii)	The two equations of asymptotes are: $x = -3$ $y = 2x - 7$

u i	u i
3	$x^2 y - \tan^{-1} y = \frac{3}{4}\pi$ Differentiate wrt x, $x^2 \frac{dy}{dx} + y(2x) - \frac{1}{1+y^2}\left(\frac{dy}{dx}\right) = 0$ $\frac{dy}{dx}\left(\frac{1}{1+y^2} - x^2\right) = 2xy$ $\frac{dy}{dx} = \frac{2xy}{\left(\frac{1}{1+y^2} - x^2\right)} = \frac{2xy(1+y^2)}{1-x^2-x^2y^2}$
	when $y = 1$ $x^2(1) - \tan^{-1} 1 = \frac{3}{4}\pi$ $x^2 - \frac{\pi}{4} = \frac{3}{4}\pi$ $x^2 = \pi$ $x = \sqrt{\pi}$ Since $x > 0$, $x = \sqrt{\pi}$ $\frac{dy}{dx} = \frac{2\sqrt{\pi}(1+1^2)}{1-\pi-\pi(1)^2} = \frac{4\sqrt{\pi}}{1-2\pi}$ (exact)
4(i)	
(ii)	The point of intersection is (2.70, 1.31) So the solution is $-1 < x < 2.70$
(iii)	Replace x by x^2: $-1 < x^2 < 2.6997$ <u>Method 1</u>  Using Graph, $-1.64 < x < 1.64$

	<u>Method 2</u> Since x^2 0, $-1 < x^2 < 2.6997$ can be simplified to $x^2 < 2.6997$ $(x - 1.64)(x + 1.64) < 0$ $-1.64 < x < 1.64$

u i	u i
5(i)	Let $y = \frac{1}{3}\left(e^{x-2} - 1\right)$ $e^{x-2} - 1 = 3y$ $x - 2 = \ln(3y + 1)$ $x = \ln(3y + 1) + 2$ $f^{-1}(x) = \ln(3x + 1) + 2,\ x > 0$
(ii)	As the graph of f^{-1} intersects the graph of f in the line $y = x$, the solution of $f(x) = f^{-1}(x)$ is the same as the solution of $f(x) = x$. Hence $\frac{1}{3}\left(e^{x-2} - 1\right) = x$ $e^{x-2} = 3x + 1$ Use GC, the solution is $x = 4.72$
(iii)	$R_f = (0,\ \ )$ and $D_g = (-\ \ ,\ \ )$ So R_f D_g, gf exists $gf(x) = g(\frac{1}{3}\left(e^{x-2} - 1\right)) = \frac{1}{9}\left(e^{x-2} - 1\right)^2 + 1,\ x > 2$  $R_{gf} = (1,\ \ )$

u i	u i
6(a)	$\mathbf{u} \bullet \mathbf{v} = 0$ **u** **v** or **u** is a zero vector or **v** is a zero vector
(b)(i)	Since N lies on l, $ON = \frac{1}{3}\mathbf{a} + \lambda(\mathbf{2b} - \mathbf{a})$ for some λ Since AN is perpendicular to l, $AN \bullet (2\mathbf{b} - \mathbf{a}) = 0$ $\left(\left(-\frac{2}{3} - \lambda\right)\mathbf{a} + 2\lambda\mathbf{b}\right) \bullet (2\mathbf{b} - \mathbf{a}) = 0$ $\left(-\frac{2}{3} - \lambda\right)\mathbf{a} \bullet 2\mathbf{b} + \left(\frac{2}{3} + \lambda\right)\mathbf{a} \bullet \mathbf{a} + 4\lambda\mathbf{b} \bullet \mathbf{b} - \mathbf{2}\lambda\mathbf{a} \bullet \mathbf{b} = 0$ Since **a** and **b** are perpendicular, $\mathbf{a} \bullet \mathbf{b} = 0$ $\left(\frac{2}{3} + \lambda\right)\mathbf{a} \bullet \mathbf{a} + 4\lambda\mathbf{b} \bullet \mathbf{b} = 0$ $\left(\frac{2}{3} + \lambda\right)\lvert\mathbf{a}\rvert^2 + 4\lambda\lvert\mathbf{b}\rvert^2 = 0$ $\left(\frac{2}{3} + \lambda\right)(4) + 4\lambda(1) = 0$ $\lambda = -\frac{1}{3}$ $ON = \frac{1}{3}\mathbf{a} - \frac{1}{3}(\mathbf{2b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$
(ii)	Area of triangle $OAN = \frac{1}{2}\lvert OA \quad ON\rvert$ $= \frac{1}{2}\left\lvert\mathbf{a} \quad \left(\frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\right)\right\rvert$ $= \frac{1}{3}\lvert\mathbf{a} \quad (\mathbf{a} - \mathbf{b})\rvert$ $= \frac{1}{3}\lvert\mathbf{a} \quad \mathbf{a} - \mathbf{a} \quad \mathbf{b}\rvert$ $= \frac{1}{3}\lvert\mathbf{0} - \mathbf{a} \quad \mathbf{b}\rvert$ $= \frac{1}{3}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\lvert\sin 90^{M}\rvert$ $= \frac{1}{3}(2)(1)(1)$ $= \frac{2}{3}$

<table>
<tr><th>u i</th><th>u i</th></tr>
<tr><td>7a(i)</td><td>$y = \mathrm{f}(2x) + 1$

</td></tr>
<tr><td>(ii)</td><td>
</td></tr>
<tr><td>(b)</td><td>$y = \dfrac{1}{x+1} \xrightarrow{\text{A}} y = \dfrac{1}{-x+1} \xrightarrow{\text{B}} y = \dfrac{1}{-(x+5)+1} = \dfrac{1}{-(x+4)}$</td></tr>
</table>

u i	u i
8(a)	$\pi r^2 + 2\pi rh + \pi r(10) = 100\pi$ $h = \dfrac{100 - 10r - r^2}{2r}$
	$V = \dfrac{1}{3}\pi r^2 kh + \pi r^2 h = \pi\left(\dfrac{1}{3}k+1\right) r^2 \left(\dfrac{100 - r^2 - 10r}{2r}\right)$ $V = \dfrac{\pi}{2}\left(\dfrac{1}{3}k+1\right)\left(100r - r^3 - 10r^2\right)$ $\dfrac{dV}{dr} = \dfrac{\pi}{2}\left(\dfrac{1}{3}k+1\right)\left(100 - 3r^2 - 20r\right) = 0$ $r = \dfrac{20 \quad \sqrt{400+4(3)(100)}}{2(-3)} = \dfrac{20 \quad 40}{-6} = -10(\text{NA}), \dfrac{10}{3}$ $\dfrac{d^2V}{dr^2} = \pi\left(\dfrac{1}{3}k+1\right)\left(-6r-20\right) < 0$ since $r > 0$ V is max when $r = \dfrac{10}{3}$ $V = \pi\left(\dfrac{1}{3}k+1\right)\left(100 - \dfrac{100}{9} - \dfrac{100}{3}\right) = \dfrac{2500}{27}\pi\left(\dfrac{1}{3}k+1\right)$
8(b)	Volume of Cone $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi r^2(2r) = \dfrac{2}{3}\pi r^3 \qquad \dfrac{dV}{dr} = 2\pi r^2$ Base area $A = \pi r^2 \qquad \dfrac{dA}{dr} = 2\pi r$ $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = \left(2\pi r^2\right)\dfrac{dr}{dt}$ $15 = 2\pi(3)^2\dfrac{dr}{dt}$ $\dfrac{dr}{dt} = \dfrac{5}{6\pi}$ $\dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = \left(2\pi r\right)\dfrac{dr}{dt} = \left(2\pi\right)(3)\left(\dfrac{5}{6\pi}\right) = 5 \text{ cm}^2 \text{ min}^{-1}$ Alternatively, $\dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dV}\dfrac{dV}{dt} = \left(2\pi r\right)\left(\dfrac{1}{2\pi r^2}\right)(15) = 5\text{cm}^2 \text{ min}^{-1}$

u i	u i
9(i)	$PQ = \begin{pmatrix}4\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\1\\-3\end{pmatrix}$ The length of projection $\dfrac{\left\|\begin{pmatrix}3\\1\\-3\end{pmatrix} \quad \begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{3}}$ $= = \dfrac{1}{\sqrt{3}}\left\|\begin{pmatrix}4\\-6\\2\end{pmatrix}\right\|$ $= \sqrt{\dfrac{56}{3}} = 2\sqrt{\dfrac{14}{3}} = \dfrac{2}{3}\sqrt{42}$
9(ii)	Let M be the point of intersection. $OM = \begin{pmatrix}1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\-3\end{pmatrix}$ for some λ $OM \bullet \begin{pmatrix}1\\1\\1\end{pmatrix} = 3 \quad \begin{pmatrix}1+3\lambda\\1+\lambda\\2-3\lambda\end{pmatrix} \bullet \begin{pmatrix}1\\1\\1\end{pmatrix} = 3$ $\lambda + 4 = 3 \quad \lambda = -1$ $OM = \begin{pmatrix}1\\1\\2\end{pmatrix} + (-1)\begin{pmatrix}3\\1\\-3\end{pmatrix} = \begin{pmatrix}-2\\0\\5\end{pmatrix}$
(iii)	Use normal vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\mathbf{r} \bullet \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\4\end{pmatrix} \bullet \begin{pmatrix}1\\0\\0\end{pmatrix} = -2$ $x = -2$

u i	u i
(iv)	$\begin{pmatrix} a \\ 7 \\ b \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \quad a+7+b=3$ $\begin{pmatrix} a \\ 7 \\ b \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -2 \quad a=-2$ $b=-2$
9(v)	Direction vector of line of intersection $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ Vector equation required is $\mathbf{r} = \begin{pmatrix} -2 \\ 7 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \in$ Alternatively, Solve $x+y+z=3$ and $x=-2$, Using GC, Obtain $\mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \in$
10(a)	$\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \mathrm{d}x$ $= e^{\sin^{-1} x} + c$ since $\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
(b)	$\mathrm{f}(x) = \frac{x^2-6x+1}{(3x+1)(x^2+3)}$ $= \frac{A}{3x+1} + \frac{Bx+C}{x^2+3}$ $= \frac{A(x^2+3)+(Bx+C)(3x+1)}{(3x+1)(x^2+3)}$ $x^2-6x+1 = Ax^2+3A+3Bx^2+3Cx+Bx+C$ $= (A+3B)x^2+(3C+B)x+3A+C$

	Comparing coefficients, $A + 3B = 1$ $B + 3C = \square{-6}$ $3A + C = 1$ From GC, $A = 1,\ B = 0,\ C = -2$ $$\mathrm{f}(x) = \frac{x^2-6x+1}{(3x+1)(x^2+3)} = \frac{1}{(3x+1)} - \frac{2}{(x^2+3)}$$ Alternatively $$\mathrm{f}(x) = \frac{x^2-6x+1}{(3x+1)(x^2+3)} = \frac{x^2+3-6x-2}{(3x+1)(x^2+3)}$$ $$= \frac{1}{3x+1} - \frac{2(3x+1)}{x^2+3} = \frac{1}{3x+1} - \frac{2}{x^2+3}$$ So $A = 1,\ B = 0,\ C = -2$ Alternatively $A(x^2+3) + (Bx+C)(3x+1) = x^2-6x+1$ $x = -\frac{1}{3}: A\left(\frac{28}{9}\right) = \frac{1}{9}+2+1 = \frac{28}{9} \quad A = 1$ $x = 0: 3 + C = 1 \quad C = -2$ $x = 1: 4 + (B-2)(4) = -4 \quad B = 0$ $$\int \mathrm{f}(x)\,\mathrm{d}x = \int \frac{1}{3x+1} - \frac{2}{x^2+3}\,\mathrm{d}x$$ $$= \frac{1}{3}\ln\lvert 3x+1\rvert - \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$$
(c)	$$\int_{\pi}^{2\pi} \sin^2\left(\frac{x}{8}\right)\mathrm{d}x$$ $$= \int_{\pi}^{2\pi} \frac{1}{2}\left(1 - \cos\left(\frac{x}{4}\right)\right)\mathrm{d}x$$ $$= \left[\frac{1}{2}x - 2\sin\left(\frac{x}{4}\right)\right]_{\pi}^{2\pi}$$ $$= \frac{\pi}{2} - 2 + \sqrt{2}$$

NANYANG JUNIOR COLLEGE

JC1 END-OF-YEAR EXAMINATION

Higher 2

CANDIDATE NAME

CT CLASS | 2 | 1 | | |

MATHEMATICS 9758/01

Paper 1 **29 September 2021**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **100**.

For examiner's use only	
Question number	Marks
1	
2	
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10	
11	
Total	

This document consists of **5** printed pages and **0** blank page.

NANYANG JUNIOR COLLEGE
Internal Examinations

1 **(a)** Differentiate $\sin^{-1} x + x\sqrt{1-x^2}$ with respect to x, expressing your answer in its simplest form. Hence, find $\int \sqrt{1-x^2}\, \mathrm{d}x$. [4]

(b) Find $\int \frac{x^2}{\sqrt{4x^3+1}}\, \mathrm{d}x$. [2]

2 **(i)** By using an algebraic method, solve the inequality $\frac{x+3}{x+4} \le \frac{5}{1-2x}$. [4]

(ii) Hence, solve the inequality $\frac{x^2+3}{x^2+4} \le \frac{5}{1-2x^2}$. [2]

3 Referred to the origin O, points A and B have position vectors **a** and **b** respectively. Point P lies on OA such that $OP = 2PA$ and point Q lies on AB such that $5AQ = 4QB$. Show that the equation of the line l passing through P and Q can be written as

$$\mathbf{r} = \frac{2}{3}\mathbf{a} + \lambda(4\mathbf{b} - \mathbf{a}),\ \text{where } \lambda \in \mathbb{R}.$$ [3]

Point X lies on l such that AX is perpendicular to l. If $|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = \frac{1}{2}$ and **a** is perpendicular to **b**, find the position vector of X in terms of **a** and **b**. [4]

4 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{x^2+1},\ x \in \mathbb{R},\ x \ge k.$$

(i) State the minimum value of k for which the function f^{-1} exists. [1]

For the rest of the question, use the value of k found in part **(i)**.

(ii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram, showing clearly the relationship between them. [3]

The function g is defined by

$$\mathrm{g}(x) = \frac{x^2+1}{x},\ x \in \mathbb{R},\ x > 0.$$

(iii) By finding fg(x) or otherwise, solve $\mathrm{g}(x) = \mathrm{f}^{-1}\left(\frac{1}{5}\right)$. [4]

5 **(i)** By writing $\frac{2-r}{r(r+1)(r+2)}$ in partial fractions, show that $\sum_{r=1}^{n} \frac{2-r}{r(r+1)(r+2)} = \frac{An}{(n+1)(n+2)}$, where A is a constant to be determined. [4]

(ii) Using the result in part **(i)**, find $\sum_{r=0}^{n} \frac{1-r}{(r+1)(r+2)(r+3)}$ in terms of n. [2]

(iii) Hence find the exact value of $\sum_{r=10}^{\infty} \frac{1-r}{(r+1)(r+2)(r+3)}$. [2]

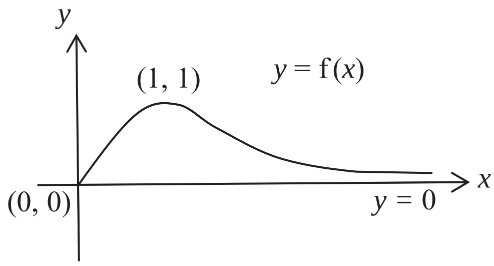
6 **(a)** The curve C_1 and C_2 have equations $y = \frac{x}{x^2+1}$ and $y = \sqrt{\frac{5}{4} - x^2}$ respectively.

(i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and stationary points, and the equation(s) of any asymptote(s). [4]

(ii) State the coordinates of the point of intersection of C_1 and C_2. [1]

(iii) Hence solve the inequality $\frac{x}{x^2+1} \geq \sqrt{\frac{5}{4} - x^2}$. [1]

(b) The diagram below shows a sketch of the graph of $y = \mathrm{f}(x)$. The graph meets the origin (0, 0), has a turning point at (1, 1) and the equation of the asymptote is $y = 0$.



On separate diagrams, draw sketches of the graphs of

(i) $y = \mathrm{f}(|x|)$, [2]

(ii) $y = \mathrm{f}'(x)$, [2]

stating the coordinates of the turning point(s), point(s) of intersection with the x-axis and equation(s) of asymptote(s) when it is possible to do so.

7 It is given that $y = \sqrt{1 + \ln(1 + \sin 2x)}$.

(i) Show that $y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos 2x}{1 + \sin 2x}$. [1]

(ii) Show that $y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{k}{(1 + \sin 2x)}$, where k is a constant to be determined. [3]

(iii) Hence show that the Maclaurin series of y is $1 + x - \frac{3}{2}x^2 + \frac{13}{6}x^3 + \dots$. [3]

(iv) Expand $\left(1 + x - \frac{3}{2}x^2 + \frac{13}{6}x^3\right)^2$ in powers of x up to and including the term in x^3, simplifying the coefficients. By using the standard series expansions of $\sin x$ and $\ln(1+x)$ from the List of Formulae

(MF26), explain briefly how the result can be used as a check on the correctness of the first four terms in the series for y. [3]

8 A curve C has parametric equations

$$x = 3t^2, \quad y = a\left(t^3 + 1\right),$$

where a is a positive constant.

(i) Sketch C, giving the coordinates of any point(s) where the curve meets the axes. [2]

The tangent to C at point $A\left(3, 2a\right)$ makes an angle of $\frac{\pi}{3}$ with the positive x-axis.

(ii) Show that $a = 2\sqrt{3}$, and find the equation of the tangent to C at A in the form $y = mx + c$, where m and c are constants to be determined. [5]

(iii) The tangent and the normal to C at A meet the x-axis at T and N respectively. Find the exact area of triangle ATN. [3]

9 The planes π_1 and π_2 have equations given by

$$\pi_1 : 4x + 3y + 5z = 7 \text{ and}$$

$$\pi_2 : \mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k}), \text{ where } \lambda, \mu \in \mathbb{R}.$$

(i) Find a vector equation of the line l_1 where π_1 and π_2 meet. Verify that point P with position vector $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ lies on l_1. [3]

(ii) The line l_2 which passes through P, lies in π_2 and is perpendicular to l_1. Find a cartesian equation of l_2. [3]

(iii) A point Q (a, b, c) lies on l_2 where a, b and c are negative constants. Given that the distance from Q to π_1 is $3\sqrt{2}$, find the coordinates of Q. Hence or otherwise, find the exact length of projection of $\overrightarrow{QP}$ on π_1. [5]

10 Mrs Tan wants to build a wooden fence using vertical planks of equal width and thickness but different lengths. In her plan, the length of the first wooden plank is 2 metres and the length of the planks forms a geometric progression. The length of the 10th plank is 1.5 metres.

(i) If the cost of wooden plank is \$18 per metre, show that the cost of the fence will never exceed \$1200. [4]

Mrs Tan realised that her plan is not feasible and now wants to build her fence using several identical panels. Each panel comprises 10 vertical planks with identical dimensions to the first 10 planks in her original plan.

(ii) Find the total length, T metres, of the planks to be used in each panel. [2]

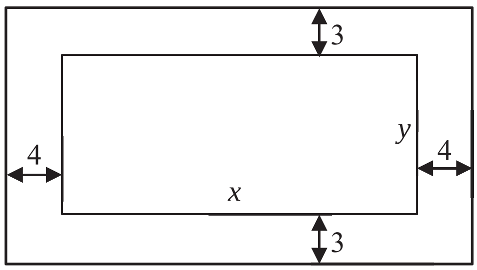
(iii) She hires a contractor to install the fence. The contractor misunderstands her instructions and uses 10 planks to construct a panel so that the lengths form an arithmetic progression with common difference d metres. If the total length of the planks to be used for one panel is still T metres and the length of the first plank is still 2 metres, find the value of d. [2]

The contractor offers to paint the fence for Mrs Tan. He buys a 5-litre can of paint to do the paint job. To save costs, he fills the can to the 5-litre mark with turpentine to form a uniform mixture when the level of paint in the can falls to the 4-litre mark. He then repeats this process whenever the level of the mixture falls to the 4-litre mark.

(iv) State, in terms of n, an expression for the volume of paint remaining in the mixture after the nth refill. [2]

(v) Find the minimum number of refills taken before the mixture is more than 80% turpentine. [2]

11 A designer wishes to create a piece of artwork with painted area of 1200 cm^2 on a rectangular piece of canvas. The painted area measures x cm by y cm and is surrounded by an unpainted border with top and bottom margins of 3 cm each, and side margins of 4 cm each on the canvas, as shown in the diagram below.



(i) By differentiation, find the dimensions of the canvas with the smallest area. [6]

(ii) What is the largest possible area of the canvas if $30 \le x \le 50$? [2]

At an exhibition, a spotlight illuminates a circular region of radius $\frac{2}{\sqrt{\pi}}$ cm on the artwork. The area of this circular region then increases at a constant rate of 20 cm^2 per minute.

(iii) Find the rate of change of the radius **after** 3 minutes. [4]

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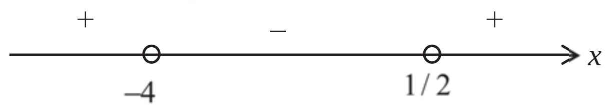
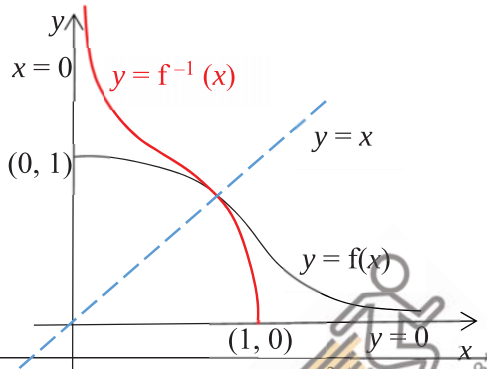
Q1	Suggested Answers	Guidance
(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) = \frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x\left(\frac{-2x}{2\sqrt{1-x^2}}\right)$ $= \frac{1-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$ $= \sqrt{1-x^2} + \sqrt{1-x^2}$ $= 2\sqrt{1-x^2}$ $\int \sqrt{1-x^2}\ \mathrm{d}x = \frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + C$	To differentiate $x\sqrt{1-x^2}$, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\sqrt{1-x^2}\right) = \sqrt{1-x^2}\,\frac{\mathrm{d}}{\mathrm{d}x}(x) + x\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-x^2}\right)$ $= \sqrt{1-x^2} + x\left[\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\right]$ Note the use of chain rule for $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-x^2}\right)$. To simplify the expression: $\frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$ Combine the first and last term.
(b)	$\int \frac{x^2}{\sqrt{4x^3+1}}\ \mathrm{d}x = \frac{1}{12}\int 12x^2\left(4x^3+1\right)^{-\frac{1}{2}}\ \mathrm{d}x$ $= \frac{1}{12}\left(\frac{\left(4x^3+1\right)^{\frac{1}{2}}}{1/2}\right) + C$ $= \frac{1}{6}\sqrt{4x^3+1} + C$	Note that the power of $\left(4x^3+1\right)$ is $-\frac{1}{2}$. Integrating it would not give $\ln\left(4x^3+1\right)$. Common algebraic error leads to a coefficient $\frac{1}{24}$.

Q2	Suggested Answers	Guidance
	$\frac{x+3}{x+4} \le \frac{5}{1-2x}, x \ne -4, \frac{1}{2}$ $\frac{x+3}{x+4} - \frac{5}{1-2x} \le 0$ $\frac{(x+3)(1-2x)-5(x+4)}{(x+4)(1-2x)} \le 0$ $\frac{-2x^2-10x-17}{(x+4)(1-2x)} \le 0$ $\frac{-2\left(x^2+5x+\frac{17}{2}\right)}{-(x+4)(2x-1)} \le 0$ $\frac{2\left(\left(x+\frac{5}{2}\right)^2+\frac{9}{4}\right)}{(x+4)(2x-1)} \le 0$ Since $\left(x+\frac{5}{2}\right)^2+\frac{9}{4} > 0$ for all $x \in \mathbb{R}$, $\frac{2}{(x+4)(2x-1)} \le 0$  $-4 < x < \frac{1}{2}$	DO NOT cross multiply at the start as $x+4$ and $1-2x$ are not always positive. There are careless mistakes made when completing the square. Expand out to check before proceeding. You MUST justify why the numerator is always positive or always negative before you omit them from your working. DO NOT directly jumped from $\frac{-2x^2-10x-17}{(x+4)(1-2x)} \le 0$ to the final answer without clear evidence that the numerator is always negative.

	$\frac{x^2+3}{x^2+4} \le \frac{5}{1-2x^2}$ Replace x with x^2 in inequality in (i), $-4 < x^2 < \frac{1}{2}$ Since $x^2 \ge 0$, $0 \le x^2 < \frac{1}{2} \Rightarrow x^2 - \frac{1}{2} < 0$ $\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right) < 0$ $\therefore -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$	It is WRONG to do $x^2 < \frac{1}{2} \Rightarrow x < \pm\sqrt{\frac{1}{2}}$

Q3	Suggested Answers	Guidance
	Since $OP = 2PA$, $\overrightarrow{OP} = \frac{2}{3}\mathbf{a}$ Using Ratio Theorem, $\overrightarrow{OQ} = \frac{5\mathbf{a}+4\mathbf{b}}{9} = \frac{5}{9}\mathbf{a} + \frac{4}{9}\mathbf{b}$ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -\frac{1}{9}\mathbf{a} + \frac{4}{9}\mathbf{b} = \frac{1}{9}(4\mathbf{b}-\mathbf{a})$ Since the line passes through P and is // to $4\mathbf{b}-\mathbf{a}$ therefore an equation of l is $\mathbf{r} = \frac{2}{3}\mathbf{a} + \lambda(4\mathbf{b}-\mathbf{a})$ where $\lambda \in \mathbb{R}$	Do take note of the presentation. $\frac{1}{9}(4\mathbf{b}-\mathbf{a}) \neq 4\mathbf{b}-\mathbf{a}$ These two vectors are parallel to each other but not equal.
	Since X lies on l, we have $\overrightarrow{OX} = \frac{2}{3}\mathbf{a} + t(4\mathbf{b}-\mathbf{a})$ for a particular value of t Since AX is perpendicular to l, $\overrightarrow{AX} \bullet (4\mathbf{b}-\mathbf{a}) = 0$ $\left(\frac{2}{3}\mathbf{a} + t(4\mathbf{b}-\mathbf{a}) - \mathbf{a}\right) \bullet (4\mathbf{b}-\mathbf{a}) = 0$ $\left(\left(-\frac{1}{3}-t\right)\mathbf{a} + 4t\mathbf{b}\right) \bullet (4\mathbf{b}-\mathbf{a}) = 0$ Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\therefore \mathbf{a}\bullet\mathbf{b} = 0$ $\therefore \left(\frac{1}{3}+t\right)\lvert\mathbf{a}\rvert^2 + 16t\lvert\mathbf{b}\rvert^2 = 0$, $\lvert\mathbf{a}\rvert = \sqrt{3}$, $\lvert\mathbf{b}\rvert = \frac{1}{2}$ $\therefore \left(\frac{1}{3}+t\right)3 + 4t = 0$,	In general: $\lambda\mathbf{b} \bullet \lambda\mathbf{b} = \lambda^2\lvert\mathbf{b}\rvert^2 \neq \lambda\lvert\mathbf{b}\rvert^2$ $\mathbf{b} \bullet \mathbf{b} = \lvert\mathbf{b}\rvert^2 \neq \lvert\mathbf{b}\rvert$ $\mathbf{a} \bullet \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta \neq \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$ $\lvert 4\mathbf{b}-\mathbf{a}\rvert \neq \left\lvert 4\left(\frac{1}{2}\right) - \sqrt{3}\right\rvert^2$

	$t=-\frac{1}{7}$ $\therefore \overrightarrow{OX}=\frac{2}{3}\mathbf{a}-\frac{1}{7}(4\mathbf{b}-\mathbf{a})=\frac{1}{21}(17\mathbf{a}-12\mathbf{b})$	

Q4	Suggested Answers	Guidance
(i)	$k = 0$	
(ii)		The graph of $y = \mathrm{f}(x)$ should show that the tangent at (0,1) is parallel to the x-axis, ie. (0,1) is a maximum point (No need to draw the tangent line). The graph of $y = \mathrm{f}^{-1}(x)$ should show that the tangent at (1,0) is parallel to the y-axis (No need to draw the tangent line). Use the **same** scale for both x-axis and y-axis so that the two graphs are **symmetrical** about $y = x$.
(iii)	$\mathrm{fg}(x) = \dfrac{1}{\left(\dfrac{x^2+1}{x}\right)^2+1} = \dfrac{x^2}{x^4+3x^2+1}$ $\mathrm{g}(x) = \mathrm{f}^{-1}\left(\dfrac{1}{5}\right)$ $\Rightarrow \mathrm{fg}(x) = \dfrac{1}{5}$ $\dfrac{x^2}{x^4+3x^2+1} = \dfrac{1}{5}$ $x^4 - 2x^2 + 1 = 0$ $(x^2-1)^2 = 0$ $x^2 = 1$	Many algebraic errors were observed in finding $\mathrm{fg}(x)$ or during the process to form the quartic equation

	Since $D_g = (0, \infty)$, $x = -1$ is rejected. $\therefore\ x = 1$	
(iii)	Alternatively, $\mathrm{fg}(x) = \dfrac{1}{\left(\dfrac{x^2+1}{x}\right)^2+1}$ $\mathrm{g}(x) = \mathrm{f}^{-1}\left(\dfrac{1}{5}\right)$ $\Rightarrow \mathrm{fg}(x) = \dfrac{1}{5}$ $\dfrac{1}{\left(\dfrac{x^2+1}{x}\right)^2+1} = \dfrac{1}{5}$ $\left(\dfrac{x^2+1}{x}\right)^2 = 4$ $\left(\dfrac{x^2+1}{x}\right) = 2$ or $\left(\dfrac{x^2+1}{x}\right) = -2$ $x^2 - 2x + 1 = 0$ or $x^2 + 2x + 1 = 0$ $(x-1)^2 = 0$ or $(x+1)^2 = 0$ $x = 1$ or $x = -1$	Many algebraic errors were observed in finding $\mathrm{fg}(x)$ or during the process to form the two quadratic equations.

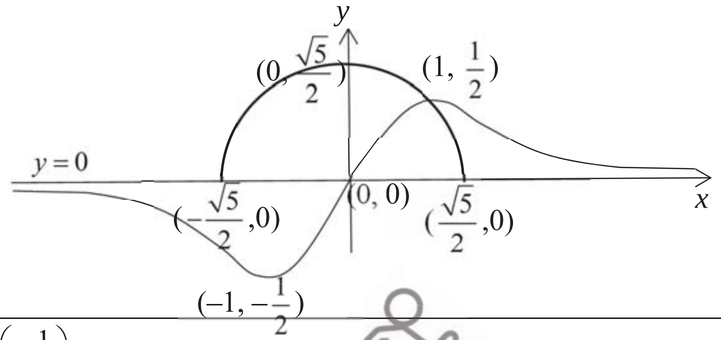
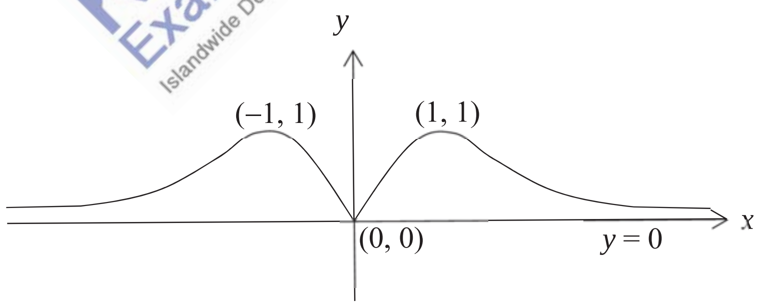
	Since $D_g = (0, \infty)$, $x = -1$ is rejected. $\therefore$ $x = 1$	
(iii)	Otherwise method: Let $y = \dfrac{1}{x^2+1}$ $x = \pm\sqrt{\dfrac{1}{y} - 1}$ Since $x \geq 0$ $x = \sqrt{\dfrac{1}{y} - 1}$ $f^{-1}(x) = \sqrt{\dfrac{1}{x} - 1}$ $f^{-1}\left(\dfrac{1}{5}\right) = \sqrt{5-1} = 2$ $\dfrac{x^2+1}{x} = 2$ $x^2 - 2x + 1 = 0$ $(x-1)^2 = 0$ $x = 1$	It is wrong to state that $f^{-1}\left(\dfrac{1}{5}\right) = \sqrt{4} = \pm 2$. It should be $\sqrt{4} = 2$, as $\sqrt{4}$ is a positive number. The graph of $y = f^{-1}(x)$ in part (ii) also shows that it lies above the x-axis, so $f^{-1}(x) \geq 0$ for all values of x.

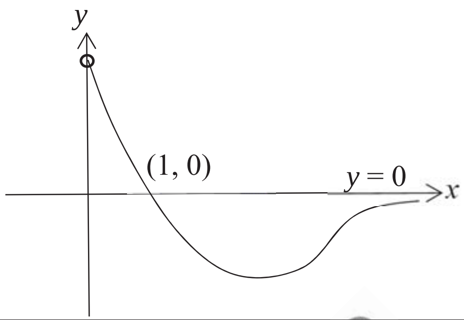
Q5	Suggested Answers	Guidance
(i)	$\dfrac{2-r}{r(r+1)(r+2)}=\dfrac{A}{r}+\dfrac{B}{r+1}+\dfrac{C}{r+2}$ Method 1: Using cover-up method, $A = 1$, $B = -3$, $C = 2$. Method 2: $\dfrac{2-r}{r(r+1)(r+2)}=\dfrac{A}{r}+\dfrac{B}{r+1}+\dfrac{C}{r+2}$ $2-r=A(r+1)(r+2)+Br(r+2)+Cr(r+1)$ Put $r = 0$: $2 = 2A$, then $A = 1$ Put $r = -1$: $3 = B(-1)(-1+2)$, then $B = -3$ Put $r = -2$: $4 = C(-2)(-2+1)$, then $C = 2$ Hence, $\dfrac{2-r}{r(r+1)(r+2)}=\dfrac{1}{r}-\dfrac{3}{r+1}+\dfrac{2}{r+2}$	

	$\sum_{r=1}^{n}\frac{2-r}{r(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{3}{r+1}+\frac{2}{r+2}\right)$ $=\frac{1}{1}-\frac{3}{2}+\frac{2}{3}$ $+\frac{1}{2}-\frac{3}{3}+\frac{2}{4}$ $+\frac{1}{3}-\frac{3}{4}+\frac{2}{5}$ ..... $+\frac{1}{n-2}-\frac{3}{n-1}+\frac{2}{n}$ $+\frac{1}{n-1}-\frac{3}{n}+\frac{2}{n+1}$ $+\frac{1}{n}-\frac{3}{n+1}+\frac{2}{n+2}$ $=\frac{2}{n+1}-\frac{3}{n+1}+\frac{2}{n+2}=-\frac{1}{n+1}+\frac{2}{n+2}$ $=\frac{-(n+2)+2(n+1)}{(n+1)(n+2)}$ $=\frac{n}{(n+1)(n+2)}, A=1$	

(ii)	**<u>Method 1</u>:** From part (ii), $\sum_{r=0}^{n}\frac{1-r}{(r+1)(r+2)(r+3)}$ (Let $r = k-1$) $=\sum_{k-1=0}^{k-1=n}\frac{1-(k-1)}{(k-1+1)(k-1+2)(k-1+3)}$ $=\sum_{k=1}^{n+1}\frac{2-k}{k(k+1)(k+2)}=\frac{n+1}{(n+2)(n+3)}$	From part (i), changing dummy variable r to k, $\sum_{k=1}^{n}\frac{2-k}{k(k+1)(k+2)}=\frac{n}{(n+1)(n+2)}$ Now replace n by $n+1$, $\sum_{k=1}^{n+1}\frac{2-k}{k(k+1)(k+2)}=\frac{n+1}{(n+2)(n+3)}$
(ii)	**<u>Method 2</u>:** From part (i), $\sum_{r=1}^{n}\frac{2-r}{r(r+1)(r+2)}=\frac{n}{(n+1)(n+2)}$ Let $r = k+1$, $\sum_{r=1}^{n}\frac{2-r}{r(r+1)(r+2)}=\sum_{k+1=1}^{k+1=n}\frac{2-(k+1)}{(k+1)(k+2)(k+3)}$ $=\sum_{k=0}^{n-1}\frac{1-k}{(k+1)(k+2)(k+3)}=\frac{n}{(n+1)(n+2)}$ Changing dummy variable k to r, $\sum_{r=0}^{n-1}\frac{1-r}{(r+1)(r+2)(r+3)}=\frac{n}{(n+1)(n+2)}$ Now replace n by $n+1$, $\sum_{r=0}^{n}\frac{1-r}{(r+1)(r+2)(r+3)}=\frac{(n+1)}{(n+2)(n+3)}$	

(iii)	$\sum_{r=10}^{\infty}\frac{1-r}{(r+1)(r+2)(r+3)}$ $=\sum_{r=0}^{\infty}\frac{1-r}{(r+1)(r+2)(r+3)}-\sum_{r=0}^{9}\frac{1-r}{(r+1)(r+2)(r+3)}$ $=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1-r}{(r+1)(r+2)(r+3)}-\frac{9+1}{(9+2)(9+3)}$ $=\lim_{n\to\infty}\frac{(n+1)}{(n+2)(n+3)}-\frac{5}{66}=0-\frac{5}{66}=-\frac{5}{66}$	The word "**hence**" in the question means that you have to use the result in either part (ii) (recommended) or even part (i). The sum in part (ii) starts from $r = 0$, so adjust the given sum in part(iii) so that it starts from this same value, ie $r = 0$.

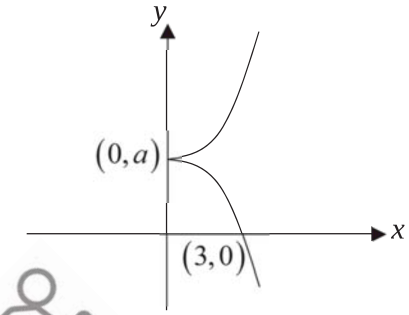
Q6	Suggested Answers	Guidance
(a)(i)		Semi-circle C_2 passes through maximum point of C_1. Note that the equation of C_2 requires only the semi-circle above the x-axis. It is not the equation of a full circle; do not draw a full circle. Label axial intercepts as coordinates as required by the question; not just $\frac{\sqrt{5}}{2}$ or $-\frac{\sqrt{5}}{2}$.
(ii)	$\left(1,\frac{1}{2}\right)$	
(iii)	$1\le x\le\frac{\sqrt{5}}{2}$	Many just gave $x\ge 1$. Note that for $x>\frac{\sqrt{5}}{2}$, it is outside the valid domain of the semi-circle, hence for $x>\frac{\sqrt{5}}{2}$ it is not defined and any answer should not include that region.
(b)(i)		

(ii)		Gradient at the origin is not defined in $y = \mathrm{f}(x)$ since the limit from the left of the origin is not defined.

Q7	Suggested Answers	Guidance
(i)	$y=\sqrt{1+\ln(1+\sin 2x)}$ $y^2=1+\ln(1+\sin 2x)$ $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cos 2x}{1+\sin 2x}$ $y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos 2x}{1+\sin 2x}$	It is easier if we square both sides of the equation before differentiation.
(ii)	$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{-2\sin 2x(1+\sin 2x)-2\cos 2x(\cos 2x)}{(1+\sin 2x)^2}$ $=\frac{-2\sin 2x-2\sin^2 2x-2\cos^2 2x}{(1+\sin 2x)^2}$ $=\frac{-2(\sin 2x+1)}{(1+\sin 2x)^2}=\frac{-2}{(1+\sin 2x)}$	
(iii)	$y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{4\cos 2x}{(1+\sin 2x)^2}$ Let $\mathrm{f}(x)=\sqrt{1+\ln(1+\sin 2x)}$ $\mathrm{f}(0)=1$ $\mathrm{f}'(0)=1$ $\mathrm{f}''(0)=-3$ $\mathrm{f}'''(0)=13$ The Maclaurin expansion of $y=\sqrt{1+\ln(1+\sin 2x)}=1+x-\frac{3x^2}{2!}+\frac{13x^3}{3!}+\ldots$ $=1+x-\frac{3}{2}x^2+\frac{13}{6}x^3+\ldots$	Use chain rule when differentiating $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ The question requires you to show the Maclaurin series; you need to SHOW your workings clearly.

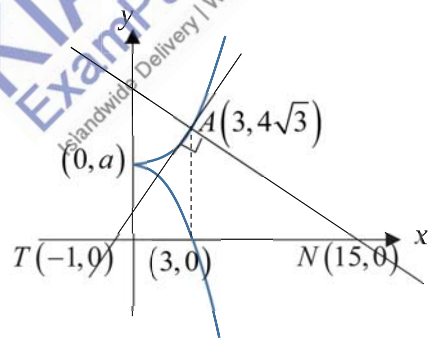
(iv)	$\left(1+x-\frac{3}{2}x^2+\frac{13}{6}x^3+\ldots\right)^2$ $=\left(1+x-\frac{3}{2}x^2+\frac{13}{6}x^3+\ldots\right)\left(1+x-\frac{3}{2}x^2+\frac{13}{6}x^3+\ldots\right)$ $=1+x-\frac{3}{2}x^2+\frac{13}{6}x^3$ $\quad +x+x^2-\frac{3}{2}x^3$ $\quad\quad -\frac{3}{2}x^2-\frac{3}{2}x^3$ $\quad\quad\quad +\frac{13}{6}x^3+\ldots$ $=1+2x-2x^2+\frac{4}{3}x^3+\ldots$ Using standard series expansion from MF 26, $y^2=1+\ln(1+\sin 2x)=1+\ln\left[1+\left(2x-\frac{(2x)^3}{3!}+\ldots\right)\right]$ $=1+\left(2x-\frac{8x^3}{6}\right)-\frac{1}{2}\left(2x-\frac{8x^3}{6}\right)^2+\frac{1}{3}\left(2x-\frac{8x^3}{6}\right)^3+\ldots$ $=1+2x-\frac{4}{3}x^3-\frac{1}{2}(2x)^2+\frac{1}{3}(2x)^3+..$ $=1+2x-2x^2+\frac{4}{3}x^3+\ldots$	Replace sin 2x by the standard series expansion from MF 26. Then apply standard series expansion of ln(1 + x), by replacing x by $\left(2x-\frac{8x^3}{6}\right)$. Note that there is no constant term in the expansion of $\left(2x-\frac{8x^3}{6}\right)^n$

	Alternatively, $1+\ln(1+\sin 2x)=1+\sin 2x-\frac{1}{2}(\sin 2x)^2+\frac{1}{3}(\sin 2x)^3+...$ $=1+\left(2x-\frac{(2x)^3}{3!}\right)-\frac{1}{2}\left(2x-\frac{(2x)^3}{3!}\right)^2+\frac{1}{3}\left(2x-\frac{(2x)^3}{3!}\right)^3+...$ $=1+2x-\frac{8}{6}x^3-\frac{1}{2}(2x)^2+\frac{1}{3}(2x)^3+..$ $=1+2x-2x^2+\frac{4}{3}x^3+...$ Since the expansion of y^2 from the standard series is same as the earlier result, therefore the first 4 terms in the series for y is correct.	Alternatively, Apply standard series expansion of ln(1 + x), replacing x by sin 2x. Then replace sin 2x by the standard series expansion. Note that there is no constant term in the expansion of $\left(2x-\frac{(2x)^3}{3!}\right)^n$.

Q8	Suggested Answers	Guidance
(i)	When $x=0$, $3t^2=0 \Rightarrow t=0$ $y=a$ When $y=0$, $a\left(t^3+1\right) \Rightarrow t=-1$ $x=3$ 	
(ii)	$x=3t^2$ $\quad$ $y=a\left(t^3+1\right)$ $\frac{dx}{dt}=6t$ $\quad$ $\frac{dy}{dt}=3at^2$ $\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{1}{2}at$ At $A(3,2a)$, $2a=a\left(t^3+1\right)$ $2=t^3+1$ $t^3=1$ $t=1$ Hence $\frac{dy}{dx}=\frac{1}{2}a$	Alternative method to solve for t: $3=3t^2$ $t=\pm 1$ When $t=1$, $y=2a$. Reject $t=-1$ as it corresponds to the point (3, 0). Give a reason to reject $t=-1$ if the above method is used.

	$\frac{1}{2}a = \tan\frac{\pi}{3} = \sqrt{3}$ Hence $a = 2\sqrt{3}$ Equation of the tangent at *A*, $y - 2a = \frac{1}{2}a(x-3)$ $y - 4\sqrt{3} = \sqrt{3}(x-3)$ $y = \sqrt{3}x + \sqrt{3}$	
(ii)	<u>Alternative method using cartesian equation:</u> $y = a(t^3+1) \quad \Rightarrow \quad t = \left(\frac{y}{a}-1\right)^{\frac{1}{3}}$ ------(1) Substitute (1) into $x = 3t^2$: $x = 3\left(\frac{y}{a}-1\right)^{\frac{2}{3}}$ ------(2) Differentiate (2) with respect to *x*: $1 = 2\left(\frac{y}{a}-1\right)^{-\frac{1}{3}}\left(\frac{1}{a}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{2}\left(\frac{y}{a}-1\right)^{\frac{1}{3}}$	Avoid converting the parametric equations to cartesian form unless the question requires you to do so. It is very tedious to work out $\frac{\mathrm{d}y}{\mathrm{d}x}$.

	At $A(3,2a)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a}{2}\left(\frac{2a}{a}-1\right)^{\frac{1}{3}}$ $=\frac{a}{2}$ $\frac{1}{2}a=\tan\frac{\pi}{3}=\sqrt{3}$ Hence $a=2\sqrt{3}$ Equation of the tangent at A, $y-2a=\frac{1}{2}a(x-3)$ $y-4\sqrt{3}=\sqrt{3}(x-3)$ $y=\sqrt{3}x+\sqrt{3}$	

(iii)	Equation of the normal at A, $y-4\sqrt{3}=-\frac{1}{\sqrt{3}}(x-3)$ $y=-\frac{1}{\sqrt{3}}x+5\sqrt{3}$ When $y=0$, $0=\sqrt{3}x+\sqrt{3} \Rightarrow x=-1 \quad T(-1,0)$ $0=-\frac{1}{\sqrt{3}}x+5\sqrt{3} \Rightarrow x=15 \quad N(15,0)$ Hence area of triangle $ATN=\frac{1}{2}(15-(-1))(4\sqrt{3})$ $=32\sqrt{3}$ sq. units  	

Q9	Suggested Answers	Guidance
(i)	Normal of $\pi_2 = \begin{pmatrix}0\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$ or $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ using G.C. $\pi_2 : \mathbf{r} \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} = 1$ Using G.C. to solve $4x+3y+5z=7$ $y-z=1$ We obtain $l_1 : \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}-2\\1\\1\end{pmatrix}, \lambda \in \mathbb{R}$ $\begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}-2\\1\\1\end{pmatrix}$ Since $\lambda = 1$ satisfy the above equation, $P(-1,2,1)$ lies on line l_1	ALWAYS use GC to check that the cross product is done correctly before proceeding to subsequent parts of the question, which rely on the accuracy of this result.
(ii)	Direction vector of $l_2 = \underbrace{\begin{pmatrix}0\\1\\-1\end{pmatrix}}_{\text{normal to } \pi_2} \times \underbrace{\begin{pmatrix}-2\\1\\1\end{pmatrix}}_{\text{direction vector of } l_1} = \begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix}$	Use GC to check if the cross product is correct.

	$\mathbf{r}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\mu\begin{pmatrix}1\\1\\1\end{pmatrix}, \mu\in\mathbb{R}$ $x+1=y-2=z-1$	
(iii)	Position vector of any point on l_2 takes the form $\begin{pmatrix}-1+\mu\\2+\mu\\1+\mu\end{pmatrix}$ Distance $\left(-1+\mu,2+\mu,1+\mu\right)$ from π_1 $=\dfrac{\left\lvert\begin{pmatrix}-1+\mu\\2+\mu\\1+\mu\end{pmatrix}\cdot\begin{pmatrix}4\\3\\5\end{pmatrix}-7\right\rvert}{\sqrt{50}}$ $=\dfrac{\lvert 4(-1+\mu)+3(2+\mu)+5(1+\mu)-7\rvert}{\sqrt{50}}=\dfrac{12\lvert\mu\rvert}{\sqrt{50}}$ When $\dfrac{12\lvert\mu\rvert}{\sqrt{50}}=3\sqrt{2}$ $\lvert\mu\rvert=\dfrac{30}{12}=\dfrac{5}{2}$ Thus $\mu=\dfrac{5}{2}$ or $-\dfrac{5}{2}$ Points on l_2 are $\left(\dfrac{3}{2},\dfrac{9}{2},\dfrac{7}{2}\right)$, $\left(-\dfrac{7}{2},-\dfrac{1}{2},-\dfrac{3}{2}\right)$ Since a, b and c are negative, thus $Q\left(-\dfrac{7}{2},-\dfrac{1}{2},-\dfrac{3}{2}\right)$ Let x be the length of projection of $\overrightarrow{QP}$ on π_1	There were a few approaches to the question. Some are longer, others are shorter. This is primarily a problem of distance between a point and a plane. There is a modulus sign which you need to handle properly.

	Using Pythagoras Theorem $\left(\left\|\overrightarrow{QP}\right\|\right)^2 = (3\sqrt{2})^2 + x^2$, where $\overrightarrow{QP} = \begin{pmatrix}-1\\2\\1\end{pmatrix} - \begin{pmatrix}-\frac{7}{2}\\-\frac{1}{2}\\-\frac{3}{2}\end{pmatrix} = \begin{pmatrix}\frac{5}{2}\\\frac{5}{2}\\\frac{5}{2}\end{pmatrix} = \frac{5}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\left\|\overrightarrow{QP}\right\| = \left\|\frac{5}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}\right\| = \frac{5}{2}\sqrt{1^2+1^2+1^2} = \frac{5}{2}\sqrt{3}$ $x = \frac{\sqrt{3}}{2}$ Alternatively, Length of projection $= \frac{\left\|\overrightarrow{PQ} \times \begin{pmatrix}4\\3\\5\end{pmatrix}\right\|}{\sqrt{4^2+3^2+5^2}} = \frac{\left\|-\frac{5}{2}\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}4\\3\\5\end{pmatrix}\right\|}{\sqrt{50}}$ $= \frac{\left\|-\frac{5}{2}\right\| \left\|\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}4\\3\\5\end{pmatrix}\right\|}{5\sqrt{2}} = \frac{\frac{5}{2}\left\|\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\|}{5\sqrt{2}} = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$	

Q10	Suggested Answers	Guidance
(i)	$u_{10}=1.5$ $ar^9=1.5$, $a=2$ $r^9=0.75$ $r=(0.75)^{\frac{1}{9}}\approx 0.96854$ $S_\infty=\dfrac{2}{1-(0.75)^{\frac{1}{9}}}\approx 63.574$ or $S_\infty=\dfrac{2}{1-0.96854}\approx 63.573$ Maximum amount to pay $<18\times S_\infty\approx \$1144.33$ or $\$1144.31$ Hence the cost will not exceed \$1200.	Since $0<r<1$, the maximum amount to pay will never exceed $18\times S_\infty$. Always use either the exact or min. 5 sig. fig. value of r for all workings.
(ii)	$T=S_{10}=\dfrac{2\left(1-0.75^{\frac{10}{9}}\right)}{1-0.75^{\frac{1}{9}}}\approx 17.394\approx 17.4\text{m}$ or $T=S_{10}=\dfrac{2\left(1-0.96854^{10}\right)}{1-0.96854}\approx 17.394\approx 17.4\text{m}$	
(iii)	**Method 1 (consider $U_1=2$):** $\frac{10}{2}\left[2(2)+(10-1)d\right]=17.394$ $d\approx -0.0579\text{m}$	Always use min. 5 sig. fig. of T for all workings.
(iii)	**Method 2 (consider $U_{10}=2$):** $\frac{10}{2}\left[2a+(10-1)d\right]=17.394$ $5\left[2a+9d\right]=17.394$--------(1)	

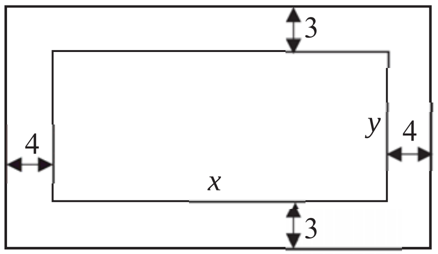
	$a+9d=2$ ------------------(2) Subst. (2) into (1): $5(a+2)=17.394$ $a=1.4788$ Hence, $\therefore d=\dfrac{2-1.4788}{9}\approx 0.0579\text{m}$	
(iv)	**Method 1:** Original volume of paint = 5 litres Let u_n denotes the volume of paint in litres remaining after the nth refill. Before the 1st refill, the quantity of paint is reduced by a factor of $\dfrac{4}{5}$. $\therefore$ the volume of paint is reduced to $u_1=\dfrac{4}{5}\times 5$ $u_2=\dfrac{4}{5}\times u_1=\left(\dfrac{4}{5}\right)^2\times 5$ $\therefore u_n=\left(\dfrac{4}{5}\right)^n\times 5=5\left(\dfrac{4}{5}\right)^n$	
(iv)	**Method 2:** Let u_n denotes the volume of paint in litres remaining after the nth refill. $u_1=4$ Before the 2nd refill, the quantity of paint is reduced by a factor of $\dfrac{4}{5}$. $\therefore$ the volume of paint is reduced to $u_2=\dfrac{4}{5}\times u_1=\dfrac{4}{5}\times 4$ $u_3=\dfrac{4}{5}\times u_2=\left(\dfrac{4}{5}\right)^2\times 4$	

	$\therefore u_n = 4\left(\frac{4}{5}\right)^{n-1}$	
(v)	If the mixture is more than 80% turpentine, then it is less than 20% paint, ie $0.2 \times 5 = 1$ litre of paint. $u_n = 5\left(\frac{4}{5}\right)^n < 1$ or $u_n = 4\left(\frac{4}{5}\right)^{n-1} < 1$ Using GC, minimum $n = 8$ The minimum number of refills taken before the mixture is more than 80% turpentine is 8.	> 80% turpentine means < 20% paint. It should be $< 0.2 \times 5$ and not just < 0.2.

Q11

(i)

Suggested Answers



$$xy = 1200 \quad \Rightarrow \quad y = \frac{1200}{x}$$

Let A be the total area of the canvas.

To minimize $A = (x+8)(y+6)$

$$= (x+8)\left(\frac{1200}{x}+6\right)$$

$$= 1200 + 6x + \frac{9600}{x} + 48$$

$$= 6x + \frac{9600}{x} + 1248$$

$$\frac{dA}{dx} = 6 - \frac{9600}{x^2}$$

$$\frac{dA}{dx} = 0 \quad \Rightarrow \quad x^2 = 1600$$

$$x = 40 \quad (\because\ x > 0)$$

$$y = 30$$

Guidance

"… create a piece of artwork with **painted area of** 1200 cm^2 <u>on a rectangular piece of canvas</u>. The **painted area measures x cm by y cm** and is **surrounded by an unpainted border…** <u>on the canvas</u>, as shown in the diagram below."

Read the question carefully to understand that the artwork consists of the painted area ($xy = 1200$) surrounded by an unpainted border and the total area of the artwork is that of the canvas.

Step 1: Construct an equation to link the three variables A, x, and y.

$$A = (x+8)(y+6)$$

Step 2: Form another equation to link the variables x and y.

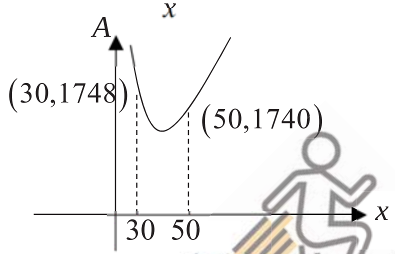
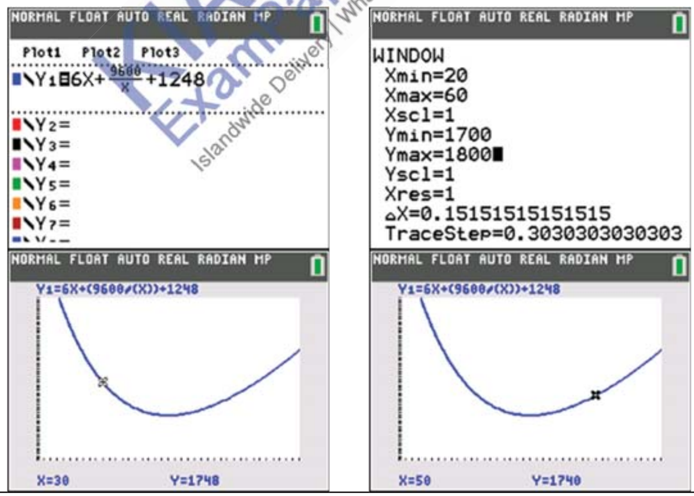
$$xy = 1200$$

As we want to minimise A, use the above relation to obtain an equation involving A and x (or A and y).

We need to do either a first or second derivative test to show that A is indeed minimised for the values of x and y obtained.

x	39.9	40	40.1
$\frac{dy}{dx}$	−0.0301	0	0.0299
	\	_	/

Show the values used in your calculation in the table.

	$\dfrac{d^2A}{dx^2}=\dfrac{19200}{x^3}>0$ since $x>0$ Hence A is minimum when $x=48$ cm and $y=36$ cm	
(ii)	Graph of $A=6x+\dfrac{9600}{x}+1248$  From graph, largest possible area of the canvas is 1748 cm^2 when $x=30$ cm. 	Since there is only a minimum point (at $x=40$), the largest possible value of A is at either end of the interval $30\le x\le 50$.

(iii)	$A=\pi r^2$ $\frac{\mathrm{d}A}{\mathrm{d}t}=2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$ --------(*) When $t=0$, $A=\pi\left(\frac{2}{\sqrt{\pi}}\right)^2=4$ When $t=3$, $A=4+3\times 20=64$ Hence $64=\pi r^2 \Rightarrow r=\frac{8}{\sqrt{\pi}}$ Sub $\frac{\mathrm{d}A}{\mathrm{d}t}=20$, $r=\frac{8}{\sqrt{\pi}}$ into (*): $20=2\pi\frac{8}{\sqrt{\pi}}\cdot\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t}=0.705$ (3 s.f.) When $t=3$, rate of change of radius is 0.705 cm/min	We are given $\frac{\mathrm{d}A}{\mathrm{d}t}=20$, and are asked to find $\frac{\mathrm{d}r}{\mathrm{d}t}$ when $t=3$. Note that $\frac{\mathrm{d}A}{\mathrm{d}t}=20$ is a constant, and $\frac{\mathrm{d}r}{\mathrm{d}t}$ is not a constant (since area of the circle increases at a constant rate, the rate at which the radius increases is actually decreasing). We need to find the value of r at the instant $t=3$ in order to substitute into the relation $\frac{\mathrm{d}A}{\mathrm{d}t}=2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$.

RAFFLES INSTITUTION
2021 YEAR 5 PROMOTION EXAMINATION

CANDIDATE NAME

CLASS 22

MATHEMATICS 9758

2.5 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper. **You may use the blank pages on page 21 and 22 if necessary and you are reminded to indicate the question number(s) clearly.**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 85.

For examiner's use only					
Q1	Q2	Q3	Q4	Q5	
/ 4	/ 5	/ 6	/ 8	/ 8	
Q6	Q7	Q8	Q9	Q10	TOTAL
/ 10	/ 10	/ 10	/ 12	/ 12	/ 85

This document consists of **20** printed pages and **2** blank pages.

RAFFLES INSTITUTION
Mathematics Department

[Turn over

1 A curve C has equation $y=\dfrac{ax+b}{cx-2}$, where a, b and c are constants. It is given that C passes through the points with coordinates $(1,\ 5)$ and $(-8,\ 0.5)$. The curve C is translated 1 unit in the positive x-direction. The new curve passes through the point with coordinates $(0,\ -0.2)$. Find the values of a, b and c. [4]

2 The curve C has equation $y^3 = 4 - \frac{xy^2}{2}$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{6y+2x}$. [2]

(ii) Find the equation of the normal to C at the point P where $y = 1$. [3]

3 The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

(a) Show that the area of triangle ABC is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$. Hence show that the shortest distance from B to AC is

$$\frac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}{|\mathbf{c}-\mathbf{a}|}.$$ [4]

(b) Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $|\mathbf{a}-\mathbf{b}|=|\mathbf{a}+\mathbf{b}|$, find the value of $\mathbf{a}\cdot\mathbf{b}$. [2]

4 A sequence $u_1, u_2, u_3, ...$ is defined by

$$u_n = \sum_{r=1}^{n} (2r + n + 1).$$

Another sequence $v_1, v_2, v_3, ...$ is given by $v_n = \frac{2}{u_n}$, where $n \in \mathbb{Z}^+$.

(i) Find u_n in terms of *n*. [2]

(ii) Show that $v_n = \frac{1}{n} - \frac{1}{n+1}$. [1]

(iii) Describe the behaviour of the sequence $v_1, v_2, v_3, ...$. [1]

(iv) Find the sum, S_N, of the first N terms of the sequence $v_1, v_2, v_3, \ldots$. [2]

(v) Give a reason why the series S_N converges, and write down the value of the sum to infinity. [2]

5 **(i)** Using standard series from the List of Formulae (MF26), expand $\dfrac{\cos 3x}{4-x}$ as far as the term in x^3. Give the coefficients as exact fractions in their simplest form. [4]

(ii) It is given that the third and fourth terms found in part **(i)** are equal to the third and fourth terms in the series expansion of $(a+bx)^5$, in ascending powers of x, respectively. Find the values of the constants a and b. [4]

6 **(i)** On the same axes, sketch the graphs of $y = x + 2 + \dfrac{1}{x-1}$ and $y = |2x+2|$, stating the coordinates of any points of intersections with the axes, turning points and the equations of any asymptotes. [5]

(ii) Hence solve the inequality

$$x + 2 + \frac{1}{x-1} > |2x+2|,$$

giving your answers in exact form. [5]

6 [Continued]

7 **(a)** An arithmetic sequence a_1, a_2, a_3,... has common difference *d*, where $d < 0$. The sum of the first *n* terms of the sequence is denoted by S_n. Given that $|a_8| = |a_{13}|$, find the value of *n* for which S_n is maximum. [4]

(b) The terms u_1, u_2 and u_3 are three consecutive terms of a geometric progression. It is given that

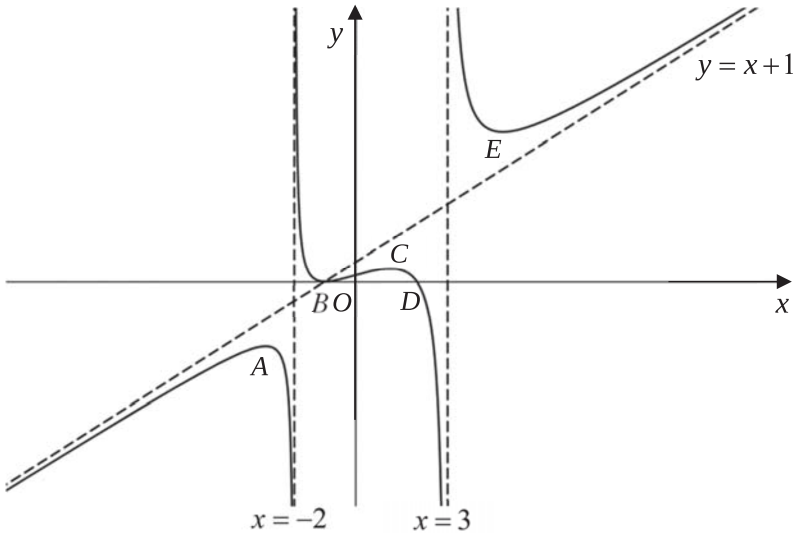
$$u_1,\ u_2 \text{ and } u_3 - 32$$

form an arithmetic progression, and that

$$u_1,\ u_2 - 4 \text{ and } u_3 - 32$$

form another geometric progression. Find the possible values of u_1, u_2 and u_3. [6]

8 The diagram below shows the graph of $y = \mathrm{f}(x)$ with asymptotes $x = -2$, $x = 3$ and $y = x + 1$. The curve intersects the x-axis at points B and D, and has turning points at points A, B, C and E. The coordinates of A, B, C, D and E are $(-3, -3)$, $(-1, 0)$, $\left(\frac{3}{2}, \frac{3}{4}\right)$, $(2, 0)$ and $(5, 8)$ respectively.



(i) By showing clearly the equations of asymptotes and the coordinates of any turning points and the points where the curve crosses the axes, where possible, sketch, on **separate diagrams**, the graphs of

(a) $y = \mathrm{f}(|x|)$, [4]

(b) $y = \dfrac{1}{\mathrm{f}(x)}$. [4]

(ii) By drawing another suitable graph on the same diagram in part **(i)(b)**, determine the number of solutions to the equation

$$\frac{x^2}{36} + \frac{1}{16\left[\mathrm{f}(x)\right]^2} = 1.$$ [2]

9 Distances in this question are in metres.

Harry and Tom's model airplanes are taking off from the horizontal ground, which is the *x*-*y* plane. Tom's airplane takes off after Harry's. The position of Harry's airplane t seconds after it takes off is given by $\mathbf{r} = (5\mathbf{i} + 6\mathbf{j}) + t(-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$. The position of Tom's airplane s seconds after it takes off is given by $\mathbf{r} = (-39\mathbf{i} + 44\mathbf{j}) + s(4\mathbf{i} - 6\mathbf{j} + 7\mathbf{k})$.

(i) State the height of Harry's airplane two seconds after it takes off and find its distance travelled in the two seconds. [3]

(ii) Find the acute angle between the path of Harry's airplane and the ground. [2]

(iii) Show that the paths of the airplanes are perpendicular. [1]

(iv) Given that the two airplanes collide, find the coordinates of the point of collision. How long after Harry's airplane takes off does Tom's airplane take off? [3]

(v) Find the cartesian equation of the plane in which both paths of the airplanes lie. [3]

10 Functions f and g are defined by

$$\mathrm{f} : x \mapsto 4x - 2k \qquad \text{for } x \in \mathbb{R}\text{, where } k \text{ is a constant,}$$
$$\mathrm{g} : x \mapsto \frac{9}{2-x} \qquad \text{for } x \in \mathbb{R},\ x \neq 2.$$

(i) Explain why gf does not exist. [1]

(ii) Find the range of values of k for which the equation $\mathrm{fg}(x) = x$ has real roots. [4]

For the rest of the question, let $k = 5$.

(iii) Sketch the graph of $y = \mathrm{fg}(x)$ for $x < 2$. Hence sketch the graph of $y = (\mathrm{fg})^{-1}(x)$ on the same diagram, showing clearly the relationship between the two graphs. [4]

10 **[Continued]**

The function h represents the height in metres of an object at time t seconds and is defined for the domain $0 \le t \le b$ by

$$\mathrm{h}(t) = \begin{cases} \frac{30}{7} + \mathrm{g}(t+9) & \text{for } 0 \le t \le a, \\ 2 - \mathrm{f}(t) & \text{for } a < t \le b, \end{cases}$$

where a and b are constants. At $t = 0$, the object was thrown up from 3 metres above the ground level. When $t = a$, the object started to descend and finally reached the ground at $t = b$.

(iv) Find the values of a and b. [2]

(v) Sketch the graph of $y = \mathrm{h}(t)$ for $0 \le t \le b$. [1]

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1 A curve C has equation $y=\dfrac{ax+b}{cx-2}$, where a, b and c are constants. It is given that C passes through the points with coordinates $(1,\ 5)$ and $(-8,\ 0.5)$. The curve C is translated 1 unit in the positive x-direction. The new curve passes through the point with coordinates $(0,\ -0.2)$. Find the values of a, b and c. [4]

Solution		Comments
1 **[4]**	$y=\dfrac{ax+b}{cx-2}$ Sub $(1,\ 5)$ and $(-8,\ 0.5)$ into equation, $a+b-5c=-10$ ----- (1) $8a-b-4c=1$ ----- (2) After transformation, the translated curve is $y=\dfrac{a(x-1)+b}{c(x-1)-2}$. Substitute $(0,\ -0.2)$, we get $5a-5b+c=-2$ ----- (3) **Alternatively,** Since $(0,\ -0.2)$ lies on the translated curve, then $(-1,-0.2)$ should lie on the original curve. Substitute $(-1,\ -0.2)$ onto the original curve, we get $-0.2=\dfrac{-a+b}{-c-2}$ $5a-5b+c=-2$ -----(3) From GC, $a=2$, $b=3$ and $c=3$, i.e. $y=\dfrac{2x+3}{3x-2}$	Most students were able to substitute the given information in the question and form the 3 equations before using the GC to solve for the values of *a*, *b* and *c*.

2 The curve C has equation $y^3 = 4 - \frac{xy^2}{2}$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{6y+2x}$. [2]

(ii) Find the equation of the normal to C at the point P where $y = 1$. [3]

<table>
<tr><th colspan="2">Solution</th><th>Comments</th></tr>
<tr><td>2(i)
[2]</td><td>
<u>Method 1</u>
$y^3 = 4 - \frac{xy^2}{2}$
Differentiate with respect to x:
$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{2} - xy\frac{\mathrm{d}y}{\mathrm{d}x}$
$\left(3y^2 + xy\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{2}$
$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{6y^2+2xy} = -\frac{y}{6y+2x}$ (shown)

<u>Method 2</u>
$y^3 = 4 - \frac{xy^2}{2} \quad \Rightarrow \quad x = \frac{8}{y^2} - 2y \quad ------(1)$
Differentiate with respect to y: $\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{16}{y^3} - 2 = -\frac{16+2y^3}{y^3}$
Then,
$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^3}{16+2y^3} = -\frac{y}{\frac{16}{y^2}+2y}$
$= -\frac{y}{2x+4y+2y}$, using (1)
$= -\frac{y}{6y+2x}$ (shown)

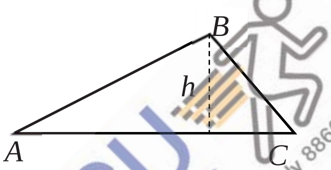
<u>Method 3</u>
$y = \left(4 - \frac{xy^2}{2}\right)^{\frac{1}{3}}$
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3\left(4-\frac{xy^2}{2}\right)^{\frac{2}{3}}}\left(-xy\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y^2}{2}\right)$
$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3y^2}\left(xy\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y^2}{2}\right)$
</td><td>Students should use implicit differentiation to solve part (i).</td></tr>
</table>

	$\frac{dy}{dx}+\frac{x}{3y}\frac{dy}{dx}=-\frac{1}{6}$ $\frac{dy}{dx}=-\frac{1}{6}\times\frac{1}{1+\frac{x}{3y}}$ $\frac{dy}{dx}=-\frac{y}{6y+2x}$ (shown)	
(ii) **[3]**	At $y=1$, $x=6$ and $\frac{dy}{dx}=-\frac{1}{18}$ Gradient of normal = 18 Hence equation of normal $y-1=18(x-6)$ $y=18x-107$	Very standard question and most students are able to get this correct.

3 The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

(a) Show that the area of triangle ABC is $\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|$. Hence show that the shortest distance from B to AC is $\dfrac{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|}{\left|\mathbf{c}-\mathbf{a}\right|}$. [4]

(b) Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $\left|\mathbf{a}-\mathbf{b}\right|=\left|\mathbf{a}+\mathbf{b}\right|$, find the value of $\mathbf{a}\cdot\mathbf{b}$. [2]

	Solution	**Comments**
3(a) **[4]**	area of triangle ABC $=\frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right|$ $=\frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times\mathbf{c}-(\mathbf{b}-\mathbf{a})\times\mathbf{a}\right|$ $=\frac{1}{2}\left|\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{a}\right|$ $=\frac{1}{2}\left|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{0}\right|$ $=\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|$ (shown) Let shortest distance from B to AC be h (which is also the perpendicular distance from B to AC). $AC=\left|\overrightarrow{AC}\right|=\left|\mathbf{c}-\mathbf{a}\right|$  area of triangle $ABC=\frac{1}{2}(AC)(h)=\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|$ Thus $h=\dfrac{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|}{\left|\mathbf{c}-\mathbf{a}\right|}$. (shown)	As this is a "show" question, students need to pen down more working to get full credit for the first part. The second part of (a) is a 'hence' question. Students need to use the result from the earlier part to show the shortest distance from B to AC.
(b) **[2]**	$\left|\mathbf{a}-\mathbf{b}\right|=\left|\mathbf{a}+\mathbf{b}\right|$ $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$ $\left|\mathbf{a}\right|^2-2\mathbf{a}\cdot\mathbf{b}+\left|\mathbf{b}\right|^2=\left|\mathbf{a}\right|^2+2\mathbf{a}\cdot\mathbf{b}+\left|\mathbf{b}\right|^2$ $\mathbf{a}\cdot\mathbf{b}=0$ *Alternative* Consider a parallelogram $OACB$ with $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. Then the lengths of its diagonals are given by $AB=\left|\mathbf{a}-\mathbf{b}\right|$ and $OC=\left|\mathbf{a}+\mathbf{b}\right|$. If $\left|\mathbf{a}-\mathbf{b}\right|=\left|\mathbf{a}+\mathbf{b}\right|$, then $OACB$ forms a rectangle and thus $\mathbf{a}\cdot\mathbf{b}=0$.	Students need to apply the properties of the dot product correctly before any credit is given.

4 A sequence $u_1, u_2, u_3, ...$ is defined by $u_n = \sum_{r=1}^{n}(2r+n+1)$. Another sequence $v_1, v_2, v_3, ...$ is given by $v_n = \frac{2}{u_n}$, where $n \in \mathbb{Z}^+$.

(i) Find u_n in terms of n. [2]

(ii) Show that $v_n = \frac{1}{n} - \frac{1}{n+1}$. [1]

(iii) Describe the behaviour of the sequence $v_1, v_2, v_3, ...$. [1]

(iv) Find the sum, S_N, of the first N terms of the sequence $v_1, v_2, v_3, ...$. [2]

(v) Give a reason why the series S_N converges, and write down the value of the sum to infinity. [2]

Solution		**Comments**
4(i) **[2]**	$u_n = \sum_{r=1}^{n}(2r+n+1) = \frac{n}{2}\left[n+3+3n+1\right] = 2n(n+1)$	u_n is the <u>sum</u> of n terms of an AP: first term $n+3$, last term $3n+1$.
(ii) **[1]**	$v_n = \frac{2}{u_n} = \frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (shown)	Show working clearly as it is a "show" question.
(iii) **[1]**	The sequence $v_1, v_2, v_3, ...$ decreases and converges to zero as $\frac{1}{n} \to 0$, and $\frac{1}{n+1} \to 0$.	Question is asking about the sequence, <u>not</u> the series.
(iv) **[2]**	$S_N = \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ $= [\ 1 - \frac{1}{2}$ $+ \frac{1}{2} - \frac{1}{3}$ $+ \frac{1}{3} - \frac{1}{4}$ $+ ...$ $+ ...$ $+ \frac{1}{N-1} - \frac{1}{N}$ $+ \frac{1}{N} - \frac{1}{N+1}\ ]$ $= 1 - \frac{1}{N+1}$	Question is asking for S_N, not S_n.
(v) **[2]**	As $N \to \infty$, $S_N = \sum_{n=1}^{N} v_n = 1 - \frac{1}{N+1} \to 1$, since $\frac{1}{N+1} \to 0$. Thus, the series S_N converges. Sum to infinity of the series = 1	Answer the question, i.e. state explicitly the sum to infinity.

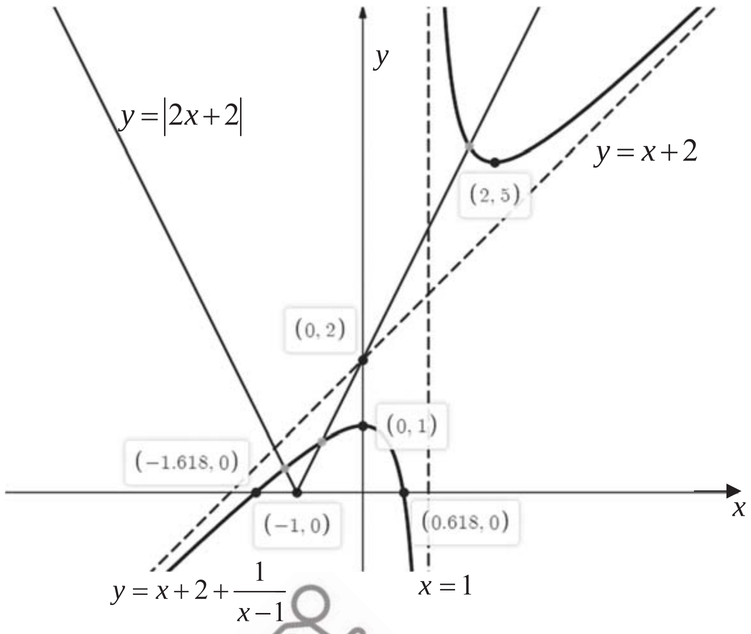
5 **(i)** Using standard series from the List of Formulae (MF26), expand $\dfrac{\cos 3x}{4-x}$ as far as the term in x^3. Give the coefficients as exact fractions in their simplest form. [4]

(ii) It is given that the third and fourth terms found in part **(i)** are equal to the third and fourth terms in the series expansion of $(a+bx)^5$, in ascending powers of x, respectively. Find the values of the constants a and b. [4]

Solution		**Comments**
5(i) **[4]**	$(4-x)^{-1}\cos 3x$ $=4^{-1}\left(1-\frac{1}{4}x\right)^{-1}\cos 3x$ $\approx\frac{1}{4}\left(1-\left(-\frac{1}{4}x\right)+\frac{-1(-2)}{2!}\left(-\frac{1}{4}x\right)^2+\frac{-1(-2)(-3)}{3!}\left(-\frac{1}{4}x\right)^3\right)\left(1-\frac{(3x)^2}{2!}\right)$ $=\frac{1}{4}\left(1+\frac{1}{4}x+\frac{1}{16}x^2+\frac{1}{64}x^3\right)\left(1-\frac{9}{2}x^2\right)$ $\approx\frac{1}{4}\left(1-\frac{9}{2}x^2+\frac{1}{4}x-\frac{9}{8}x^3+\frac{1}{16}x^2+\frac{1}{64}x^3\right)$ $=\frac{1}{4}\left(1+\frac{1}{4}x-\frac{71}{16}x^2-\frac{71}{64}x^3\right)$ $=\frac{1}{4}+\frac{1}{16}x-\frac{71}{64}x^2-\frac{71}{256}x^3$	When applying the standard series from MF 26, replace the 'x' in the standard series correctly.
(ii) **[4]**	$(a+bx)^5=a^5+5a^4bx+10a^3b^2x^2+10a^2b^3x^3+\ldots b^5x^5$ $10a^3b^2=-\frac{71}{64}$ -----(1) $\quad 10a^2b^3=-\frac{71}{256}$ -----(2) $(1)\div(2),\ \frac{a}{b}=\frac{256}{64}=4$ $\therefore a=4b$ Substituting into (1) $10(4b)^3b^2=-\frac{71}{64}$ $640b^5=-\frac{71}{64}$ $b^5=-\frac{71}{40960}$ $b=-0.280$ (3s.f.) $\quad a=-1.12$(3.s.f.)	

6 **(i)** On the same axes, sketch the graphs of $y = x + 2 + \dfrac{1}{x-1}$ and $y = |2x + 2|$, stating the coordinates of any points of intersections with the axes, turning points and the equations of any asymptotes. [5]

(ii) Hence solve the inequality $x + 2 + \dfrac{1}{x-1} > |2x + 2|$, giving your answers in exact form. [5]

	Solution		**Comments**
6(i) **[5]**	 		Take note that the *y*-intercepts of both oblique asymptote and modulus function are the same. Maximum point is on the *y*-axis. Drawing of graphs can definitely be improved. Many students drew graphs with all the points so close to each other that it is difficult to decipher the handwriting and the details.
(ii) **[5]**	Intersection between $y = x + 2 + \frac{1}{x-1}$ and $y = 2x + 2$ $2x + 2 = x + 2 + \frac{1}{x-1}$ $x(x-1) = 1$ $x^2 - x - 1 = 0$ $x = \frac{1 \pm \sqrt{5}}{2}$	Intersection between $y = x + 2 + \frac{1}{x-1}$ and $y = -2x - 2$ $-2x - 2 = x + 2 + \frac{1}{x-1}$ $-3x - 4 = \frac{1}{x-1}$ $(3x+4)(1-x) = 1$ $-3x^2 - x + 3 = 0$ $x = \frac{-1 \pm \sqrt{37}}{6}$ From the graph, the intersection occurs at $x < -1$. So, $x = \frac{-1-\sqrt{37}}{6}$	This is a "Hence" question. The expectation is to solve points of intersection with the correct equations. There is no need to solve with the inequality signs. The question also specified exact solution – means GC answers will not be accepted.
	From the graphs in **(i)**, the solution is $\frac{-1-\sqrt{37}}{6} < x < \frac{1-\sqrt{5}}{2}$ or $1 < x < \frac{1+\sqrt{5}}{2}$.		

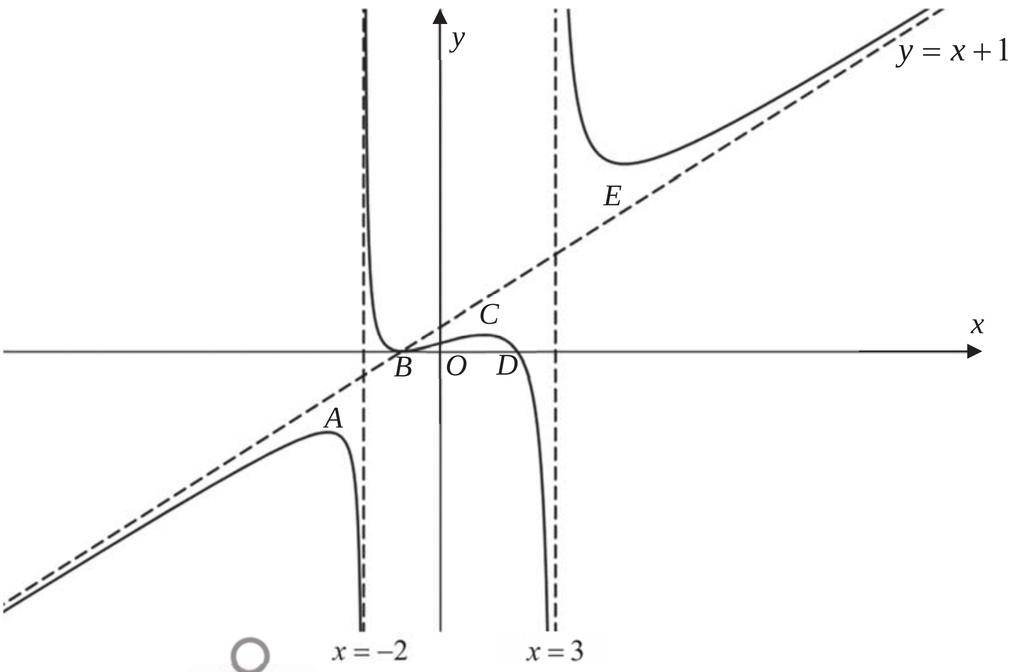
7 **(a)** An arithmetic sequence $a_1, a_2, a_3,...$ has common difference d, where $d<0$. The sum of the first n terms of the sequence is denoted by S_n. Given that $|a_8|=|a_{13}|$, find the value of n for which S_n is maximum. [4]

(b) The terms u_1, u_2 and u_3 are three consecutive terms of a geometric progression. It is given that u_1, u_2 and u_3-32 form an arithmetic progression, and that u_1, u_2-4 and u_3-32 form another geometric progression. Find the possible values of u_1, u_2 and u_3. [6]

Solution		Comments
7(a) [4]	$|a_8|=|a_{13}|$ $|a_1+7d|=|a_1+12d|$ $a_1+7d=a_1+12d$ or $a_1+7d=-a_1-12d$ $7d=12d$ $2a_1+19d=0$ $d=0$ $a_1=-\frac{19}{2}d$ (rejected since $d<0$) **Or** $|a_8|=|a_{13}|$ $(a_1+7d)^2=(a_1+12d)^2$ $(a_1+12d)^2-(a_1+7d)^2=0$ $(2a_1+19d)(5d)=0$ $2a_1+19d_1=0$, since $d<0$ $a_1=-\frac{19}{2}d$ Thus, $S_n=\frac{n}{2}(2a_1+(n-1)d)=\frac{n}{2}(-19d+nd-d)=\frac{d}{2}n(n-20)$ <u>To find n for max S_n</u> $S_n=\frac{d}{2}n(n-20)$ is a quadratic expression with negative coefficient of n^2 $\left(\frac{d}{2}<0\right)$. When $S_n=0$, $n=0$ or $n=20$. Hence, S_n is maximum at $n=\frac{0+20}{2}=10$. **OR** $\frac{dS_n}{dn}=nd-10d$ When $\frac{dS_n}{dn}=0$, $(n-10)d=0$ $n=10$ since $d<0$ $\frac{d^2S_n}{dn^2}=d<0$ Hence, maximum S_n at $n=10$. **OR** S_n will keep increasing when each term added is positive, until a maximum, and decrease when the next term added is negative. Consider $a_n>0$, then $-\frac{19}{2}d+(n-1)d>0$ $-\frac{19}{2}+n-1<0$ since $d<0$ $n<10.5$	When using the differentiation method, it is necessary to use either second derivative test or a complete first derivative test to show that the stationary value is a maximum value. Other methods should be accompanied with complete and thorough explanations, showing and explaining $a_1>0$, $a_{10}>0$ and

	S_n is maximum when $n=10$ since $a_{10}=-0.5d>0$ and $a_{11}=a_{10}+d=0.5d<0$. **Note that** a_1 has to be positive. If a_1 is negative (and with negative *d*), then a_r will just get smaller and smaller (more and more negative) as *r* increases. Then, it would not be possible that $\lvert a_8\rvert=\lvert a_{13}\rvert$. And so, a_1 has to be positive. In fact, a_8 is positive and a_{13} is negative, and we can say that $a_8=-a_{13}$.	$a_{11}<0$, and drawing link to how this affects S_n in order to obtain full credit.
(b) **[6]**	Let *a* and *r* be the first term and common ratio of the geometric progression. Then, $u_1=a,\ u_2=ar, u_3=ar^2$ $ar-a=ar^2-32-ar \quad ------(*)$ $ar^2-2ar+a=32$ $a(r-1)^2=32$ $(r-1)^2=\dfrac{32}{a}$ -------------- (1) $\dfrac{ar^2-32}{ar-4}=\dfrac{ar-4}{a}$ ------------(**) $(ar^2-32)a=a^2r^2-8ar+16$ $-32a=-8ar+16$ $a(8r-32)=16$ $a=\dfrac{2}{r-4}$ --------------- (2) Substituting (2) into (1), $r^2-2r+1=16r-64$ $r^2-18r+65=0$ $r=5$ or $r=13$ $a=2$ or $a=\dfrac{2}{9}$ $\left(\dfrac{2}{a}+3\right)^2=\dfrac{32}{a}$ $(2+3a)^2=32a$ $9a^2-20a+4=0$ $a=2$ or $a=\dfrac{2}{9}$ Hence $u_1=2,\ u_2=10$ and $u_3=50$ or $u_1=\dfrac{2}{9},\ u_2=\dfrac{26}{9}$ and $u_3=\dfrac{338}{9}$.	One of the key learning points in this question is that students should strive to reduce the number of variables they are dealing with. Those that started with only *a* and *r* have more success than those who dealt with u_1, u_2 and u_3. Also, do read the question carefully, there is no mention of the terms needing to be integers. Solutions that used the guess and check method did not give the second set of answer and the calculators are usually set to show integer values only.

8 The diagram below shows the graph of $y = \mathrm{f}(x)$ with asymptotes $x = -2$, $x = 3$ and $y = x + 1$. The curve intersects the x-axis at points B and D, and has turning points at points A, B, C and E. The coordinates of A, B, C, D and E are $(-3,-3)$, $(-1,0)$, $\left(\frac{3}{2},\frac{3}{4}\right)$, $(2,0)$ and $(5,8)$ respectively.



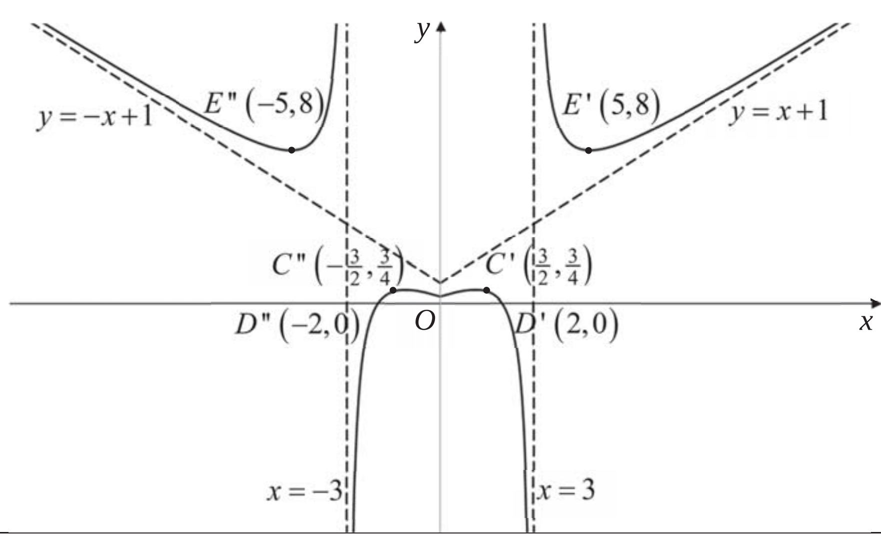
(i) By showing clearly the equations of asymptotes and the coordinates of any turning points and the points where the curve crosses the axes, where possible, sketch, on **separate diagrams**, the graphs of

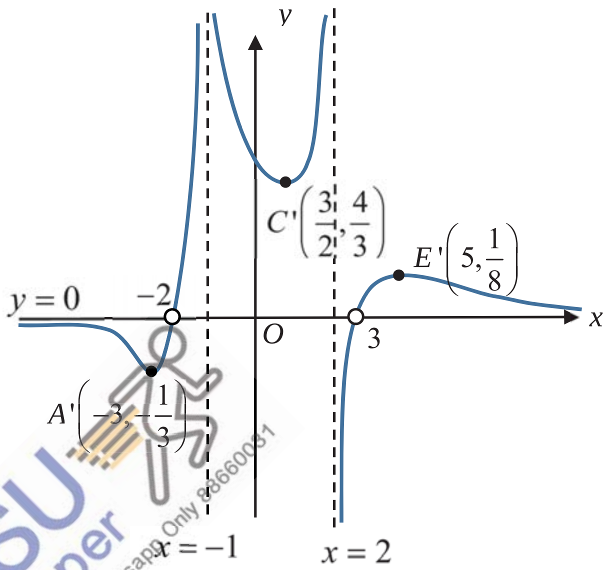
(a) $y = \mathrm{f}(|x|)$, [4]

(b) $y = \dfrac{1}{\mathrm{f}(x)}$. [4]

(ii) By drawing another suitable graph on the same diagram in part **(i)(b)**, determine the number of solutions to the equation $\dfrac{x^2}{36} + \dfrac{1}{16[\mathrm{f}(x)]^2} = 1$. [2]

Solution	**Comments**

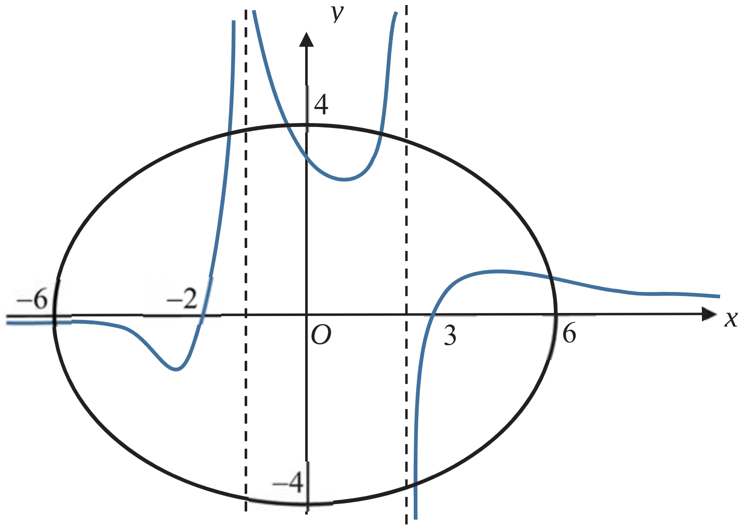
<table>
<tr>
<td>8(i)
(a)
[4]</td>
<td>


</td>
<td>Note that there is a sharp point at the y-intercept, and not a stationary point.</td>
</tr>
<tr>
<td>(b)
[4]</td>
<td>


</td>
<td>The graph should be drawn such that it approaches the x-axis as $x \to \infty$ and $x \to -\infty$.

Since the point C is lower than the point E, C' should be higher than E'</td>
</tr>
<tr>
<td>(ii)
[2]</td>
<td>$\frac{x^2}{36}+\frac{1}{\left[\mathrm{f}(x)\right]^2 16}=1$

By drawing $\frac{x^2}{36}+\frac{y^2}{16}=1$ and $y=\frac{1}{\mathrm{f}(x)}$, number of solution is 6</td>
<td>Since we're drawing on the diagram for the graph of $y=\frac{1}{\mathrm{f}(x)}$, we should substitute $\frac{1}{\mathrm{f}(x)}$ in the given equation with y to get the equation of the ellipse. It should not be $y=\mathrm{f}(x)$.</td>
</tr>
</table>

		When drawing 2 graphs on the same diagram, we need to consider their relative position to each other. Here, all 3 turning points are inside the ellipse.

9 Distances in this question are in metres.

Harry and Tom's model airplanes are taking off from the horizontal ground, which is the x-y plane. Tom's airplane takes off after Harry's. The position of Harry's airplane t seconds after it takes off is given by $\mathbf{r}=(5\mathbf{i}+6\mathbf{j})+t(-4\mathbf{i}+2\mathbf{j}+4\mathbf{k})$. The position of Tom's airplane s seconds after it takes off is given by $\mathbf{r}=(-39\mathbf{i}+44\mathbf{j})+s(4\mathbf{i}-6\mathbf{j}+7\mathbf{k})$.

(i) State the height of Harry's airplane two seconds after it takes off and find its distance travelled in the two seconds. [3]

(ii) Find the acute angle between the path of Harry's airplane and the ground. [2]

(iii) Show that the paths of the airplanes are perpendicular. [1]

(iv) Given that the two airplanes collide, find the coordinates of the point of collision. How long after Harry's airplane takes off does Tom's airplane take off? [3]

(v) Find the cartesian equation of the plane in which both paths of the airplanes lie. [3]

Solution		**Comments**
9(i) **[3]**	When $t=2$, $\mathbf{r}=(5\mathbf{i}+6\mathbf{j})+2(-4\mathbf{i}+2\mathbf{j}+4\mathbf{k})=-3\mathbf{i}+10\mathbf{j}+8\mathbf{k}$ Height of Harry's airplane = 8 m Distance travelled = $\left\lvert\begin{pmatrix}-3\\10\\8\end{pmatrix}-\begin{pmatrix}5\\6\\0\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}-8\\4\\8\end{pmatrix}\right\rvert=\sqrt{(-8)^2+4^2+8^2}$ = 12 m	Since the ground is the x-y plane, the height is given by the z-component. Note that the airplane did not start flying from the origin but at point (5, 6, 0).
(ii) **[2]**	Let the angle of takeoff of Harry's airplane from the ground be θ. $\sin\theta=\dfrac{\begin{pmatrix}-4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{6}=\dfrac{2}{3}$ $\theta=41.8°$	Vector perpendicular to the x-y plane is **k**.
(iii) **[1]**	The paths of Harry's airplane and Tom's airplane are parallel to $\begin{pmatrix}-4\\2\\4\end{pmatrix}$ and $\begin{pmatrix}4\\-6\\7\end{pmatrix}$ respectively. Since $\begin{pmatrix}-4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}4\\-6\\7\end{pmatrix}=-16-12+28=0$, the paths of the airplanes are perpendicular.	
(iv) **[3]**	Since the airplanes collide,	

	$\begin{pmatrix}5\\6\\0\end{pmatrix}+t\begin{pmatrix}-4\\2\\4\end{pmatrix}=\begin{pmatrix}-39\\44\\0\end{pmatrix}+s\begin{pmatrix}4\\-6\\7\end{pmatrix}$ for some $s,t\in\mathbb{R}$ $-4t-4s=-44$ --- (1) $2t+6s=38$ --- (2) $4t-7s=0$ --- (3) From the GC, $t = 7$, $s = 4$. Position vector of the point of collision $=\begin{pmatrix}5\\6\\0\end{pmatrix}+7\begin{pmatrix}-4\\2\\4\end{pmatrix}=\begin{pmatrix}-23\\20\\28\end{pmatrix}$ Thus coordinates of the point of collision = $(-23, 20, 28)$. Tom's airplane takes off 3 seconds after Harry's airplane takes off.	Note that question requires the answer to be given in coordinates.
(v) **[3]**	Vector perpendicular to the plane $=\begin{pmatrix}-4\\2\\4\end{pmatrix}\times\begin{pmatrix}4\\-6\\7\end{pmatrix}=\begin{pmatrix}38\\44\\16\end{pmatrix}=2\begin{pmatrix}19\\22\\8\end{pmatrix}$ Equation of the plane containing both paths of the airplanes is $\mathbf{r}\cdot\begin{pmatrix}19\\22\\8\end{pmatrix}=\begin{pmatrix}5\\6\\0\end{pmatrix}\cdot\begin{pmatrix}19\\22\\8\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}19\\22\\8\end{pmatrix}=227$ Cartesian equation of the plane is $19x+22y+8z=227$.	

10 Functions f and g are defined by

$$\mathrm{f}: x \mapsto 4x-2k \quad \text{for } x\in\mathbb{R}\text{, where } k \text{ is a constant,}$$
$$\mathrm{g}: x \mapsto \frac{9}{2-x} \quad \text{for } x\in\mathbb{R},\ x\neq 2.$$

(i) Explain why gf does not exist. [1]

(ii) Find the range of values of k for which the equation $\mathrm{fg}(x)=x$ has real roots. [4]

For the rest of the question, let $k=5$.

(iii) Sketch the graph of $y=\mathrm{fg}(x)$ for $x<2$. Hence sketch the graph of $y=(\mathrm{fg})^{-1}(x)$ on the same diagram, showing clearly the relationship between the two graphs. [4]

The function h represents the height in metres of an object at time t seconds and is defined for the domain $0\le t\le b$ by

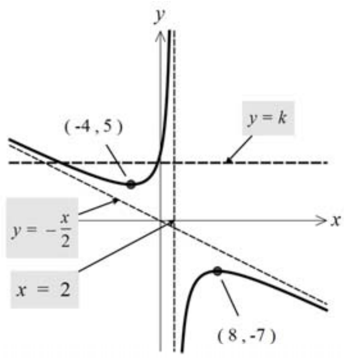
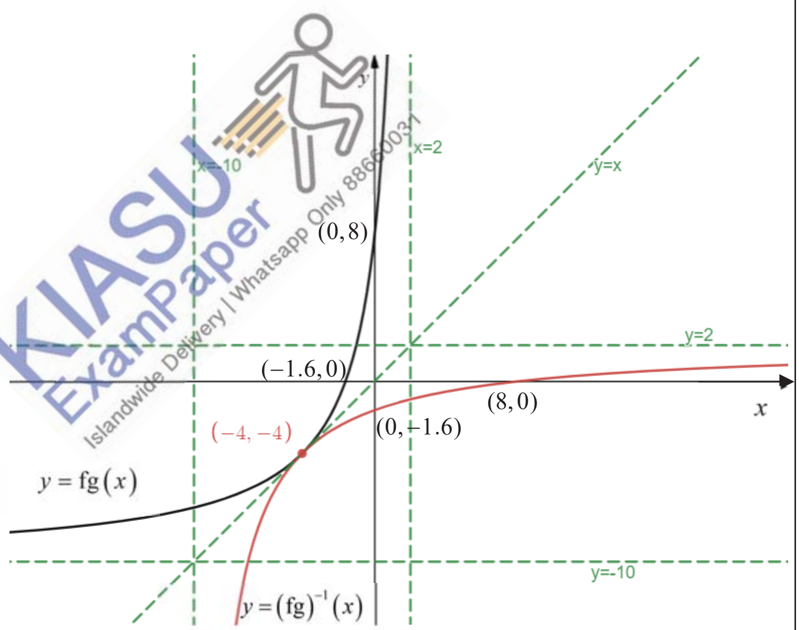
$$\mathrm{h}(t)=\begin{cases}\frac{30}{7}+\mathrm{g}(t+9) & \text{for } 0\le t\le a,\\ 2-\mathrm{f}(t) & \text{for } a<t\le b,\end{cases}$$

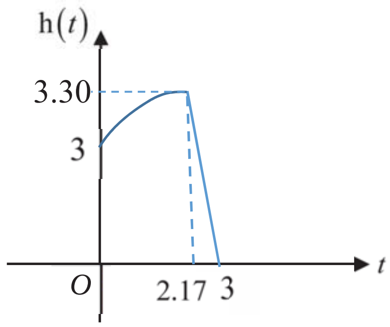
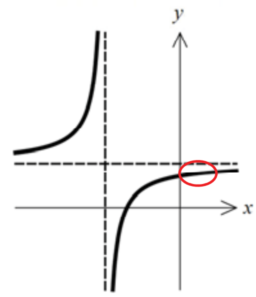
where a and b are constants. At $t=0$, the object was thrown up from 3 metres above the ground level. When $t=a$, the object started to descend and finally reached the ground at $t=b$.

(iv) Find the values of a and b. [2]

(v) Sketch the graph of $y=\mathrm{h}(t)$ for $0\le t\le b$. [1]

	Solution	Comments
10 (i) [1]	$\mathrm{R_f}=(-\infty,\infty)$ $\mathrm{D_g}=(-\infty,\infty)\setminus\{2\}$ Since $\mathrm{R_f}\not\subseteq \mathrm{D_g}$, gf does not exist.	Some only state down the condition and did **not** explicitly give the range and domain, thus, not awarded any marks. Common errors in writing sets: • $\mathrm{R_f}\in\mathbb{R}$ • $\mathrm{R_f}=x\in\mathbb{R}$ • $\mathrm{D_g}=(-\infty,\infty)/\{2\}$ • $\mathrm{D_g}=x\neq 2$
(ii) [4]	$\mathrm{f}(x)=4x-2k$; $\mathrm{g}(x)=\frac{9}{2-x}$ So, $\mathrm{fg}(x)=4\left(\frac{9}{2-x}\right)-2k=\frac{36}{2-x}-2k$ **Method 1** $\mathrm{fg}(x)=x$ $\frac{36}{2-x}-2k=x$ $36-2k(2-x)=x(2-x)$ $36-4k+2kx=2x-x^2$ $x^2+(2k-2)x+36-4k=0$	There are a few who gave $\mathrm{fg}(x)=\frac{9}{2-(4x-2k)}$ when this is actually gf(x). There are quite a few careless mistakes in arriving at this quadratic equation, either the sign is wrong or 36 is mission or -4k is missing.

	For real roots, discriminant ≥ 0 , $(2k-2)^2-4(36-4k)\geq 0$ $k^2+2k-35\geq 0$ $(k+7)(k-5)\geq 0$ Therefore $k\leq -7$ or $k\geq 5$ **Method 2** $\mathrm{fg}(x)=x$ $\frac{36}{2-x}-2k=x$ $-\frac{1}{2}x+\frac{18}{2-x}=k$  By considering the intersection between $y=-\frac{1}{2}x+\frac{18}{2-x}$ and $y=k$, $\mathrm{fg}(x)=x$ has real roots is equivalent to the graphs of $y=-\frac{1}{2}x+\frac{18}{2-x}$ and $y=k$ intersect. The range of the graph of $y=-\frac{1}{2}x+\frac{18}{2-x}$ is $(-\infty,-7])\cup[5,\infty)$. Thus, the range of k is $(-\infty,-7]\cup[5,\infty)$.	Note that if the quadratic equation has 2 real and different roots, then discriminant > 0. If the quadratic equation has 2 real and equal roots, then discriminant = 0. Thus, if the quadratic equation has real roots, then discriminant ≥ 0 .
(iii) **[4]**	$\mathrm{fg}(x)=4\left(\frac{9}{2-x}\right)-10=-10+\frac{36}{2-x}=\frac{16+10x}{2-x}$ $\Rightarrow y=\mathrm{fg}(x)$ has asymptotes $x=2$ and $y=-10$. 	**1)** $y=\mathrm{fg}(x)$ is a rational function of the form linear/linear, and so, it has vertical and horizontal asymptotes. However, many did not give the equation of the horizontal asymptote. **2)** Although the domain of fg is $(\infty,2)$, quite a few thought that the domain of $(\mathrm{fg})^{-1}$ is also $(\infty,2)$. However, recall that $\mathrm{D}_{(\mathrm{fg})^{-1}}=\mathrm{R}_{\mathrm{fg}}=(-10,\infty)$. **3)** When $k=5$, $\mathrm{fg}(x)=x$ has equal roots. Thus, the line $y=x$ is a tangent to the graph of $y=\mathrm{fg}(x)$. **4)** It is crucial that when we draw the graphs of $y=$ function and $y=\text{function}^{-1}$ on the same diagram, the x and y scale should be the same. **5)** The lines $y=x$, $y=2$ and $x=2$ should be concurrent. Likewise $y=x$, $y=-10$ and $x=-10$.

(iv) **[2]**	$\mathrm{h}(t)=\begin{cases}\dfrac{30}{7}+\dfrac{9}{2-(t+9)} & \text{for } 0\le t\le a\\ 2-(4t-10) & \text{for } a<t\le b\end{cases}$ When $t=a$, $\dfrac{30}{7}+\dfrac{9}{2-(a+9)}=2-(4a-10)$ $-\dfrac{9}{7+a}=\dfrac{54}{7}-4a$ Using GC, $a=2.1738=2.17$ (3s.f.) At $t=b$, $h(t)=0$. Thus, $2-(4b-10)=0 \Rightarrow b=3$.	As the height h is a continuous function, $\dfrac{30}{7}+\mathrm{g}(t+9)$ at $t=a$ is equal to $2-\mathrm{f}(t)$ at $t=a$.
(v) **[1]**		Very few are able to draw the correct graph. Note that when $0\le t\le a$, $y=\dfrac{30}{7}+\mathrm{g}(t+9)=\dfrac{30}{7}-\dfrac{9}{t+7}$ and the graph should be concave downwards. It can be seen from the graph ($y=\dfrac{30}{7}-\dfrac{9}{x+7}$) below:  Note that when $a<t\le b$, $y=2-\mathrm{f}(t)=12-4t$ is a straight line.

2021 SAJC H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**
Attempt all questions.

1 **(a)** Let u_n and S_n denote the n^{th} term and the sum of the first n terms of a sequence respectively. Given that where $u_1 = 1$, $u_2 = 5$, the sum of the first twelve terms is 276 and S_n is a quadratic polynomial in n, find S_n in terms of n. [3]

(b) A sequence $\{v_n\}$ is defined by $v_n = v_{n-1} - \frac{3}{5}\left(\frac{2}{5}\right)^{n-2}$, $n \geq 2$ and $v_1 = 4$. By considering $\sum_{n=2}^{N}(v_n - v_{n-1})$, find v_N. Hence, explain, with a reason, whether the sequence is convergent or divergent. [5]

2 The sum of the first n terms of a series, S_n, is given by $\frac{a^n}{2^{n-1}} - 2.$, where a is a non-zero constant and $a \neq 2$.

(i) Show that T_n, the n^{th} term of the series, is $(a-2)\left(\frac{a}{2}\right)^{n-1}$. Hence show that the given series is a geometric series. [4]

(ii) Find the range of values of a for the sum to infinity to exist. [2]

(iii) Given that $a = 1$, find the least value of n for S_n to be within $\pm\,0.2$ of the value of the sum to infinity. [3]

3 The function f is defined by

$$\mathrm{f}(x) = \ln\left(\frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1}\right), \qquad x > 0.$$

(i) Show that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

(ii) Find $\mathrm{f}^2(x)$ and hence evaluate $\mathrm{f}^{2021}(3)$, leaving your answer in exact form. [3]

4 **(i)** Given that $\mathbf{p} \times \mathbf{q} = \mathbf{0}$, what can be deduced about the vectors $\mathbf{p}$ and $\mathbf{q}$? [1]

(ii) Find the unit vector $\mathbf{r}$ such that $\mathbf{r} \times (-6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = \mathbf{0}$. [2]

(iii) Find the sine of the acute angle between $-6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and the x-axis, leaving your answer in exact form. [3]

5 **(a)** **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(3^{2x}\right)=9^x \ln 9$. [1]

(ii) Hence, differentiate $\tan^{-1}\left(3^{2x}\right)$, $x>0$ with respect to x. [2]

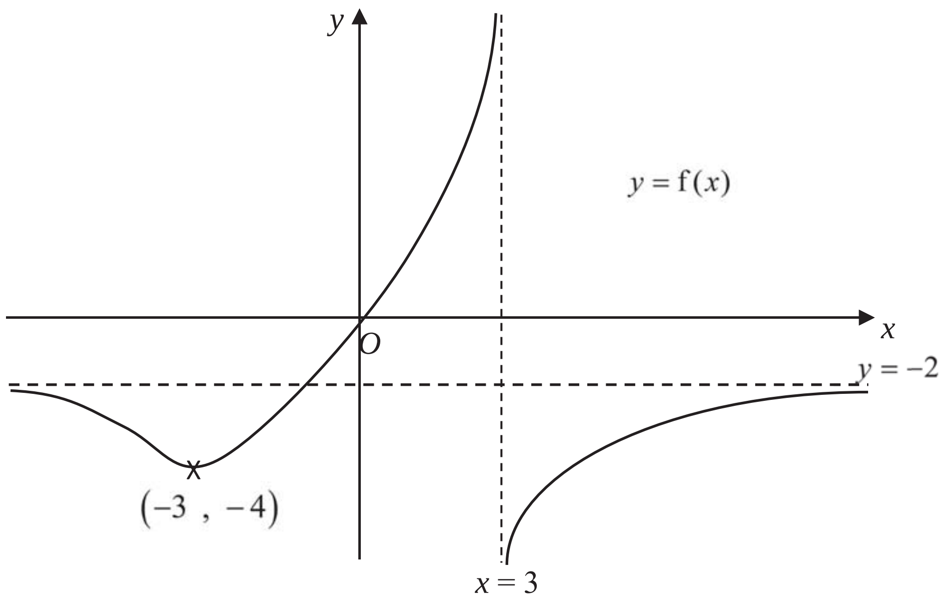
(b) Given that $y^{2y}=\mathrm{e}^{3x+2\mathrm{e}}$, where $y>\mathrm{e}^{-1}$, show that $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{2}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$. [5]

6 **(a)** The graph of $y=\mathrm{h}(x)$, where $\mathrm{h}(x)=\mathrm{e}^{3x}$, was transformed to a new graph with equation $y=\mathrm{g}(x)$.

(i) Given that $\mathrm{g}(x)=\frac{1}{\mathrm{h}'(x)}$, express $\mathrm{g}(x)$ in the form of $k\mathrm{e}^{mx}$, where k and m are exact real constants. [1]

(ii) Describe a sequence of two transformations which would transform the graph of $y=\mathrm{h}(x)$ to the graph of $y=\mathrm{g}(x)$. [2]

(b) The diagram below shows the curve $y = \mathrm{f}(x)$. The curve has a minimum point $(-3\ ,\ -4)$ and passes through the origin. The lines $x=3$ and $y=-2$ are the vertical and horizontal asymptotes to the curve respectively.



(i) Sketch, including the coordinates of the point(s) of intersections with the axes, turning point(s) and equation(s) of asymptote(s), if any, the following:

(a) $y=-\frac{1}{2}\mathrm{f}(x-3)$ [4]

(b) $y=\frac{1}{\mathrm{f}(x)}$. [3]

(ii) State the coordinates of the point(s) where the curve $y=\mathrm{f}'(x)$ cuts the axes. [1]

7 A curve C has equation $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=\dfrac{2x^2+6x+k}{x+3}$, k is a non-zero real constant. It is given that the gradient of the curve C is always positive.

(i) Find the range of values of k. [3]

(ii) Sketch the curve C for the range of values of k in **(i)**, showing clearly, if any, the equation(s) of the asymptote(s) and axial intercept(s). [3]

(iii) By adding a suitable curve to the graph of $y=\mathrm{f}(x)$ in **(ii)**, deduce the number of distinct real roots of the equation $\dfrac{1}{4}\left[\dfrac{\left(2x^2+6x+k\right)^2}{(x+3)^2}\right]-x^2=1$. [2]

(iv) Given that $k=-1$ and drawing two suitable graphs, solve the inequality $\mathrm{f}(x)>|x-3|$. [4]

8 A curve C has parametric equations

$$x=2\sin\theta+2\ ,\ \ y=3\cos\theta-1\ ,$$

where $\theta\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$.

(i) The point $P\left(\sqrt{2}+2\ ,\ \dfrac{3}{2}\sqrt{2}-1\right)$ lies on the curve C. **Without converting the parametric equations into cartesian form**, show that the equation of the normal to the curve C at P is $6y=4x-14+5\sqrt{2}$. [5]

(ii) Find the area of the quadrilateral bounded by the y-axis, x-axis, the normal at the point P and the horizontal line passing through P, correct to 3 decimal places. [3]

(iii) What can be said about the tangents to C as $\theta\to\pm\dfrac{\pi}{2}$? [1]

(iv) Draw the curve C, showing clearly the features of the curve at the points where $\theta=-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. [2]

9 When referred to the origin O, the points A and B have position vectors $5\mathbf{j}+15\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ respectively.

The line l_1 has equation $\mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(i) Find a vector equation of line l_2 passing through the points A and B. [2]

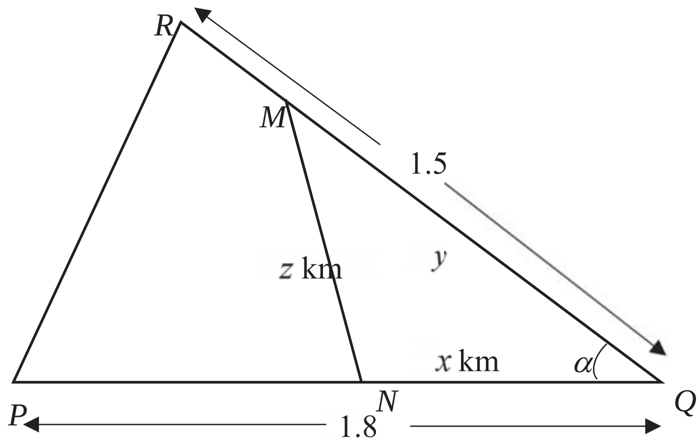
(ii) Find the coordinates of the point C, where l_1 and l_2 intersect. [3]

(iii) Find the position vector of the point F, the foot of the perpendicular from A to the line l_1. [4]

(iv) Find the vector equation of the line of reflection of l_2 in the line l_1. [3]

(v) Find a cartesian equation of the plane that contains the lines l_1 and l_2. [3]

10 A property developer wants to develop a triangular plot of land PQR as shown in the diagram below.
One section, NQM, is to be used for residential development and the other section, $PNMR$, is to be used for commercial development where M is on RQ and N is on PQ.
It is given that $NQ=x$ km, $QM=y$ km, $MN=z$ km, $RQ=1.5$ km, $PQ=1.8$ km, and a fixed angle $\angle NQM=\alpha$ radians where $\alpha\in\left(0,\dfrac{\pi}{2}\right)$.



(i) To achieve the requirements set out by the government on the use of the plot, the developer plans the use such that the residential development and commercial development takes up the same area each in the plot PQR.

Show that $z^2=x^2+\dfrac{1.8225}{x^2}-2.7\cos\alpha$. [4]

The developer wants to build a fence on the boundary MN. In order to minimize the construction costs, he decides that the boundary MN should be of minimum length.

(ii) Using differentiation, find the value of x which will minimise the length MN, giving your answers correct to 3 decimal places. [7]

(iii) Given that $\angle NQM=\alpha=\dfrac{\pi}{3}$, sketch the graph showing the relationship of the square of the length MN as the length of NQ varies. [3]

2021 SAJC H2 Maths Promo Paper Solution

1(a)	Let $S_n = an^2 + bn + c$ When $n = 1$, $1 = a + b + c$ --- (1) When $n = 2$, $S_2 = u_1 + u_2$ $4a + 2b + c = 6$ ---- (2) When $n = 12$, $276 = 144a + 12b + c$ --- (3) Using GC to solve equations (1), (2) and (3), $a = 2, b = -1, c = 0$. $\therefore S_n = 2n^2 - n$
(b)	$\sum_{n=2}^{N}(v_n - v_{n-1}) = \left(-\frac{3}{5}\right)\sum_{n=2}^{N}\left(\frac{2}{5}\right)^{n-2}$ LHS: $\sum_{n=2}^{N}(v_n - v_{n-1})$ $= v_2 - v_1$ $+ v_3 - v_2$ $+ v_4 - v_3$ $+ \ldots$ $+ v_{N-1} - v_{N-2}$ $+ v_N - v_{N-1}$ $= v_N - 4$ RHS: $\left(-\frac{3}{5}\right)\sum_{n=2}^{N}\left(\frac{2}{5}\right)^{n-2} = \left(-\frac{3}{5}\right)\frac{1\left[1-\left(\frac{2}{5}\right)^{N-1}\right]}{1-\frac{2}{5}} = \left(\frac{2}{5}\right)^{N-1} - 1$ Therefore $v_N - 4 = \left(\frac{2}{5}\right)^{N-1} - 1$ $v_N = \left(\frac{2}{5}\right)^{N-1} + 3$ As $N \to \infty, \left(\frac{2}{5}\right)^{N-1} \to 0$. Hence $v_n \to 3$, a **constant/unique finite value**. Therefore the sequence is convergent.

2(i)	$S_n = \frac{a^n}{2^{n-1}} - 2$; $S_{n-1} = \frac{a^{n-1}}{2^{n-2}} - 2$ For $n \geq 2$, $T_n = S_n - S_{n-1} = \frac{a^n}{2^{n-1}} - 2 - \left(\frac{a^{n-1}}{2^{n-2}} - 2\right)$ $= \left(\frac{a}{2}\right)^{n-1}\left(a - \frac{1}{2^{-1}}\right) = (a-2)\left(\frac{a}{2}\right)^{n-1}$ When $n = 1$, $T_1 = S_1 = \frac{a}{2^0} - 2 = a - 2 = (a-2)\left(\frac{a}{2}\right)^{1-1}$ which follows the form of $T_n = (a-2)\left(\frac{a}{2}\right)^{n-1}$ when $n = 1$. Thus, $T_n = (a-2)\left(\frac{a}{2}\right)^{n-1}$ for $n \geq 1$ (shown) $\frac{T_n}{T_{n-1}} = \frac{(a-2)\left(\frac{a}{2}\right)^{n-1}}{(a-2)\left(\frac{a}{2}\right)^{n-2}} = \frac{a}{2}$ (constant independent of n) Series is a geometric series. (shown)
(ii)	For the sum to infinity to exist, $\left\lvert\frac{a}{2}\right\rvert < 1$ $\therefore -2 < a < 2, a \neq 0$ [or $\therefore \lvert a \rvert < 2, a \neq 0$]
(iii)	Need to find the least value of n such that $\lvert S_n - S_\infty \rvert < 0.2$ $T_1 = -1$, $S_n = \frac{1}{2^{n-1}} - 2$, Common ratio$=\frac{1}{2}$; $S_\infty = \frac{-1}{1-\frac{1}{2}} = -2$ For $\lvert S_n - S_\infty \rvert < 0.2$ $\left\lvert \frac{1}{2^{n-1}} - 2 - (-2) \right\rvert < 0.2$ $\left\lvert \frac{1}{2^{n-1}} \right\rvert < 0.2$ $\lvert 2^{n-1} \rvert > 5$ $2^{n-1} > 5$ $n - 1 > \frac{\ln 5}{\ln 2}$ $n > 3.32$ Least $n = 4$

Alternatively,

$T_1 = -1,\ S_n = \dfrac{1}{2^{n-1}} - 2$, Common ratio$=\dfrac{1}{2}$

$S_\infty = \dfrac{-1}{1-\frac{1}{2}} = -2$

For $\left|S_n - S_\infty\right| < 0.2$

$\left|\dfrac{1}{2^{n-1}} - 2 - (-2)\right| < 0.2$

$\left|\dfrac{1}{2^{n-1}}\right| < 0.2$

n	$\left\lvert\dfrac{1}{2^{n-1}}\right\rvert$
3	0.25>0.2
4	0.125<0.2
5	0.0625<0.2

Hence, the least $n = 4$

3(i)

Let $y = \ln\left(\dfrac{e^x+1}{e^x-1}\right)$

$e^y = \dfrac{e^x+1}{e^x-1}$

$e^y e^x - e^y = e^x + 1$

$e^y e^x - e^x = e^y + 1$

$e^x\left(e^y - 1\right) = e^y + 1$

$e^x = \dfrac{e^y+1}{e^y-1}$

$x = \ln\left(\dfrac{e^y+1}{e^y-1}\right)$

Since $x = f^{-1}(y) = \ln\left(\dfrac{e^y+1}{e^y-1}\right)$,

$f^{-1}(x) = \ln\left(\dfrac{e^x+1}{e^x-1}\right) = f(x)$

(ii)

Since $f(x) = f^{-1}(x)$,

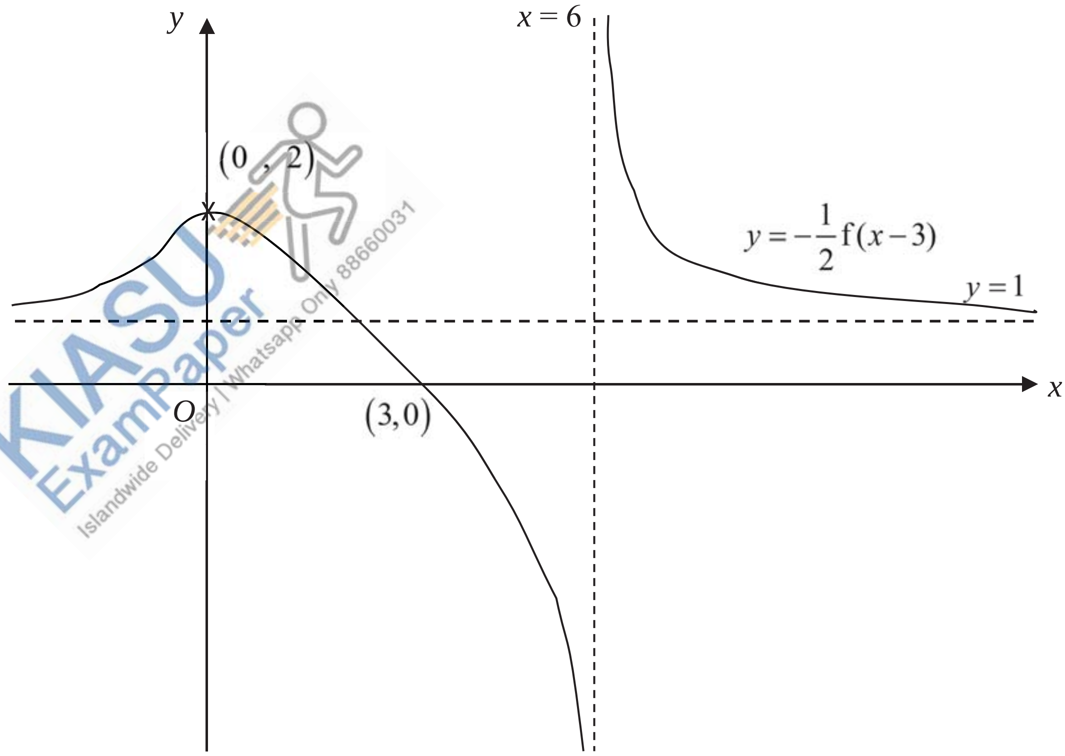
$ff(x) = ff^{-1}(x) = x$. Thus, $f^2(x) = x$.

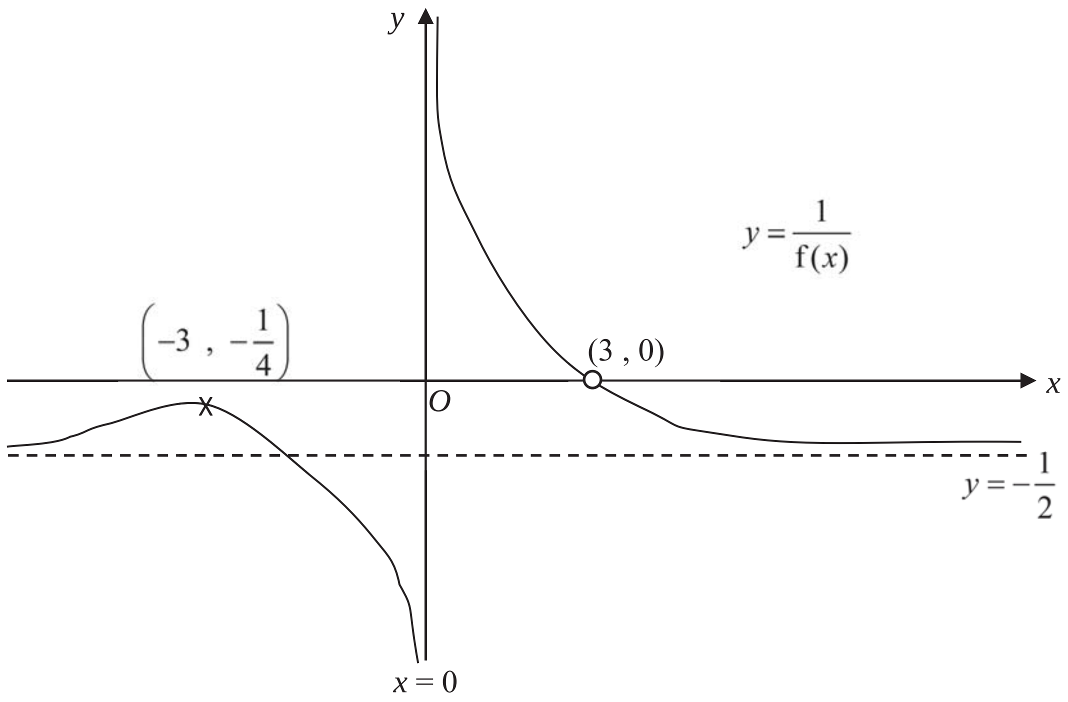
$f^{2021}(3) = f(3) = \ln\left(\dfrac{e^3+1}{e^3-1}\right)$

4(i)	Vectors **p** and **q** are parallel or either **p** or **q** (or both) is a zero vector.
(ii)	Since $\mathbf{r}\times\begin{pmatrix}-6\\3\\2\end{pmatrix}=\mathbf{0}$ $\mathbf{r}$ is parallel to the vector $\begin{pmatrix}-6\\3\\2\end{pmatrix}$. $\therefore\quad \mathbf{r}=\frac{1}{\sqrt{(-6)^2+3^2+2^2}}\begin{pmatrix}-6\\3\\2\end{pmatrix}$ $=\frac{1}{7}\begin{pmatrix}-6\\3\\2\end{pmatrix}$

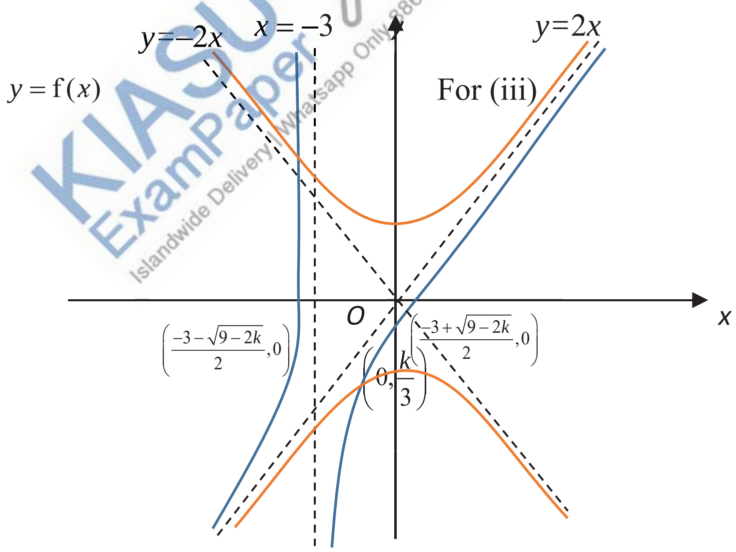
(iii)	$\cos\theta=\frac{\left\lvert\begin{pmatrix}-6\\3\\2\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rvert}{\lvert\sqrt{49}\rvert\lvert\sqrt{1}\rvert}=\frac{\lvert-6\rvert}{\lvert\sqrt{49}\rvert\lvert\sqrt{1}\rvert}=\frac{6}{7}$ Using the trigonometric identity, $\sin^2\theta+\cos^2\theta=1$ $\sin\theta=\sqrt{1-\cos^2\theta}$ $=\sqrt{1-\left(\frac{6}{7}\right)^2}$ $=\frac{1}{7}\sqrt{49-36}$ $=\frac{\sqrt{13}}{7}$ $\sin\theta>0$ since θ is an acute angle.	*Alternatively* Let θ be the required acute angle. $\sin\theta=\frac{\left\lvert\begin{pmatrix}-6\\3\\2\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rvert}{\lvert\sqrt{49}\rvert\lvert\sqrt{1}\rvert}=\frac{\left\lvert\begin{pmatrix}0\\2\\-3\end{pmatrix}\right\rvert}{\lvert\sqrt{49}\rvert\lvert\sqrt{1}\rvert}$ $=\frac{\sqrt{13}}{7}$

5(a) (i)	$\frac{d}{dx}\left(3^{2x}\right)=3^{2x}(2)(\ln 3)$ $=9^x \ln 3^2$ $=9^x \ln 9$ (Shown)	Alternatively, Let $y=3^{2x}$. Then, $\ln y=2x\ln 3$ Differentiate both sides with respect to x, $\frac{1}{y}\frac{dy}{dx}=2\ln 3$ $\frac{dy}{dx}=2y\ln 3=2\left(3^{2x}\right)\ln 3=3^{2x}\ln 9$
(a) (ii)	Let $y=\tan^{-1}\left(3^{2x}\right)$ $\frac{dy}{dx}=\frac{1}{1+\left(3^{2x}\right)^2}\left(9^x \ln 9\right)=\frac{9^x \ln 9}{1+3^{4x}}$	
(b)	$y^{2y}=e^{3x+2e}$ Taking ln on both sides, $2y\ln y=3x+2e$ Differentiating both sides with respect to x, $2\left[y\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right)+(\ln y)\left(\frac{dy}{dx}\right)\right]=3$ $2\left(\frac{dy}{dx}\right)(1+\ln y)=3---(1)$ Differentiating (1) with respect to x, $2\left[\left(\frac{dy}{dx}\right)\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right)+(1+\ln y)\left(\frac{d^2y}{dx^2}\right)\right]=0$ $\left(\frac{dy}{dx}\right)^2\left(\frac{1}{y}\right)+(1+\ln y)\left(\frac{d^2y}{dx^2}\right)=0$ $y\left(\frac{d^2y}{dx^2}\right)=-\frac{1}{(1+\ln y)}\left(\frac{dy}{dx}\right)^2$ From (1): $\frac{2}{3}\left(\frac{dy}{dx}\right)=\frac{1}{(1+\ln y)}$ Hence, $y\left(\frac{d^2y}{dx^2}\right)=-\frac{1}{(1+\ln y)}\left(\frac{dy}{dx}\right)^2=-\frac{2}{3}\left(\frac{dy}{dx}\right)^3$	

6(a) (i)	$\mathrm{h}'(x)=3\mathrm{e}^{3x}$ $\mathrm{g}(x)=\dfrac{1}{\mathrm{h}'(x)}=\dfrac{1}{3}\mathrm{e}^{-3x}$
(a) (ii)	$y=\mathrm{e}^{3x}$ $\downarrow$ Replace x with $-x$ $y=\mathrm{e}^{-3x}$ $\downarrow$ Replace y with $3y$ $y=\dfrac{1}{3}\mathrm{e}^{-3x}$ The graph of $y = \mathrm{h}(x)$ undergoes the transformations: 1. Reflection about the y-axis, followed by 2. A scaling parallel to the y axis with a scale factor of $\dfrac{1}{3}$ to obtain the graph of $y=\dfrac{1}{\mathrm{h}'(x)}$.
(b) (i) (a)	$y=\mathrm{f}(x)\xrightarrow{\text{replace } x \text{ by } x-3} y=\mathrm{f}(x-3)\xrightarrow{\text{replace } y \text{ by } -y} y=-\mathrm{f}(x-3)\xrightarrow{\text{replace } y \text{ by } 2y} y=-\dfrac{1}{2}\mathrm{f}(x-3)$ 

(b) (i) (b)	

(b) (ii)	$(-3\ ,\ 0)$ **[Note: (-3,0) on $y=f'(x)$ corresponds to the stationary point on $y=f(x)$, and there are no other *x*-intercept on $y=f'(x)$. However, there would be a *y*-intercept on $y=f'(x)$, but there is insufficient information from the question for us to determine the value of the *y*-intercept since we do not know the gradient of the graph $y=f(x)$ at *x* = 0.]**

7(i)	$y=\dfrac{2x^2+6x+k}{x+3}$ $\dfrac{dy}{dx}=\dfrac{(4x+6)(x+3)-(2x^2+6x+k)}{(x+3)^2}$ $=\dfrac{2x^2+12x+18-k}{(x+3)^2}>0$ $\Rightarrow 2x^2+12x+18-k>0$ for all real values of $x, x\neq -3$ $\Rightarrow$ Discriminant <0 $12^2-4(2)(18-k)<0$ $\Rightarrow k<0$	Alternatively, $y=\dfrac{2x^2+6x+k}{x+3}$ $y=2x+\dfrac{k}{x+3}$ $\dfrac{dy}{dx}=2-\dfrac{k}{(x+3)^2}$ For $\dfrac{dy}{dx}=2-\dfrac{k}{(x+3)^2}>0$, $k<0$
(ii)	$y=\dfrac{2x^2+6x+k}{x+3}$ $y=2x+\dfrac{k}{x+3}, k<0$ Equations of asymptotes: $y=2x,\ x=-3$ Intercepts: $x=0\Rightarrow y=\dfrac{k}{3}$ $y=0\Rightarrow 2x^2+6x+k=0$ $\Rightarrow x=\dfrac{-6\pm\sqrt{36-8k}}{4}=\dfrac{-3\pm\sqrt{9-2k}}{2}$ 	

(iii)

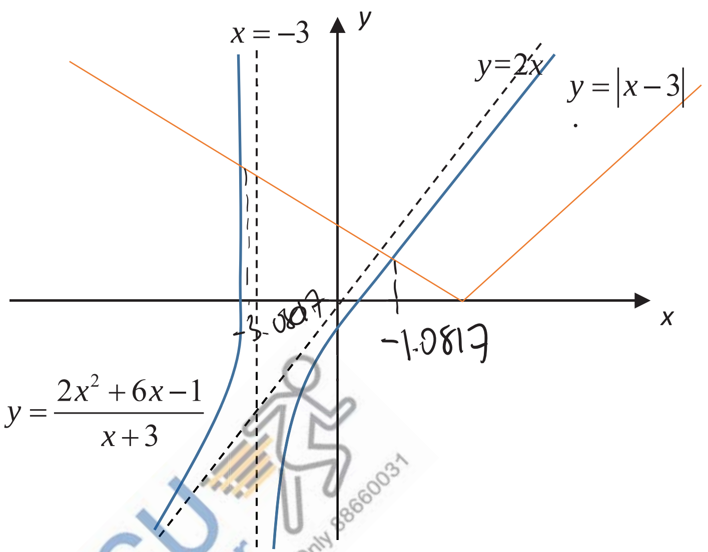
$$\frac{\left(\frac{2x^2+6x+k}{x+3}\right)^2}{4}-x^2=1 \text{ -----(*)}$$

Add the hyperbola $\frac{y^2}{2^2}-\frac{x^2}{1^2}=1$.

Asymptotes are $y=\pm 2x$ and y-intercepts are $(0,\pm 2)$

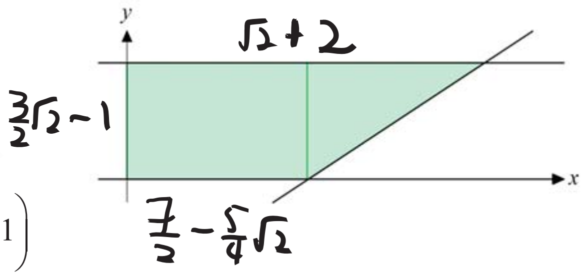
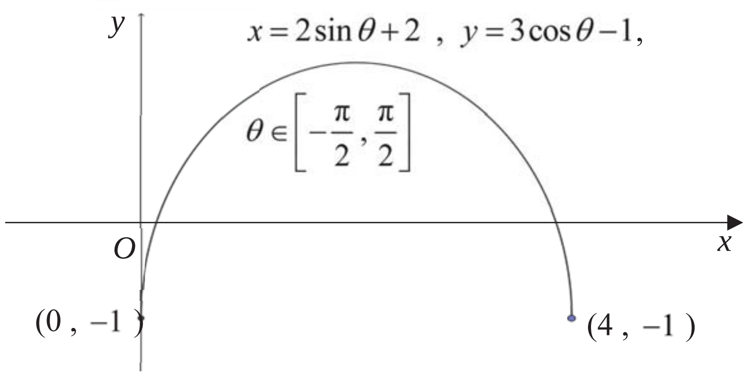
Since the graphs $y=\mathrm{f}(x)$ and $\frac{y^2}{2^2}-\frac{x^2}{1^2}=1$ have two intersection points, the equation (*) has two real roots.

(iv)



The x-coordinate of the intersection points are between the graphs $y=\frac{2x^2+6x-1}{x+3}$ and $y=|x-3|$ are –3.0817 and 1.0817.

For $\frac{2x^2+6x-1}{x+3}>|x-3|$, $\quad -3.08<x<-3$ or $x>1.08$.

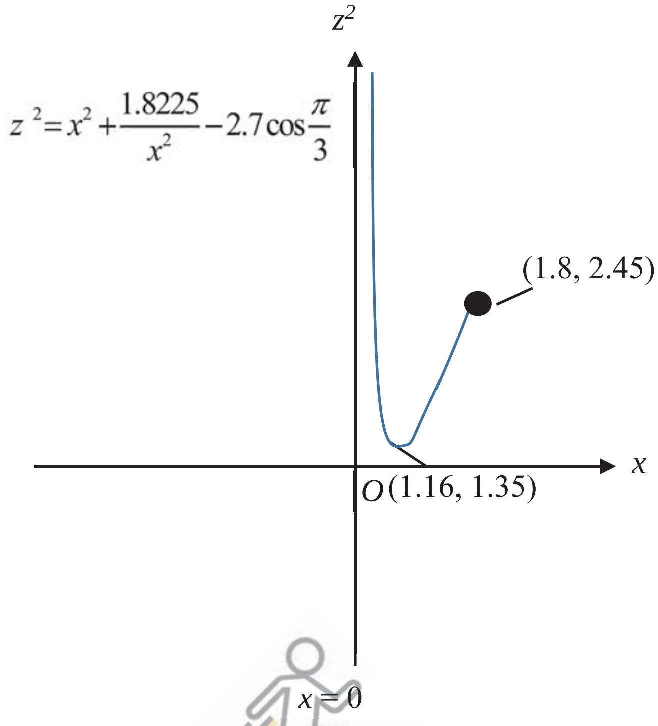
8(i)	$\frac{dx}{d\theta} = 2\cos\theta$, $\frac{dy}{d\theta} = -3\sin\theta$ $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{1}{\frac{dx}{d\theta}} = (-3\sin\theta) \times \frac{1}{2\cos\theta} = -\frac{3}{2}\tan\theta$ At P, $\sqrt{2}+2 = 2\sin\theta + 2 \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{2}}{2} \quad \Rightarrow \theta = \frac{\pi}{4}$ Check: $y = 3\cos\frac{\pi}{4} - 1 = \frac{3\sqrt{2}}{2} - 1$ Gradient of normal at P $= -\frac{1}{\left(-\frac{3}{2}\tan\frac{\pi}{4}\right)} = \frac{2}{3}$ Equation of normal at P: $y - \left(\frac{3}{2}\sqrt{2} - 1\right) = \frac{2}{3}\left[x - \left(\sqrt{2}+2\right)\right]$ $y = \frac{2}{3}x - \frac{4}{3} + \frac{5}{3}\left(\frac{\sqrt{2}}{2}\right) - 1 = \frac{2}{3}x - \frac{7}{3} + \frac{5}{6}\sqrt{2}$ $6y = 4x - 14 + 5\sqrt{2}$
(ii)	When the normal at point P intersects the x-axis, $y = 0$ $0 = 4x - 14 + 5\sqrt{2}$ $x = \frac{14}{4} - \frac{5}{4}\sqrt{2} = \frac{7}{2} - \frac{5}{4}\sqrt{2}$  Area of quadrilateral $= \frac{1}{2}\left[\left(\frac{7}{2} - \frac{5}{4}\sqrt{2}\right) + \left(\sqrt{2}+2\right)\right]\left(\frac{3}{2}\sqrt{2} - 1\right)$ $= 2.8854 = 2.885$ units^2 (3 d.p.)
(iii)	As $\theta \to \pm\frac{\pi}{2}$, $\frac{dy}{dx} = -\frac{3}{2}\tan\theta \to \pm\infty$. Hence, the tangents will become/approach vertical lines.
(iv)	

9(i)	$\overrightarrow{OA}=\begin{pmatrix}0\\5\\15\end{pmatrix}$; $\overrightarrow{OB}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}1\\-1\\3\end{pmatrix}-\begin{pmatrix}0\\5\\15\end{pmatrix}=\begin{pmatrix}1\\-6\\-12\end{pmatrix}$ Equation of line l_2 : $\mathbf{r}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\mu\begin{pmatrix}1\\-6\\-12\end{pmatrix}$, $\mu\in\mathbb{R}$
(ii)	Let the point of intersection between l_1 and l_2 be X. $\overrightarrow{OX}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$ and $\overrightarrow{OX}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\mu\begin{pmatrix}1\\-6\\-12\end{pmatrix}$ for some $\lambda,\mu\in\mathbb{R}$ Since l_1 and l_2 intersect, $\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\5\\15\end{pmatrix}+\mu\begin{pmatrix}1\\-6\\-12\end{pmatrix}$ $-2-\lambda=\mu \Rightarrow \lambda+\mu=-2 \ldots (1)$ $1+2\lambda=5-6\mu \Rightarrow 2\lambda+6\mu=4 \cdots (2)$ $3+3\lambda=15-12\mu \Rightarrow 3\lambda+12\mu=12 \ldots (3)$ Solving (1) , (2) and (3) using GC, $\lambda=-4; \mu=2$ $\overrightarrow{OC}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+(-4)\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\-7\\-9\end{pmatrix}$ Coordinates of C is $(2,-7,-9)$
(iii)	Since F is a point on l_1 : $\overrightarrow{OF}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{AF}=\overrightarrow{OF}-\overrightarrow{OA}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}-\begin{pmatrix}0\\5\\15\end{pmatrix}=\begin{pmatrix}-2-\lambda\\-4+2\lambda\\-12+3\lambda\end{pmatrix}$ Since $\overrightarrow{AF}$ is perpendicular to l_1, $\overrightarrow{AF}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix}=0$

	$\begin{pmatrix}-2-\lambda\\-4+2\lambda\\-12+3\lambda\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix}=0$ $2+\lambda-8+4\lambda-36+9\lambda=0$ $14\lambda=42$ $\lambda=3$ Therefore $\overrightarrow{OF}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+3\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}-5\\7\\12\end{pmatrix}$	
(iv)	Let A' be the point of reflection of A along the line l_1 Using Ratio Theorem, $\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$ $=2\begin{pmatrix}-5\\7\\12\end{pmatrix}-\begin{pmatrix}0\\5\\15\end{pmatrix}=\begin{pmatrix}-10\\9\\9\end{pmatrix}$ Let l_3 be the line of reflection of l_2 in the line l_1 $\overrightarrow{CA'}=\overrightarrow{OA'}-\overrightarrow{OC}$ $=\begin{pmatrix}-10\\9\\9\end{pmatrix}-\begin{pmatrix}2\\-7\\-9\end{pmatrix}$ $=\begin{pmatrix}-12\\16\\18\end{pmatrix}=2\begin{pmatrix}-6\\8\\9\end{pmatrix}$ Direction vector of $l_3=\begin{pmatrix}-6\\8\\9\end{pmatrix}$ equation of the line l_3: $\mathbf{r}=\begin{pmatrix}2\\-7\\-9\end{pmatrix}+\alpha\begin{pmatrix}-6\\8\\9\end{pmatrix},\ \alpha\in\mathbb{R}$	Alternatively, Let A' be the point of reflection of A along the line l_1 Using Ratio Theorem, $\overrightarrow{CF}=\dfrac{\overrightarrow{CA}+\overrightarrow{CA'}}{2}$ $\overrightarrow{CA'}=2\overrightarrow{CF}-\overrightarrow{CA}$ $=2\left[\overrightarrow{OF}-\overrightarrow{OC}\right]-\left[\overrightarrow{OA}-\overrightarrow{OC}\right]$ $=2\overrightarrow{OF}-\overrightarrow{OC}-\overrightarrow{OA}$ $=2\begin{pmatrix}-5\\7\\12\end{pmatrix}-\begin{pmatrix}2\\-7\\-9\end{pmatrix}-\begin{pmatrix}0\\5\\15\end{pmatrix}$ $=\begin{pmatrix}-12\\16\\18\end{pmatrix}=2\begin{pmatrix}-6\\8\\9\end{pmatrix}$ Let l_3 be the line of reflection of l_2 in the line l_1 Direction vector of $l_3=\begin{pmatrix}-6\\8\\9\end{pmatrix}$ equation of the line l_3: $\mathbf{r}=\begin{pmatrix}2\\-7\\-9\end{pmatrix}+\alpha\begin{pmatrix}-6\\8\\9\end{pmatrix},\ \alpha\in\mathbb{R}$

(v)	The normal of plane = $\begin{pmatrix}-1\\2\\3\end{pmatrix}\times\begin{pmatrix}1\\-6\\-12\end{pmatrix}=\begin{pmatrix}-6\\-9\\4\end{pmatrix}$ Using $\overrightarrow{OA}=\begin{pmatrix}0\\5\\15\end{pmatrix}$ as a point on the plane $\mathbf{r}\bullet\begin{pmatrix}-6\\-9\\4\end{pmatrix}=\begin{pmatrix}0\\5\\15\end{pmatrix}\bullet\begin{pmatrix}-6\\-9\\4\end{pmatrix}$ Cartesian of the required plane is $-6x-9y+4z=15$

10(i)	By cosine rule, $z^2 = x^2 + y^2 - 2xy\cos\alpha$ ---------- (1) By considering the area of the triangle and the quadrilateral, $2\left(\frac{1}{2}xy\sin\alpha\right) = \frac{1}{2}(1.5)(1.8)\sin\alpha$ $xy = 1.35$ ---------- (2) $y = \frac{1.35}{x}$ --------- (3) Substitute (2) and (3) in (1): $z^2 = x^2 + \left(\frac{1.35}{x}\right)^2 - 2(1.35)\cos\alpha$ $z^2 = x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha$ (Shown)
(ii)	Differentiating with respect to x, $2z\frac{dz}{dx} = 2x - \frac{3.645}{x^3}$ For the stationary values of z, $\frac{dz}{dx} = 0$ $2x - \frac{3.645}{x^3} = 0$ $2x^4 - 3.645 = 0$ $x^4 = 1.8225$ $x = \sqrt[4]{1.8225}$ $= 1.16189 = 1.162$ m (3 dp) or $x = -\sqrt[4]{1.8225}$ (N.A, $x > 0$) **<u>Alternative solution</u>** $z = \left(x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha\right)^{\frac{1}{2}}$ $\frac{dz}{dx} = \frac{1}{2}\left(x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha\right)^{-\frac{1}{2}}\left(2x - \frac{2(1.8225)}{x^3}\right)$ $= \frac{1}{2}\frac{1}{\sqrt{x^2 + \frac{1.8225}{x^2} - 2.7\cos\alpha}}\left(2x - \frac{2(1.8225)}{x^3}\right)$ For stationary values of z, $\frac{dz}{dx} = 0$ $2x - \frac{3.645}{x^3} = 0$ $2x^4 - 3.645 = 0$ $x^4 = 1.8225$ $x = \sqrt[4]{1.8225} = 1.16189 = 1.162$ m (3 dp) or $x = -\sqrt[4]{1.8225}$ (N.A, $x > 0$)
	<u>Method</u> (Second Derivative Test) Differentiating with respect to x: $2\left(\frac{dz}{dx}\right)^2 + 2z\frac{d^2z}{dx^2} = 2 + \frac{10.935}{x^4}$ For the value of z to be minimum, $\frac{dz}{dx} = 0$ $2z\frac{d^2z}{dx^2} = 2 + \frac{10.935}{x^4}$

	$\frac{d^2z}{dx^2}=\frac{1}{2z}\left(2+\frac{10.935}{x^4}\right)$ For $x=\sqrt[4]{1.8225}$, $\frac{d^2z}{dx^2}=\frac{1}{2z}\left(2+\frac{10.935}{1.8225}\right)>0$ since $z>0$ given that z is a length. Hence the length of MN is a minimum when $x=1.162$ m (3 decimal places)
(iii)	

1 **(a)** Describe a sequence of transformations which would transform a curve $y = \mathrm{f}(x)$ onto the curve $y = \mathrm{f}\left(1-2x\right)$. [3]

(b) Sketch the graph of $y = \left|2-3\mathrm{e}^{x}\right|$, showing the relevant features. [2]

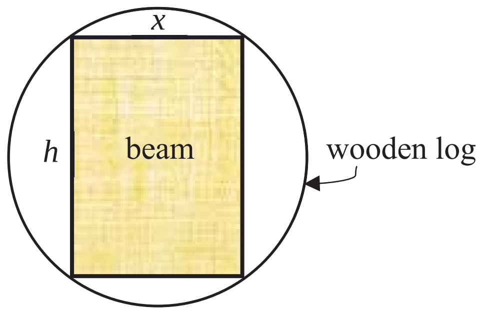
Without the use of a calculator, hence find the exact range of values of x that satisfy the inequality $\left|2-3\mathrm{e}^{x}\right| > \mathrm{e}^{2x}$, for $x \geq 0$. [4]

2 The position vectors of the points A and B relative to O are **a** and **b** respectively, where **a** and **b** are non-zero and non-parallel vectors. The point C, with position vector **c**, is the reflection of A in the line OB. Given that **b** is a unit vector,

(i) show that $\mathbf{c} = 2(\mathbf{a}.\mathbf{b})\,\mathbf{b} - \mathbf{a}$, [2]

(ii) show that $\mathbf{c} \times \mathbf{a} = 2(\mathbf{a}.\mathbf{b})(\mathbf{c} \times \mathbf{b})$. [2]

3 In the Emazon forest, trees are chopped down and the wooden logs are transported to the timber factory where they are cut into beams. In the figure shown below, it is assumed that the cross-section of each wooden log is a circle. Each beam is cut such that the cross-section is a rectangle and the 4 corners of the rectangle touch the circumference of the log to reduce wastage. It is given that the strength, S, of a beam is proportional to the product of its height h and the square of its width x.

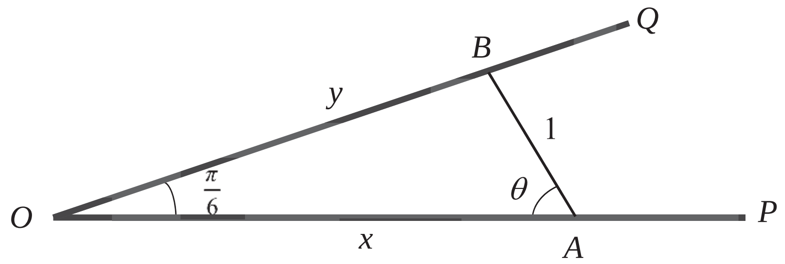


A particular wooden log of radius r has been sent for cutting.

(i) Show that $S = kx^2\sqrt{4r^2 - x^2}$, where k is a real constant. [2]

(ii) Using differentiation, find the exact width, in terms of r, of the strongest beam that can be cut from the log. [5]

4 In the figure (not drawn to scale), POQ is a rail and $\angle POQ = \frac{\pi}{6}$. AB is a rod of fixed length 1 m which is free to slide on the rail with end A on OP and end B on OQ. It is given that $OA = x$ m, $OB = y$ m and $\angle OAB = \theta$ radians.



(i) Express x and y in terms of θ. [2]

(ii) Given that S is the area of triangle OAB, find S in terms of θ. [1]

Given that B is moving towards O at a rate of 0.2 ms^{-1} at the instant when $\theta = \frac{\pi}{4}$,

(iii) find the rate of change of S at this instant. [4]

5 A curve C has equation $y = \frac{\alpha - x^2}{1 - x}$ where α is a real constant and $\alpha \neq 1$.

(i) Find the equations of the asymptotes of C. [2]

It is given that C has positive gradient for all $x \in \mathbb{R}$.

(ii) Find the range of values of α. [4]

(iii) Sketch C, giving the equations of the asymptotes and the coordinates of the axial intercepts. [2]

6 **(a)** Find $\int \sin px \sin qx \, \mathrm{d}x$, where p and q are real constants. [2]

(b) Find the exact value of $\int_{-\ln 2}^{0} \left(\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} + 1} \right) \mathrm{d}x$, giving your answer as a single logarithm.

[2]

(c) By considering $x - 1 = A(1 - 2x) + B$, where A and B are constants to be determined, find $\int \frac{x - 1}{\sqrt{6 + x - x^2}} \, \mathrm{d}x$. [5]

7 **(a)** The function g is defined by

$$g : x \mapsto x + \frac{1}{x}, \quad x > 0.$$

(i) Explain why g does not have an inverse. [2]

(ii) If the domain of g is restricted to the subset of $\mathbb{R}$ for which $x \geq p$, find the minimum value of p such that g^{-1} exists. Using this value of p, find an expression for $g^{-1}(x)$, stating the domain. [5]

(b) The functions f and h are defined by

$$f : x \mapsto k - (x+1)^2, \ -2 < x < 2 \text{ and } k \text{ is a constant,}$$
$$h : x \mapsto \ln(x+3), \quad x > 0.$$

Find the minimum value of k such that the composite function hf exists. [2]

8 **(a)** Find $\int x e^{\frac{1}{2}x} \, dx$. [2]

(b) A curve has parametric equations

$$x = e^t - 2t, \quad y = e^{\frac{1}{2}t}, \quad \text{where } 0 \leq t \leq 2.$$

(i) Sketch the curve, giving the exact coordinates of the end-points. [2]

(ii) Find the exact x-coordinate of the point on the curve where the tangent to the curve is parallel to the y-axis. [4]

(iii) Find the exact area of the region bounded by the curve, the lines $y = 1$ and $y = e$, and the y-axis. [4]

9 An ellipse C has equation $x^2 + 3y^2 = a^2$ where a is a positive constant.

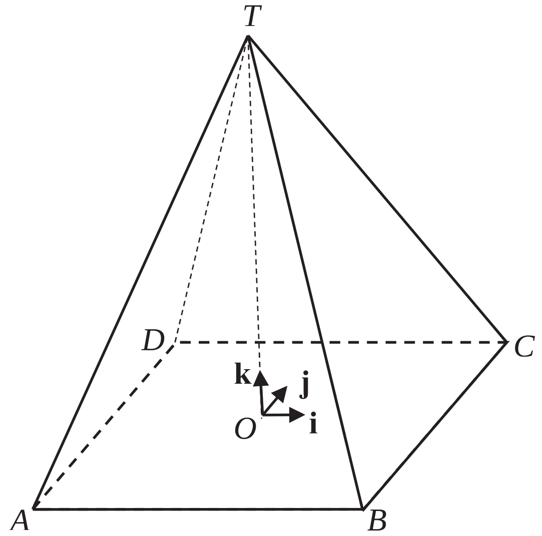
(i) Sketch C. [1]

(ii) The region enclosed by C, the line $y = x$ and the positive x-axis is denoted by R. By using the substitution $x = a\sin\theta$, find the exact area of R in the form $k\pi a^2$ where k is a constant to be determined. [7]

It is given that $a = \sqrt{10}$. The region enclosed by C and the line $x = 2$, where $x > 0$ is denoted by S.

(iii) Write down the equation of the curve obtained when C is translated 2 units in the negative x-direction. Hence or otherwise, find the volume of the solid formed when S is rotated through 2π radians about the line $x = 2$, giving your answer to 3 significant figures. [4]

10



A pyramid has a horizontal rectangular base *ABCD*, where $AB = 4$ units and $AD = 2$ units. The vertex *T* is 5 units vertically above the centre of the base, *O*. Perpendicular unit vectors **i**, **j** and **k** are parallel to *AB, AD* and *OT* respectively. The point *N* is on *TB* such that $TN : NB = 5 : 1$.

(i) Find a vector equation of the line *AB*. [1]

(ii) *P* is a point on *AB* such that *PN* is perpendicular to *TB*. Find the position vector of *P*. [4]

(iii) Find a cartesian equation of the plane *TBC*. [3]

A line *l* has cartesian equation $\dfrac{x+2}{5} = y = \dfrac{z}{k}$, where *k* is a positive real constant.

(iv) Find the exact range of values of *k* such that *l* is inclined at an angle of less than 45° to the horizontal plane. [3]

11 **(a)** The sum, S_n, of the first *n* terms of a sequence $u_1, u_2, \ldots, u_n, \ldots$ is given by $\dfrac{2}{3}\left(1-\dfrac{1}{k^n}\right)$, where *k* is a non-zero constant and $k \neq 1$.

(i) Show that the sequence is a geometric progression, and state the values of the common ratio and the first term in terms of *k*. [4]

(ii) The sum of the first *n* terms of another geometric series is given by $\dfrac{1}{u_1}+\dfrac{1}{u_2}+\ldots+\dfrac{1}{u_n}$. State the range of values of *k* for the sum to infinity of this series to exist and find the sum to infinity in terms of *k*. [3]

(b) An arithmetic progression has first term *a* and common difference *d*. The sum of the first *n* terms is one-third the sum of the next *n* terms of the arithmetic progression.

(i) Show that $d = 2a$. [3]

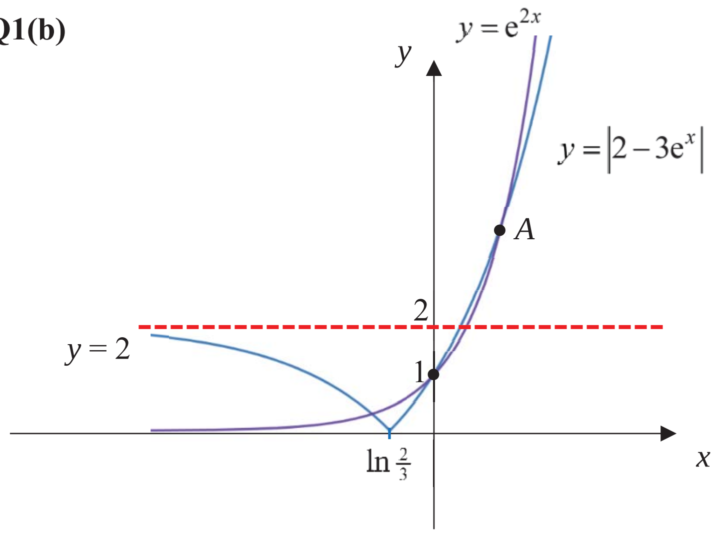
It is also given that *a* and *d* are positive integers, and $n \neq 1$.

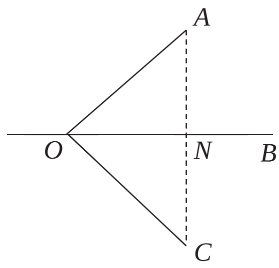
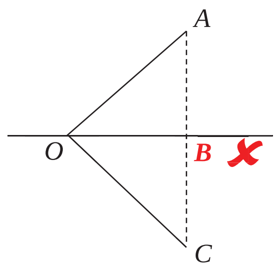
(ii) Hence find the value of *n* if the sum of the first *n* terms is 98. [2]

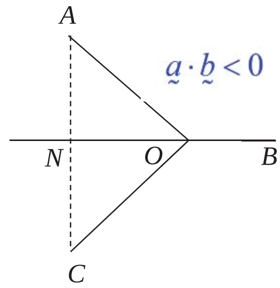
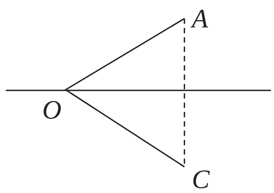
TEMASEK JUNIOR COLLEGE

2021 JC 1 H2 Mathematics Promotional Examination (Suggested solutions)

[Solution]	Comments
Q1(a) $y=\mathrm{f}(x)\xrightarrow{T} y=\mathrm{f}(x+1)$ $\xrightarrow{S} y=\mathrm{f}(2x+1)$ $\xrightarrow{R} y=\mathrm{f}(-2x+1)$ A sequence of transformations is: 1 A translation of 1 unit in the negative direction of the x-axis 2 A scaling parallel to the x-axis by factor $\frac{1}{2}$ 3 A reflection about the y-axis	- Use proper phrasing, eg translation instead of shift, reflection instead of flip, scaling…by factor $\frac{1}{2}$ instead of $\frac{1}{2}$ units. - Generally well done for students who did translation first, some errors made are those highlighted in yellow.
OR $y=\mathrm{f}(x)\xrightarrow{S} y=\mathrm{f}(2x)$ $\xrightarrow{T} y=\mathrm{f}\left(2\left(x+\frac{1}{2}\right)\right)=\mathrm{f}(2x+1)$ $\xrightarrow{R} y=\mathrm{f}(-2x+1)$ A sequence of transformations is: 1 A scaling parallel to the x-axis by factor $\frac{1}{2}$ 2 A translation of $\frac{1}{2}$ units in the negative direction of the x-axis 3 A reflection about the y-axis	- Students who did scaling first tend to get the translation in the second step wrong. Note that we only replace x as highlighted in green.
OR $y=\mathrm{f}(x)\xrightarrow{R} y=\mathrm{f}(-x)$ $\xrightarrow{T} y=\mathrm{f}(-(x-1))=\mathrm{f}(-x+1)$ $\xrightarrow{S} y=\mathrm{f}(-2x+1)$ A sequence of transformations is: 1 A reflection about the y-axis 2 A translation of 1 unit in the positive direction of the x-axis 3 A scaling parallel to the x-axis by factor $\frac{1}{2}$	- Students who did reflection first tend to get the translation in the second step wrong. Note that we only replace x as highlighted in green.

[Solution]	Comments
Q1(b) 	Let $x=0$, $y=1$ Let $y=\|2-3e^x\|=0 \Rightarrow x=\ln\frac{2}{3}$ Asymptote: $y=2$ (As $x\to-\infty$, $e^x\to 0 \therefore y\to 2$) Things to take note: 1. Horizontal asymptote $y=2$ must be drawn. 2. Sharp point at $(\ln\frac{2}{3},0)$.
Add the graph of $y=e^{2x}$ on the same diagram. At the intersection point A, $-(2-3e^x)=e^{2x}$ $e^{2x}-3e^x+2=0$ $(e^x-2)(e^x-1)=0$ $e^x=2$ or $e^x=1$ $x=\ln 2$ or $x=0$ (rej. since $x>0$ for A) For $\|2-3e^x\|>e^{2x}$, where $x\geq 0$, the solution is $0<x<\ln 2$	- Hence means need to make use of the graph above to solve. - Since $x\geq 0$, only need to solve for the intersection point A. - Algebraic errors made in the process of solving for intersection, eg $3e^x-e^{2x}=e^x(3-e^2)$ ✗ - Note that the solution must be given in exact form.

Q2	[Solution]	Comments
(i)	Let N be the foot of perpendicular from A on line OB. $\overrightarrow{ON}=\lambda \underset{\sim}{b}$ for some $\lambda \in \mathbb{R}$ $\overrightarrow{AN}\cdot \underset{\sim}{b}=0$ $(\lambda \underset{\sim}{b}-\underset{\sim}{a})\cdot \underset{\sim}{b}=0$ $\lambda \underset{\sim}{b}\cdot \underset{\sim}{b}-\underset{\sim}{a}\cdot \underset{\sim}{b}=0$ $\lambda=\frac{\underset{\sim}{a}\cdot \underset{\sim}{b}}{\lvert \underset{\sim}{b}\rvert^2}$ $=\underset{\sim}{a}\cdot \underset{\sim}{b}$ since $\lvert \underset{\sim}{b}\rvert=1$ Therefore, $\overrightarrow{ON}=(\underset{\sim}{a}\cdot \underset{\sim}{b})\underset{\sim}{b}$ Note that λ can be negative $\underset{\sim}{b}\cdot \underset{\sim}{b}=\lvert \underset{\sim}{b}\rvert^2$  Since N is the mid-point of AC, $\overrightarrow{ON}=\frac{\overrightarrow{OA}+\overrightarrow{OC}}{2}$ $\underset{\sim}{c}=2\overrightarrow{ON}-\overrightarrow{OA}$ $=2(\underset{\sim}{a}\cdot \underset{\sim}{b})\underset{\sim}{b}-\underset{\sim}{a}$	Use **correct notation:** Vector $\underset{\sim}{b}$ or $\overrightarrow{ON}$ ON without $\rightarrow$ is treated as the length of ON. Use the usual procedure to find **c** 1) first find the position vector of foot of perpendicular from A to OB 2) then use ratio theorem to find the point of reflection of A about OB As 'numbers' are not given, you **need to use vector algebra and notation well.**
	Alternative to find $\overrightarrow{ON}$: $\overrightarrow{ON}$ is the projection vector of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ Since $\underset{\sim}{b}$ is a unit vector, $\overrightarrow{ON}=(\underset{\sim}{a}\cdot \underset{\sim}{b})\underset{\sim}{b}$	Use the alternative method only if you know the direct formula for **projection vector** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ **(not to be confused with length of projection)**. Note this *formula is often poorly understood or wrongly quoted.*
	Note: 1) We cannot assume that point B is the foot of the perpendicular from A to the line OB as it is not stated in the question. In fact B is not the foot of the perpendicular. Instead we have to let N be the foot of the perpendicular and find it as given in the above solution.	

	2) Many students used $\overrightarrow{ON}$ = (length of projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$) $\dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $= \dfrac{\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert}{\lvert\underset{\sim}{b}\rvert}\dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $= \lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert\ \underset{\sim}{b}$ since $\lvert\underset{\sim}{b}\rvert = 1$ $\neq (\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}$ incorrect as $\underset{\sim}{a}\cdot\underset{\sim}{b}$ can be negative Cannot assume that $\overrightarrow{ON}$ is in the same direction as $\overrightarrow{OB}$. As $\overrightarrow{ON}$ can be in the opposite direction to OB, hence it is possible $\underset{\sim}{a}\cdot\underset{\sim}{b} < 0$ and $(\underset{\sim}{a}\cdot\underset{\sim}{b}) \neq \lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert$ and the above method is not complete. The above method works only if angle AOB is acute and $\overrightarrow{ON}$ is strictly in the same direction as $\overrightarrow{OB}$.	
(ii)	From (i) $\underset{\sim}{c} = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{a}$ Rearranging $\underset{\sim}{a} = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{c}$ $\underset{\sim}{c}\times\underset{\sim}{a} = \underset{\sim}{c}\times\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{c}\right]$ $= \underset{\sim}{c}\times\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}\right] - \underset{\sim}{c}\times\underset{\sim}{c}$ $= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\ (\underset{\sim}{c}\times\underset{\sim}{b}) - \underset{\sim}{0}$ ($\underset{\sim}{0}$ is zero vector, not 0 (scalar)) $= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\ (\underset{\sim}{c}\times\underset{\sim}{b})$ (shown) **Alternative** $\underset{\sim}{c}\times\underset{\sim}{a} = \left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{a}\right]\times\underset{\sim}{a}$ from (i) $= \left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}\right]\times\underset{\sim}{a} - \underset{\sim}{a}\times\underset{\sim}{a}$ expand $= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})(\underset{\sim}{b}\times\underset{\sim}{a}) - \underset{\sim}{0}$ ($\underset{\sim}{0}$ is zero vector, not 0 (scalar)) $= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})(\underset{\sim}{b}\times\underset{\sim}{a})$ ----- (1) $2(\underset{\sim}{a}\cdot\underset{\sim}{b})(\underset{\sim}{c}\times\underset{\sim}{b}) = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\left(\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{a}\right]\times\underset{\sim}{b}\right)$ from (i) $= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})\left(\left[2(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{b}\times\underset{\sim}{b}\right] - \underset{\sim}{a}\times\underset{\sim}{b}\right)$ expand $= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})(\underset{\sim}{0} - \underset{\sim}{a}\times\underset{\sim}{b})$ $= 2(\underset{\sim}{a}\cdot\underset{\sim}{b})(\underset{\sim}{b}\times\underset{\sim}{a})$ ----- (2) From (1) & (2), $\underset{\sim}{c}\times\underset{\sim}{a} = 2(\underset{\sim}{a}\cdot\underset{\sim}{b})(\underset{\sim}{c}\times\underset{\sim}{b})$ (shown)	Cross product is a vector: $\underset{\sim}{a}\times\underset{\sim}{a} = \underset{\sim}{0}$ Common mistakes: 1) **Wrongly** concluding that because $\underset{\sim}{c}$ is the reflection of $\underset{\sim}{a}$ about OB $\underset{\sim}{a} = -\underset{\sim}{c}$. As seen in diagram below, $\underset{\sim}{a} \neq -\underset{\sim}{c}$.  2) Wrongly equating vector to area (scalar): $\frac{1}{2}(\underset{\sim}{c}\times\underset{\sim}{a}) \neq$ Area of triangle OAC vector $\neq$ scalar Formula should be $\frac{1}{2}\lvert\underset{\sim}{c}\times\underset{\sim}{a}\rvert$ = Area of triangle OAC

Q3 | **[Solution]** | **Comments**

(i)

Using Pythagoras theorem, $x^2 + h^2 = (2r)^2$

$\Rightarrow\ h = \sqrt{4r^2 - x^2}$

$S = khx^2 = kx^2\sqrt{4r^2 - x^2}$ where $k > 0$, k is a real const

Comments: As this is a "show" question, do not skip steps.

(ii)

$S = k\sqrt{4r^2x^4 - x^6}$

$$\frac{dS}{dx} = \frac{1}{2}k\left(4r^2x^4 - x^6\right)^{-\frac{1}{2}}\left(16r^2x^3 - 6x^5\right)$$

Let $\dfrac{dS}{dx} = \dfrac{k\left(8r^2x^3 - 3x^5\right)}{\sqrt{4r^2x^4 - x^6}} = \dfrac{kx^3\left(8r^2 - 3x^2\right)}{\sqrt{4r^2x^4 - x^6}} = 0$

$x^3\left(8r^2 - 3x^2\right) = 0$

$x = 0$ (rej. since $x > 0$) or $x = \sqrt{\dfrac{8}{3}}\,r$

<u>Alternative method</u>

$S = kx^2\sqrt{4r^2 - x^2}$

$$\frac{dS}{dx} = 2xk\sqrt{4r^2 - x^2} + kx^2\,\frac{1}{2}\left(4r^2 - x^2\right)^{-\frac{1}{2}}(-2x)$$

$$= 2xk\sqrt{4r^2 - x^2} - \frac{kx^3}{\sqrt{4r^2 - x^2}}$$

$$= \frac{2xk\left(4r^2 - x^2\right) - kx^3}{\sqrt{4r^2 - x^2}}$$

$$= \frac{8kxr^2 - 3kx^3}{\sqrt{4r^2 - x^2}} = \frac{kx\left(8r^2 - 3x^2\right)}{\sqrt{4r^2 - x^2}}$$

$x\left(8r^2 - 3x^2\right) = 0$

$x = 0$ (rej. since $x > 0$) or $x = \sqrt{\dfrac{8}{3}}\,r$

x	$\left(\sqrt{\frac{8}{3}}\,r\right)^-$	$\sqrt{\frac{8}{3}}\,r$	$\left(\sqrt{\frac{8}{3}}\,r\right)^+$
$\dfrac{dS}{dx} = \dfrac{kx^3\left(8r^2 - 3x^2\right)}{\sqrt{4r^2x^4 - x^6}}$	**> 0** since $\left(8r^2 - 3x^2\right) > 0$, $x^3 > 0$ and $k > 0$	**0**	**< 0** since $\left(8r^2 - 3x^2\right) < 0$, $x^3 > 0$ and $k > 0$
shape			

Hence S is max i.e the beam is the strongest when the width $x = \sqrt{\dfrac{8}{3}}\,r$

Comments:

Common mistakes:

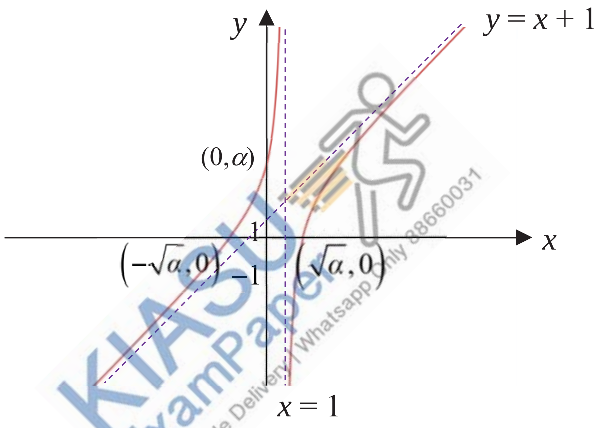
-Forget chain rule

-treat r as a variable instead of constant (see the word particular highlighted in blue)

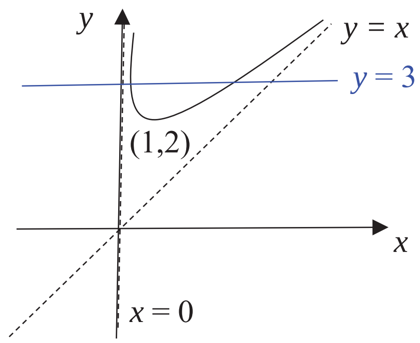
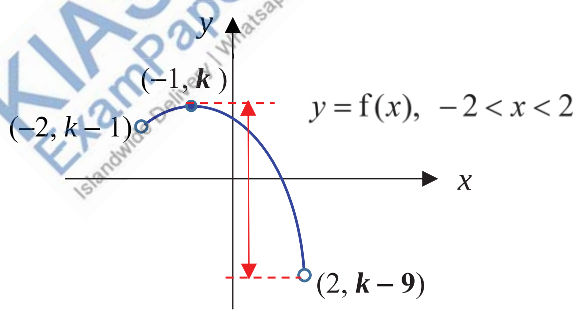
-careless mistakes in algebraic manipulation

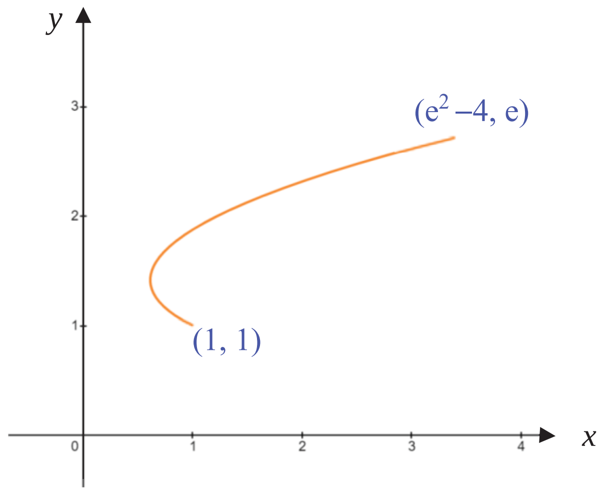
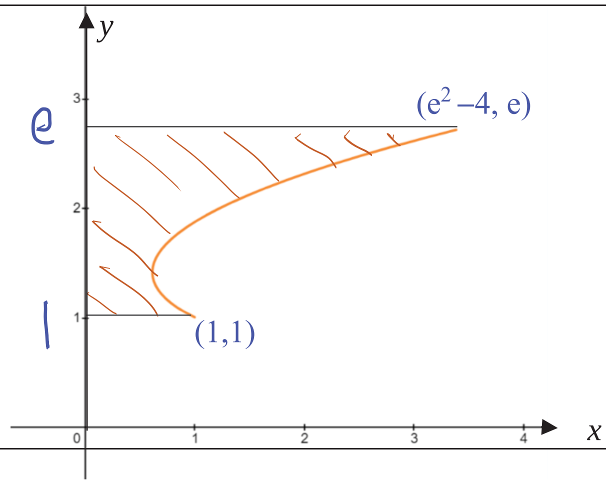
-Not enough explanation is given for using the first derivative test

Q4	[Solution]	Comments
(i)	Using sine rule, $\dfrac{y}{\sin\theta} = \dfrac{x}{\sin\left(\frac{5\pi}{6}-\theta\right)} = \dfrac{1}{\sin\frac{\pi}{6}}$. (Note that $\sin\frac{\pi}{6}=\frac{1}{2}$) $y = 2\sin\theta$ $x = 2\sin\left(\frac{5\pi}{6}-\theta\right)$	Note: Sum of angles in a triangle is π not 2π Use **sine rule** to find x and y, choose appropriate equality: $\dfrac{y}{\sin\theta}=2$ $\dfrac{x}{\sin\left(\frac{5\pi}{6}-\theta\right)}=2$
(ii)	$S=\frac{1}{2}(x)(1)\sin\theta=\frac{1}{2}\left(2\sin\left(\frac{5\pi}{6}-\theta\right)\right)\sin\theta=\sin\left(\frac{5\pi}{6}-\theta\right)\sin\theta$	There is no necessary to further breakdown $\sin\left(\frac{5\pi}{6}-\theta\right)$.
(iii)	At the instant when $\theta=\frac{\pi}{4}$, $\frac{dy}{dt}=-0.2$ $\frac{dy}{d\theta}=2\cos\theta$ Using $\frac{dy}{dt}=\frac{dy}{d\theta}\times\frac{d\theta}{dt}$, $-0.2=2\cos\frac{\pi}{4}\times\frac{d\theta}{dt}$ $\therefore\ \frac{d\theta}{dt}=-\frac{1}{5\sqrt{2}}$ or -0.141 $\frac{dS}{d\theta}=-\sin\theta\cos\left(\frac{5\pi}{6}-\theta\right)+\sin\left(\frac{5\pi}{6}-\theta\right)\cos\theta$ Using $\frac{dS}{dt}=\frac{dS}{d\theta}\times\frac{d\theta}{dt}$, $\frac{dS}{dt}=\left[-\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{7\pi}{12}\right)+\sin\left(\frac{7\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)\right]\times\left(-\frac{1}{5\sqrt{2}}\right)$ $=-0.122$ Thus the rate of change of S is $-0.122\ \text{ms}^{-1}$	Given B is moving towards O $\Rightarrow y$ is decreasing. Therefore, $\frac{dy}{dt}$ is negative *Thinking:* We need to find $\frac{dS}{dt}$ and we have S in terms of θ. Therefore, we need to use $\frac{dS}{dt}=\frac{dS}{d\theta}\times\frac{d\theta}{dt}$ Hence, we must get $\frac{d\theta}{dt}$ first. Since we have $\frac{dy}{dt}$ and $\frac{dy}{d\theta}$, we can use find $\frac{d\theta}{dt}$ from $\frac{dy}{dt}=\frac{dy}{d\theta}\times\frac{d\theta}{dt}$.

Q5	[Solution]	Comments
(i)	$y=\dfrac{\alpha-x^2}{1-x}=\underline{x+1}+\dfrac{\alpha-1}{1-x}$ Equations of asymptotes: $x=1$, $\underline{y=x+1}$	Perform long division till the degree of the remainder is <u>less than</u> the degree of the divisor.
(ii)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+\dfrac{\alpha-1}{(1-x)^2}$ Given C has positive gradient for all $x\in\mathbb{R}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}>0$ $1+\dfrac{\alpha-1}{(1-x)^2}>0$ $\dfrac{(1-x)^2+\alpha-1}{(1-x)^2}>0$ $\dfrac{x^2-2x+\alpha}{(1-x)^2}>0$ Since $(1-x)^2>0$, $x^2-2x+\alpha>0$ for all $x\in\mathbb{R}$, $x\neq 1$ Discriminant = $2^2-4(1)(\alpha)<0$ $\therefore\ \alpha>1$	Students must differentiate y w.r.t. x correctly to obtain the inequality $x^2-2x+\alpha>0$ for all $x\in\mathbb{R}$, $x\neq 1$ Remember the objective is to find the range of value of α, not x.
(iii)		Note: Since the question requested for 'the **coordinates** of the axial intercepts', it is required to labelled the axial intercepts with the correct coordinates form. Also it must be shown that the graph approaches to the asymptotes appropriately.

Q6	[Solution]	Comments
(a)	$\int \sin px \sin qx \, dx$ $= -\frac{1}{2}\int \cos(p+q)x - \cos(p-q)x \, dx$ $= -\frac{1}{2}\left[\frac{\sin(p+q)x}{p+q} - \frac{\sin(p-q)x}{p-q}\right] + c$	• Use **Factor Formula** $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$ • Be familiar with integration techniques $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c$
(b)	$\int_{-\ln 2}^{0} \left(\frac{e^{2x}}{e^{2x}+1}\right) dx = \frac{1}{2}\int_{-\ln 2}^{0}\left(\frac{2e^{2x}}{e^{2x}+1}\right) dx$ $= \frac{1}{2}\left[\ln\left(e^{2x}+1\right)\right]_{-\ln 2}^{0}$ $= \frac{1}{2}\left[\ln\left(e^{0}+1\right) - \left(e^{-2\ln 2}+1\right)\right]$ $= \frac{1}{2}\left(\ln 2 - \ln\left(\frac{1}{4}+1\right)\right)$ $= \frac{1}{2}\ln\frac{8}{5}$	$\frac{d}{dx}\left(e^{2x}+1\right) = 2e^{2x}$ Use $\int \frac{f'(x)}{f(x)}\, dx = \ln\lvert f(x)\rvert + c$ $\ln\lvert e^{2x}+1\rvert = \ln\left(e^{2x}+1\right)$ since $e^{2x}+1>0$ Use result $e^{\ln x} = x$ $\therefore\ e^{-2\ln 2} = e^{\ln\left(2^{-2}\right)} = 2^{-2} = \frac{1}{4}$
(c)	$x-1 = A(1-2x)+B = -2Ax+(A+B)$ Equating coefficients, $-2A = 1 \Rightarrow A = -\frac{1}{2}$ $A+B = -1 \Rightarrow B = -1-A = -\frac{1}{2}$ $\therefore\ x-1 = -\frac{1}{2}(1-2x) - \frac{1}{2}$ $\int \frac{x-1}{\sqrt{6+x-x^2}}\, dx$ $= \int \frac{-\frac{1}{2}(1-2x)-\frac{1}{2}}{\sqrt{6+x-x^2}}\, dx$ $= -\frac{1}{2}\int (1-2x)\left(6+x-x^2\right)^{-\frac{1}{2}} dx - \frac{1}{2}\int \frac{1}{\sqrt{6+x-x^2}}\, dx$ $= -\frac{1}{2}\cdot\frac{\left(6+x-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}\, dx$ $= -\sqrt{6+x-x^2} - \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{5}\right) + C$	Split into 2 integrals **using the suggested form** $\int f'(x)\left[f(x)\right]^n\, dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$ Note that $\sqrt{6+x-x^2} = \sqrt{-1}\sqrt{x^2-x-6}$ **BUT** $\sqrt{-1}$ **is undefined**. $\int \frac{1}{\sqrt{a^2-(px+q)^2}}\, dx = \frac{1}{p}\sin\left(\frac{px+q}{a}\right) + c$

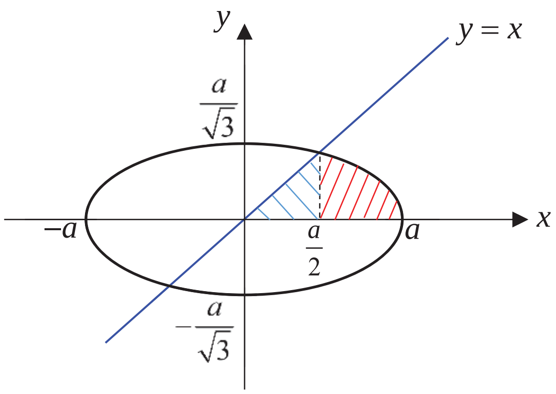
Q7	[Solution]	Comments
(a)(i)	 Since the **horizontal line** $y = 3$ cuts the **graph of** $y = \mathrm{g}(x)$ twice, **g is not one-one**. Thus g does not have an inverse.	Your sketch should indicate the coordinates of the min point (1, 2). Give **proper argument**! Give **equation of a particular line** eg, $y = 3$ to show that g is not 1-1.
(a)(ii)	For g to be 1-1, $x \geq 1$, **Minimum value** of p is 1. Let $y = x + \frac{1}{x}$ $x^2 - yx + 1 = 0$ $x = \frac{y \pm \sqrt{y^2 - 4}}{2}$ Since $x \geq 1$, $x = \frac{y + \sqrt{y^2 - 4}}{2}$ $\mathrm{g}^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$, $\mathrm{D}_{\mathrm{g}^{-1}} = \mathrm{R}_\mathrm{g} = [2, \infty)$	Need to answer to the question on what is the min value of p. Use **quadratic formula** to express x in terms of y. Always **state the reason** why we rejected one of the answers.
(b)	For hf to exist, $\mathrm{R}_\mathrm{f} \subseteq \mathrm{D}_\mathrm{h}$.  $\mathrm{R}_\mathrm{f} = (k - 9, k]$ $\mathrm{D}_\mathrm{h} = (0, \infty)$ Hence, the **minimum value** of k is 9.	**Sketch the graph of f to obtain the range of f** (for the smallest and largest value of y) in the given domain. Unless the function is strictly increasing or decreasing, the range should not be obtained by substituting the min/max x-values. Need to answer to the question on what is the min value of k.

Q8	[Solution]	Comments
(a)	$\int xe^{\frac{1}{2}x}\,dx = x\left(2e^{\frac{1}{2}x}\right) - \int 2e^{\frac{1}{2}x}\,dx$ $= 2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x} + c$ or $2e^{\frac{1}{2}x}(x-2)+c$	Integration by parts: ensure you know how to decide which term to integrate and which term to differentiate. Be careful of careless mistakes such as 1) writing '+' instead of '–' 2) forgetting constant of integration ' $+c$ '
(b)(i)		Ensure you answer the question by giving the **coordinates** of the **end-points** in **exact form.**
(ii)	$x = e^t - 2t$, $y = e^{\frac{1}{2}t}$ Tangent is parallel to y-axis: $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{1}{2}e^{\frac{1}{2}t}}{e^t-2} = \frac{e^{\frac{1}{2}t}}{2(e^t-2)}$ is undefined $2(e^t-2)=0$ $e^t = 2$ $t = \ln 2$ $\therefore\ x = e^{\ln 2} - 2\ln 2 = 2 - 2\ln 2$	When the tangent is parallel to the y-axis, we have a 'vertical line', i.e. the gradient approaches $\pm\infty$. i.e. put denominator of $\frac{dy}{dx}$ to 0 to solve for t. Do not confuse with tangents parallel to the x-axis where we put $\frac{dy}{dx} = 0$.
(iii)		

	When $y = \mathrm{e}$, $\mathrm{e} = \mathrm{e}^{\frac{1}{2}t} \Rightarrow 1 = \frac{1}{2}t \Rightarrow t = 2$ When $y = 1$, $1 = \mathrm{e}^{\frac{1}{2}t} \Rightarrow \ln 1 = \frac{1}{2}t \Rightarrow t = 0$ Area $= \int_1^{\mathrm{e}} x \, \mathrm{d}y$ $= \int_0^2 \left(\mathrm{e}^t - 2t\right)\left(\frac{1}{2}\mathrm{e}^{\frac{1}{2}t}\right)\mathrm{d}t$ $= \int_0^2 \left(\frac{1}{2}\mathrm{e}^{\frac{3}{2}t} - t\,\mathrm{e}^{\frac{1}{2}t}\right)\mathrm{d}t$ $= \frac{1}{2} \times \frac{2}{3}\left[\mathrm{e}^{\frac{3}{2}t}\right]_0^2 - \left[2\mathrm{e}^{\frac{1}{2}t}(t-2)\right]_0^2$ using **(a)** $= \frac{1}{3}\left(\mathrm{e}^3 - 1\right) - 2\left[0 - (-2)\right]$ $= \frac{1}{3}\left(\mathrm{e}^3 - 13\right)$ units^2	Students to be careful to **use the correct formula** $\int x \, \mathrm{d}y$ or $\int y \, \mathrm{d}x$ as using the wrong formula from the start may mean not being awarded any mark at all. For areas involving parametric equations, use **integration by substitution** to find the integral. There is no need to find the cartesian equation. DO NOT FORGET to change the limits to values of t as well. Students can **make use of the result in (a)** without having to integrate by parts again.

[Solution]	Comments

Q9(i)



Note: the line $y = x$ is for part (ii) only.
Remember to always label all vertices (calculate the coordinates carefully!), and try to draw a symmetrical diagram.

$$x^2+3y^2=a^2 \Rightarrow y=\pm\frac{\sqrt{a^2-x^2}}{\sqrt{3}}$$

(ii) Area $= \frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)+\frac{1}{\sqrt{3}}\int_{\frac{a}{2}}^{a}\sqrt{a^2-x^2}\,dx$

$$=\frac{a^2}{8}+\frac{1}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta)\,d\theta$$

$$=\frac{a^2}{8}+\frac{1}{\sqrt{3}}a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=\frac{a^2}{8}+\frac{a^2}{2\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\cos 2\theta+1)\,d\theta$$

$$=\frac{a^2}{8}+\frac{a^2}{2\sqrt{3}}\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=\frac{a^2}{8}+\frac{a^2}{2\sqrt{3}}\left[\frac{\pi}{2}-\left(\frac{\sqrt{3}}{4}+\frac{\pi}{6}\right)\right]$$

$$=\frac{\pi a^2}{6\sqrt{3}}$$

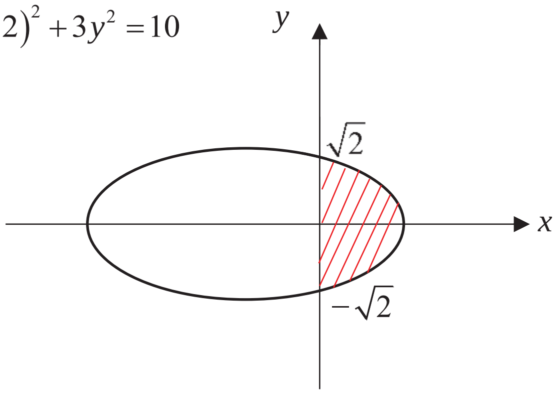
Thus, $k=\frac{1}{6\sqrt{3}}$

At intersection: sub $y = x$ into $x^3 + 3y^2 = a^2 \Rightarrow 4x^2 = a^2$.
Since $x > 0$, $x=\frac{1}{2}a$ and $y=\frac{1}{2}a$
Use the diagram in (i) to see that the first portion on the left is a triangle (use $\frac{1}{2}\times$ base $\times$ height), and to ensure that the limits of integration are correct.

When $x=\frac{1}{2}a$, $\frac{1}{2}=\sin\theta \Rightarrow \theta=\frac{\pi}{6}$
When $x=a$, $1=\sin\theta \Rightarrow \theta=\frac{\pi}{2}$

A lot of careless seen here with the integral of $\cos 2\theta$ and also using +/− signs.

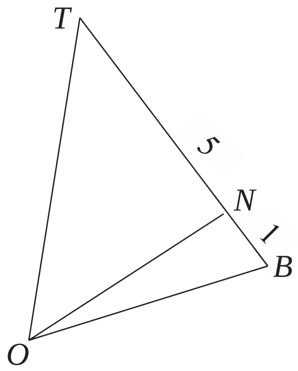
(iii) $(x+2)^2+3y^2=10$



It is given that $a=\sqrt{10}$, so do not continue using a in the equation.

$(x+2)^2+3y^2=10$
$\Rightarrow x=-2\pm\sqrt{10-3y^2}$
Since $x > 0$, $x=-2+\sqrt{10-3y^2}$
When $y = 0$, $x=\pm\sqrt{2}$

Volume of solid formed (= Volume of solid formed when the region between $\left(x+2\right)^2+3y^2=10$ and the <u>*y*-axis</u> is rotated about the <u>*y*-axis</u>) $= \pi \int_{-\sqrt{2}}^{\sqrt{2}} x^2 \mathrm{d}y$ or $2\times\pi \int_{0}^{\sqrt{2}} x^2 \mathrm{d}y$ $= 2\pi \int_{0}^{\sqrt{2}} \left(\left(-2+\sqrt{10-3y^2}\right)^2\right) \mathrm{d}y$ = 6.80 units^3	

Q10	[Solution]	Comments
(i)	$\overrightarrow{OB}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$, $\overrightarrow{OA}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}$ Since $\overrightarrow{AB}$ is in the direction of $\mathbf{i}$, a vector equation of the line AB is $\underset{\sim}{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix},\ \lambda\in\mathbb{R}$	Observe the direction of $\mathbf{i}$ and $\mathbf{j}$ carefully: For example, the y-coordinate of B is negative not positive. Also, $\overrightarrow{AB}$ is in the direction of $\mathbf{i}$ and not $-\mathbf{i}$.
(ii)	$\overrightarrow{ON}=\dfrac{5\overrightarrow{OB}+\overrightarrow{OT}}{6}$ $=\dfrac{1}{6}\left[5\begin{pmatrix}2\\-1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\5\end{pmatrix}\right]$ $=\dfrac{1}{6}\begin{pmatrix}10\\-5\\5\end{pmatrix}$  Since P lies on line AB, $\overrightarrow{OP}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{PN}=\dfrac{1}{6}\begin{pmatrix}10\\-5\\5\end{pmatrix}-\left[\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}\right]=\begin{pmatrix}-\frac{1}{3}\\\frac{1}{6}\\\frac{5}{6}\end{pmatrix}-\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\overrightarrow{TB}=\begin{pmatrix}2\\-1\\0\end{pmatrix}-\begin{pmatrix}0\\0\\5\end{pmatrix}=\begin{pmatrix}2\\-1\\-5\end{pmatrix}$	Students must learn to use **ratio theorem** to find $\overrightarrow{ON}$ as this can help to save a lot of time during exam. DO NOT use $\overrightarrow{OP}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ unless you recognise that in this case the y-coordinate of P is -1 and z-coordinate of P is 0, i.e. $\overrightarrow{OP}=\begin{pmatrix}x\\-1\\0\end{pmatrix}$.

	$\overrightarrow{PN} \perp \overrightarrow{TB} \Rightarrow \overrightarrow{PN} \cdot \overrightarrow{TB} = 0$ $\Rightarrow \left[\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} = 0$ $-5 - 2\lambda = 0 \Rightarrow \lambda = -\frac{5}{2}$ $\overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \frac{5}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$ or $-\frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ **Alternative method:** Since P lies on line AB (y-coordinate is -1) and is on the horizontal plane $ABCD$ (so z-coordinate is 0), we can let $\overrightarrow{OP} = \begin{pmatrix} x \\ -1 \\ 0 \end{pmatrix}$. $\overrightarrow{PN} = \frac{1}{6}\begin{pmatrix} 10 \\ -5 \\ 5 \end{pmatrix} - \begin{pmatrix} x \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} - x \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}$ $\overrightarrow{PN} \perp \overrightarrow{TB} \Rightarrow \overrightarrow{PN} \cdot \overrightarrow{TB} = 0$ $\Rightarrow \begin{pmatrix} \frac{5}{3} - x \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} = 0$ $\frac{10}{3} - 2x - \frac{1}{6} - \frac{25}{6} = 0$ $x = -\frac{1}{2}$, $\therefore \overrightarrow{OP} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$	

(iii)

Since $\overrightarrow{BC} \ // \ \mathbf{j}$,

A normal to plane TBC is $\overrightarrow{BC} \times \overrightarrow{TB} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = 10$$

A Cartesian equation of the plane TBC is $5x + 2z = 10$

Remember to simplify your Cartesian equation (remove common factor).

(iv)

$$l: \ \frac{x+2}{5} = \frac{y}{1} = \frac{z}{k}$$

Note that when $\frac{x+2}{5} = y = \frac{z}{k} = \mu$, we have

$x = -2 + 5\mu$

$y = 0 + 1\mu$

$z = 0 + k\mu$

So direction vector of line l is $\begin{pmatrix} 5 \\ 1 \\ k \end{pmatrix}$. It is NOT $\begin{pmatrix} 5 \\ 0 \\ k \end{pmatrix}$.

You must define the symbol θ or ϕ that you use.

Let θ be the acute angle between l and the horizontal plane with equation $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$.

$$\sin\theta = \frac{\left| \begin{pmatrix} 5 \\ 1 \\ k \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{25+1+k^2}\sqrt{1}} = \frac{k}{\sqrt{26+k^2}}$$

Since sine function is an **increasing function for acute angles,**

$0^\circ < \theta < 45^\circ \ \Rightarrow \ 0 < \sin\theta < \sin 45^\circ$

$$0 < \frac{k}{\sqrt{26+k^2}} < \frac{\sqrt{2}}{2}$$

$$2k < \sqrt{2}\sqrt{26+k^2}$$

$$4k^2 < 52 + 2k^2$$

$$k^2 < \frac{52}{2} = 26$$

$$\left(k - \sqrt{26}\right)\left(k + \sqrt{26}\right) < 0$$

$$-\sqrt{26} < k < \sqrt{26}$$

Since $k > 0$, $\therefore 0 < k < \sqrt{26}$

Note that $k < \pm\sqrt{26}$ is WRONG!

Alternatively method 1

Let ϕ be the acute angle between *l* and the normal **k** to the horizontal plane.

$$\cos\phi = \frac{\left|\begin{pmatrix}5\\1\\k\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{25+1+k^2}\sqrt{1}} = \frac{k}{\sqrt{26+k^2}}$$

Since cosine function is a **decreasing function for acute angles,**

$45^\circ < \phi < 90^\circ \Rightarrow\ 0 < \cos\phi < \cos 45^\circ$

$$0 < \frac{k}{\sqrt{26+k^2}} < \frac{\sqrt{2}}{2}$$

$$2k < \sqrt{2}\sqrt{26+k^2}$$

$$4k^2 < 52 + 2k^2$$

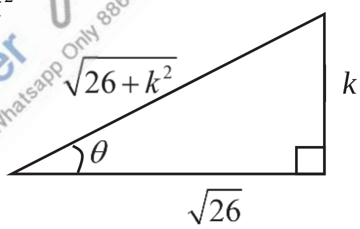
$$k^2 < \frac{52}{2} = 26$$

$$\left(k-\sqrt{26}\right)\left(k+\sqrt{26}\right) < 0$$

Since $k > 0$, $\quad 0 < k < \sqrt{26}$

Alternative Method 2

Since $\sin\theta = \frac{k}{\sqrt{26+k^2}}$, using the right-angled triangle



Since **tangent function is an increasing function for acute angles**.

$$0^\circ < \theta < 45^\circ$$

$$\Rightarrow 0 < \tan\theta < \tan 45^\circ$$

$$\therefore\quad 0 < \frac{k}{\sqrt{26}} < 1$$

$$\Rightarrow 0 < k < \sqrt{26}$$

[Solution]	Comments For Students
Q11(a)(i) n^{th} term, $\boxed{u_n = S_n - S_{n-1}}$ $= \frac{2}{3}\left(1-\frac{1}{k^n}\right) - \frac{2}{3}\left(1-\frac{1}{k^{n-1}}\right)$ $= \frac{2}{3}\left(\frac{1}{k^n}\frac{1}{k^{-1}} - \frac{1}{k^n}\right) = \frac{2}{3}\left(\frac{k}{k^n} - \frac{1}{k^n}\right)$ $= \frac{2(k-1)}{3}\left(\frac{1}{k}\right)^n$ $\frac{u_n}{u_{n-1}} = \frac{\frac{2(k-1)}{3}\left(\frac{1}{k}\right)^n}{\frac{2(k-1)}{3}\left(\frac{1}{k}\right)^{n-1}} = \left(\frac{1}{k}\right)^{n-(n-1)} = \frac{1}{k} \quad (k \neq 0)$ which is a constant independent of *n*. Hence the sequence is geometric with common ratio $r = \frac{1}{k}$ and first term, $u_1 = \frac{2(k-1)}{3}\left(\frac{1}{k}\right)^1 = \frac{2(k-1)}{3k}$ or $u_1 = S_1 = \frac{2}{3}\left(1-\frac{1}{k}\right)$	To prove the existence of a GP, we need to show that the ratio of $\frac{u_n}{u_{n-1}}$ or $\frac{u_{n+1}}{u_n}$ is a constant independent of *n*. This means that the working should start in terms of *n* and ultimately show that ratio obtained does not contain *n*. Showing $\frac{u_2}{u_1} = \frac{u_3}{u_2}$ is insufficient as it merely shows that the first 3 terms are geometric. It is a good practice to simplify where possible e.g. it is more desirable to simplify $r = \frac{k+1}{k^2+k} = \frac{k+1}{k(k+1)} = \frac{1}{k}$ Many candidates missed out the last sub-part of finding u_1. When dealing with a question with many sub-parts that are not explicitly labelled, it is important to tick off a part when the solution to that part has been completed. This will minimize the likelihood of missing out on a sub-part.
(a)(ii) The new series $\frac{1}{u_1} + \frac{1}{u_2} + \ldots + \frac{1}{u_n}$ is geometric with common ratio $= \frac{1}{\frac{1}{k}} = k$ Sum to infinity exists when $\lvert k \rvert < 1,\ k \neq 0$, $\Rightarrow\ -1 < k < 1,\ k \neq 0$ First term $= \frac{1}{u_1} = \frac{3k}{2(k-1)}$	When the reciprocal of the terms of a given GP are used to form another GP in the same successive order, the common ratio will be the reciprocal of the common ratio of the original GP. Hence this can be stated as a deduction directly instead incurring unnecessary working to find the common ratio. In addition, since the common ratio is derived from the original common ratio, it must obey all conditions (both explicit and inherent) of the original ratio.

Sum to infinity, $S_\infty = \dfrac{\dfrac{3k}{2(k-1)}}{1-k}$ $= \dfrac{3k}{2(k-1)} \times \dfrac{1}{1-k}$ $= -\dfrac{3k}{2(1-k)^2}$	
(b)(i) $S_n = \frac{1}{3}\left(S_{2n} - S_n\right)$ $4S_n = S_{2n}$ $4 \times \frac{n}{2}\left(2a + (n-1)d\right) = \frac{2n}{2}\left(2a + (2n-1)d\right)$ $4na + 2n^2 d - 2nd = 2na + 2n^2 d - nd$ $2na = nd$ $2a = d$ (shown)	Simplify $S_n = \frac{1}{3}\left(S_{2n} - S_n\right)$ to $4S_n = S_{2n}$ will make subsequent working easier.
(b)(ii) Subst $d = 2a$ into S_n: $\frac{n}{2}\left(2a + (n-1)2a\right) = 98$ $n^2 a = 98$ Since a and n are positive integers, $a = 2,\ n = 7$	By guess and check

2021 TMJC H2 Maths Promo Paper **Duration: 3 hrs** **Marks: 100**
Attempt all questions.

1 A local tour agent brought 4 groups of 5 tourists each to a durian shop for a durian tasting tour. He paid for the durians ordered by each group *A* - *D* as shown in the following table:

	A	*B*	*C*	*D*
Type of durian	**Weight of durians**			
Mao Shan Wang (in kg)	4.5	3	6	7.5
Red Prawn (in kg)	4	4.5	3	3
D24 (in kg)	4	2	3	6
Total amount paid (in \$)	218.50	162.50	219	*k*

(i) The tour agent could not remember the price per kilogram for each type of durian and he lost the receipt for group *D*. Find *k*, the amount paid for group *D*. [4]

The tour agent found a new durian shop which offers a durian buffet. Durian lovers can feast on as many durians (Mao Shan Wang, Red Prawn, and D24 durians) as they want within 90 minutes. The buffet is priced at \$50 per person.

(ii) Give a reason why the tour agent should not bring future tourist groups to the new durian shop for durian tasting. [1]

2 Find, by differentiation, the coordinates of the stationary point of the curve

$$y = 3x^2 - k^2 \ln\left(\frac{x}{4}\right),$$

where $x > 0$ and k is a positive constant. Hence determine the nature of the stationary point. [6]

3 **(i)** Without using a calculator, find the exact solution set for the inequality

$$\frac{7}{x^2 - 2x - 6} \geq -1.$$ [4]

(ii) Hence solve the inequality $\dfrac{7}{x^2 - 2|x| - 6} \geq -1$. [3]

4 **(i)** Expand $\dfrac{\sqrt{1-3x}}{2+4x}$ in ascending powers of x, up to and including the term in x^2. [4]

(ii) Find the range of values of x for which the expansion is valid. [2]

(iii) By putting $x = -\dfrac{1}{4}$ into your result in part **(i)**, show that $\sqrt{7} \approx \dfrac{p}{q}$, where p and q are integers to be determined. [2]

5 Given that $a > 0$, functions f and g are defined by

$$\mathrm{f} : x \mapsto x + 2 + \frac{4}{x-1} \qquad \text{for } x \in \mathbb{R}, x \neq 1,$$

$$\mathrm{g} : x \mapsto \ln(x + a) \qquad \text{for } x \in \mathbb{R}, x > 0.$$

(i) Explain clearly why gf does not exist. [2]

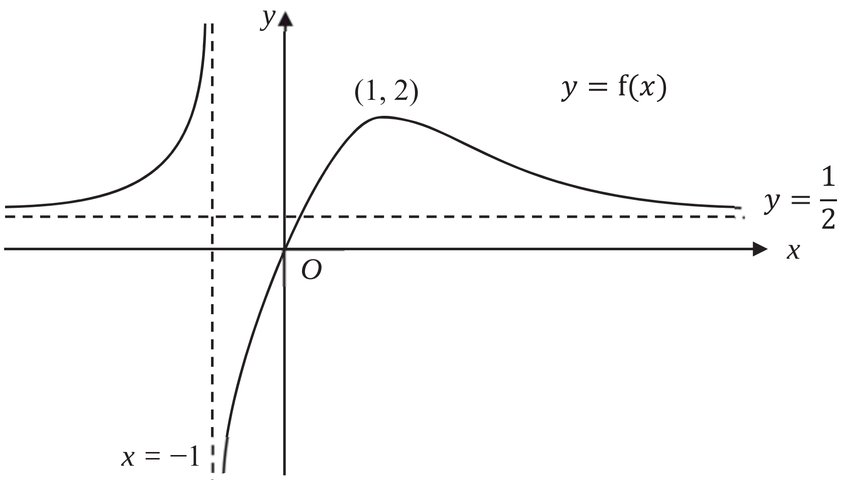
(ii) Find the range of values of a, in exact form, such that fg exists. [2]

For the rest of the question, assume that the function fg exists.

(iii) Define fg in a similar form, in terms of a. [2]

(iv) Given that $a > \mathrm{e}^3$, find the range of fg, in terms of a. [2]

6



The diagram shows the graph of $y = \mathrm{f}(x)$. The curve has an axial intercept at $(0,0)$, a turning point at $(1,2)$ and asymptotes $x = -1$ and $y = \frac{1}{2}$.

Sketch, on separate clearly labelled diagrams, the graphs of

(i) $y = \mathrm{f}(|x| - 1)$, [3]

(ii) $y = \mathrm{f}'(x)$, [3]

(iii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

giving the equations of any asymptotes and coordinates of any x-intercepts and turning points, where applicable.

7 Given that $y = \mathrm{f}(x) = \ln(1 + \sin x)$, show that $\mathrm{e}^y \left[\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x} \right)^2 \right] = -\sin(x)$. [2]

(i) By further differentiation of this result, find the Maclaurin series of y, up to and including the term in x^3. [4]

(ii) Let the answer in part **(i)** be g(x). Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ on the same diagram, where $0 < x < 3$. Hence or otherwise, explain why the approximation of $\mathrm{f}(x)$ using $\mathrm{g}(x)$ is not accurate when $x = 2$. [4]

8 A curve C has parametric equations

$$x=\sqrt{2}\sin\theta-1,\quad y=2+3\cos\theta,\quad \text{for } -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$$

(i) Using differentiation, find the coordinates of the point(s) on the curve at which the tangent is parallel to the y-axis. [7]

(ii) Sketch C, showing clearly the features of the curve at the points where $\theta=-\frac{\pi}{2}$ and $\theta=\frac{\pi}{2}$ and coordinates of any turning point(s). [3]

(iii) Find the cartesian equation of C in the form $y=\mathrm{f}(x)$. [3]

9 The plane π_1 contains the point A with coordinates $(1,0,3)$ and the line l_1 with equation $x+1=\frac{z-2}{2}, y=2.$

(i) Show that the cartesian equation of π_1 is $-4x-3y+2z=2$. [2]

The line l_2 passes through the point B with coordinates $(5,2,8)$ and is parallel to $2\mathbf{i}+\mathbf{k}$.

(ii) Find the position vector of the point of intersection between π_1 and l_2. [3]

(iii) Find the position vector of the foot of the perpendicular from B to π_1. [3]

The plane π_2 has equation $\mathbf{r}\bullet\begin{pmatrix}8\\6\\-4\end{pmatrix}=-62$.

(iv) Find the exact distance between π_1 and π_2. [2]

(v) Determine whether the origin is in between the two planes π_1 and π_2. [2]

10 Referred to origin O, the points P, Q, R and S have position vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ respectively.

(i) Given that $\alpha\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$ and $\alpha+\beta+1=0$, where α and β are non-zero constants. Show that P, Q and R are collinear. [3]

It is given that the point R lies on PQ produced. The point T lies on line RS produced such that $RT:ST=3:2$.

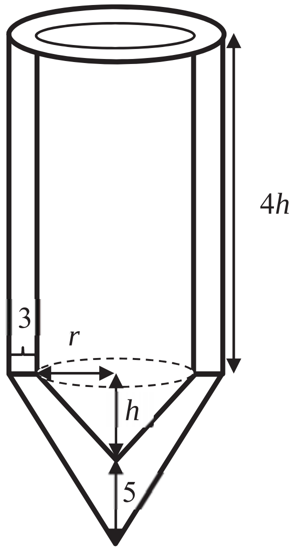
(ii) Find the position vector of the point T in terms of $\mathbf{r}$ and $\mathbf{s}$. [2]

(iii) Give a geometrical interpretation of $\frac{|(\mathbf{p}-\mathbf{r})\cdot(\mathbf{s}-\mathbf{r})|}{|\mathbf{s}-\mathbf{r}|}$. [1]

(iv) Find the area of triangle PRT in the form $\gamma|\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}|$, where γ is a constant to be determined. [3]

[Turn Over

11



The diagram shows a storage well with fixed capacity k m^3. The storage well is made up of two parts, a right circular cone and a cylinder. The right circular cone has radius r m and height h m. The cylinder has radius r m and height $4h$ m. The walls of the storage well are made of a special material. The walls of the cylindrical part of the storage well are 3 m thick. The height from the vertex of the inner wall to the outer wall of the right circular cone is 5 m. The volume of the special material used to make the storage well is V m^3.

(i) Show that the volume of the special material used to make the storage well is given by $V = (r+3)^2\left(\frac{k}{r^2}\right) + \frac{5}{3}\pi(r+3)^2 - k$. [2]

[It is given that the volume of a right circular cone with radius r and height h is $\frac{1}{3}\pi r^2 h$.]

(ii) Use differentiation to find the exact value of r where V is minimum. (You need not show that the volume is a minimum.) [4]

(iii) It is given that the capacity of the storage well is 700 m^3 and $2 < r \le 10$. Sketch the graph showing V as r varies. [2]

(iv) It is given instead that $h = 10$ and $r = 5$. Liquid is poured into the storage well at a constant rate of 0.1 m^3 per minute. The depth of the liquid in the storage well at time t minutes, is x m. Find the rate of change of the depth of the liquid in the storage well when the volume of liquid in the storage well is 200 m^3. [5]

End of Paper

2021 TMJC H2 Maths Promo Paper Solution

Qn	Solution
1	**System of Linear Equations**
(i)	Let x, y and z be the price per kg for Mao Shan Wang, Red Prawn and D24 durians respectively. $4.5x+4y+4z=218.5$ $3x+4.5y+2z=162.5$ $6x+3y+3z=219$ Using GC, $x=21,\ y=15,\ z=16$ The total amount paid for Group D is 7.5(21) + 3(15) + 6(16) = \$298.50. Hence k = 298.50
(ii)	Possible answers: • The average expenditure for each tourist is \$44.93, which is less than that at the new durian shop. • The total expenditure for the four groups is \$898.50. Total expenditure will be higher at \$1000 if he engages the new durian shop. • The average cost for each group is \$224.63, while the cost for the new durian shop is higher at \$250 per group.

Qn	Solution
2	**Differentiation**
	$y=3x^2-k^2\ln\left(\frac{x}{4}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=6x-k^2\left(\frac{4}{x}\right)\left(\frac{1}{4}\right)=6x-\frac{k^2}{x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $6x-\frac{k^2}{x}=0$ $6x=\frac{k^2}{x}$ $x^2=\frac{k^2}{6}$ $x=\pm\frac{k}{\sqrt{6}}\ (\text{since } k>0)$ $x=\frac{k}{\sqrt{6}}\ (\text{since } x>0)$ $y=3\left(\frac{k}{\sqrt{6}}\right)^2-k^2\ln\left(\frac{\frac{k}{\sqrt{6}}}{4}\right)=\frac{k^2}{2}-k^2\ln\left(\frac{k}{4\sqrt{6}}\right)=k^2\left(\frac{1}{2}-\ln\left(\frac{\sqrt{6}k}{24}\right)\right)$ Coordinates of stationary point is $\left(\frac{\sqrt{6}k}{6},k^2\left(\frac{1}{2}-\ln\sqrt{6}k+\ln 24\right)\right)$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6+\frac{k^2}{x^2}>0$ since $\frac{k^2}{x^2}>0$ Hence the stationary point is a minimum point.

Qn	Solution
3	**Equations and Inequalities**

(i)

$$\frac{7}{x^2-2x-6} \geq -1$$

$$\frac{7}{x^2-2x-6}+1 \geq 0$$

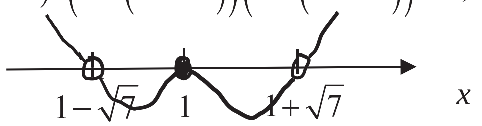
$$\frac{7+x^2-2x-6}{x^2-2x-6} \geq 0$$

$$\frac{x^2-2x+1}{x^2-2x-6} \geq 0$$

$$\frac{(x-1)^2}{(x-1)^2-7} \geq 0$$

$$\frac{(x-1)^2}{\left(x-\left(1-\sqrt{7}\right)\right)\left(x-\left(1+\sqrt{7}\right)\right)} \geq 0$$

$$(x-1)^2\left(x-\left(1-\sqrt{7}\right)\right)\left(x-\left(1+\sqrt{7}\right)\right) \geq 0, \quad x \neq 1-\sqrt{7}, 1+\sqrt{7}$$



$$\left\{x \in \mathbb{R}: \ x<1-\sqrt{7} \quad \text{or} \quad x=1 \quad \text{or} \quad x>1+\sqrt{7}\right\}$$

(ii) Replacing x by $|x|$,

$|x|<1-\sqrt{7}$ or $|x|=1$ or $|x|>1+\sqrt{7}$

(NA since $|x| \geq 0$) $\Rightarrow x=\pm 1$ $\Rightarrow x>1+\sqrt{7}$ or $x<-1-\sqrt{7}$

So, the solution is $x=\pm 1$ or $x>1+\sqrt{7}$ or $x<-1-\sqrt{7}$

Qn	Solution
4	**Integration Techniques**

(i)

$$\frac{\sqrt{1-3x}}{2+4x}=(1-3x)^{\frac{1}{2}}\left[2^{-1}(1+2x)^{-1}\right]$$

$$\therefore (1-3x)^{\frac{1}{2}}\left[2^{-1}(1+2x)^{-1}\right]$$

$$=\frac{1}{2}\left[1+\frac{1}{2}(-3x)-\frac{1}{8}(-3x)^2+\cdots\right]\left[1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2+\cdots\right]$$

$$=\frac{1}{2}\left(1-\frac{3}{2}x-\frac{9}{8}x^2+\cdots\right)\left(1-2x+4x^2+...\right)$$

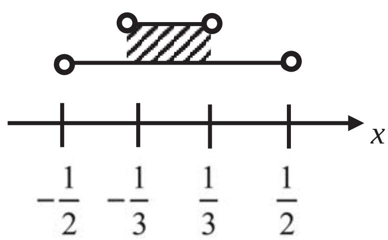
$$=\frac{1}{2}\left(1-\frac{7x}{2}+\frac{47}{8}x^2+\cdots\right)=\frac{1}{2}-\frac{7}{4}x+\frac{47}{16}x^2+\cdots$$

(ii) For expansion to be valid,

$|-3x|<1$ and $|2x|<1$

$-\frac{1}{3}<x<\frac{1}{3}$ and $-\frac{1}{2}<x<\frac{1}{2}$

$\therefore -\frac{1}{3}<x<\frac{1}{3}$



(iii)

Since $\dfrac{\sqrt{1-3x}}{2+4x} \simeq \dfrac{1}{2} - \dfrac{7}{4}x + \dfrac{47}{16}x^2$,

Let $x = -\dfrac{1}{4}$, $\quad \dfrac{\sqrt{1-3\left(-\frac{1}{4}\right)}}{2+4\left(-\frac{1}{4}\right)} = \dfrac{1}{2} - \dfrac{7}{4}\left(-\dfrac{1}{4}\right) + \dfrac{47}{16}\left(-\dfrac{1}{4}\right)^2 + \cdots$

$\sqrt{\dfrac{7}{4}} = \dfrac{1}{2} - \dfrac{7}{4}\left(-\dfrac{1}{4}\right) + \dfrac{47}{16}\left(-\dfrac{1}{4}\right)^2 + \cdots \approx \dfrac{287}{256}$

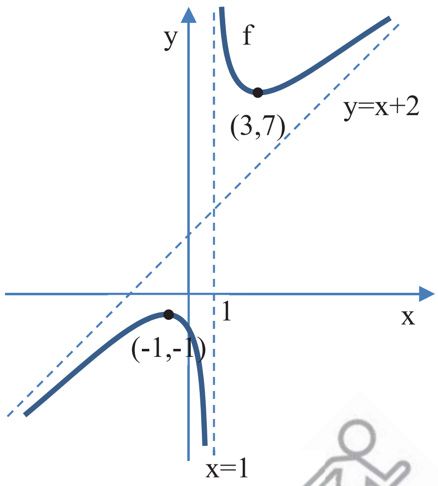
$\sqrt{7} \approx \dfrac{287}{128}$

$p = 287, \quad q = 128$

Qn	Solution
5	**Functions**

(i)

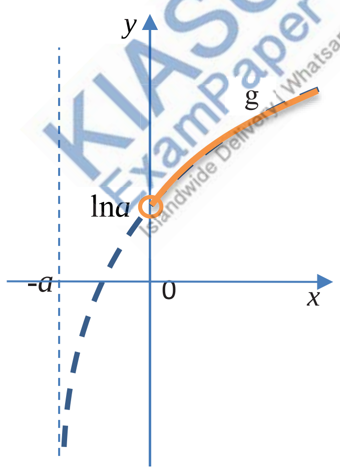
$\mathrm{f}(x) = x + 2 + \dfrac{4}{x-1}$



$R_f = (-\infty,-1] \cup [7,\infty) \not\subseteq D_g = (0,\infty)$.

$\therefore$ gf does not exist regardless of the value of a.

(ii)



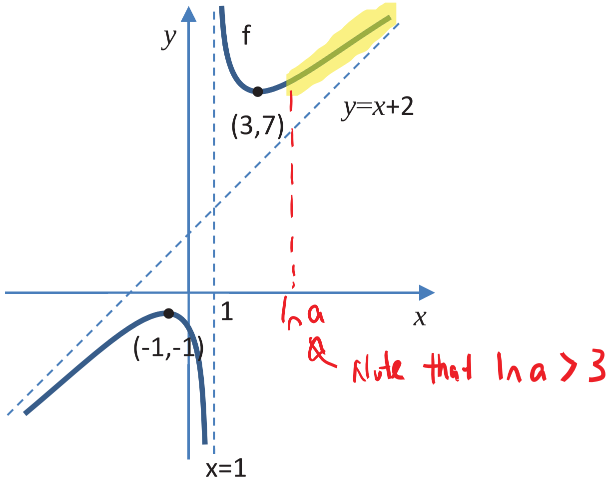
For fg to exist, $R_g \subseteq D_f$.

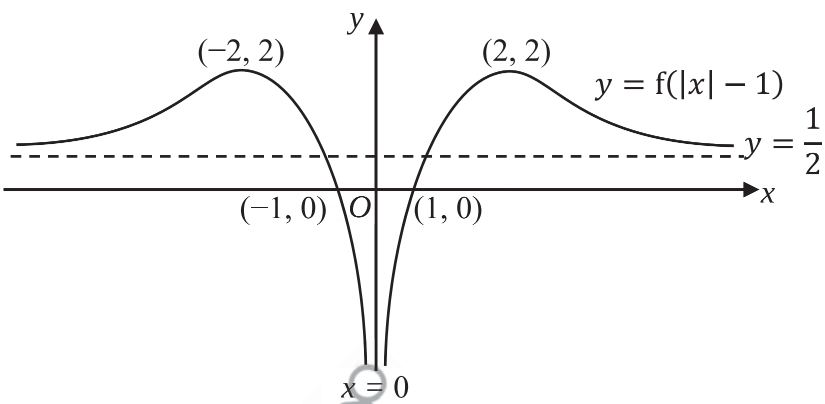
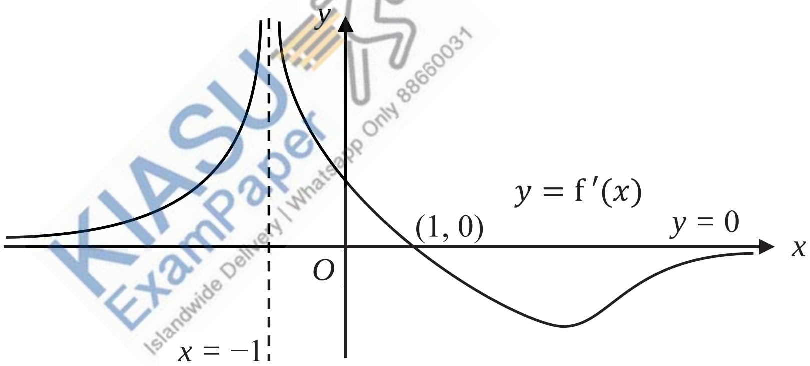
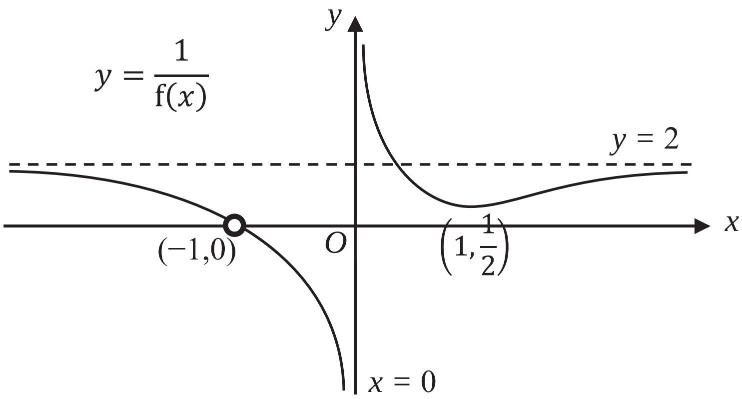
$R_g = (\ln a, \infty)$ and $D_f = (-\infty,1) \cup (1,\infty)$.

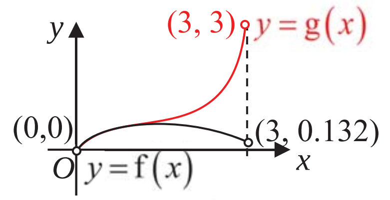
$\therefore$ fg exists if $\ln a \geq 1 \Rightarrow a \geq \mathrm{e}$

(iii)

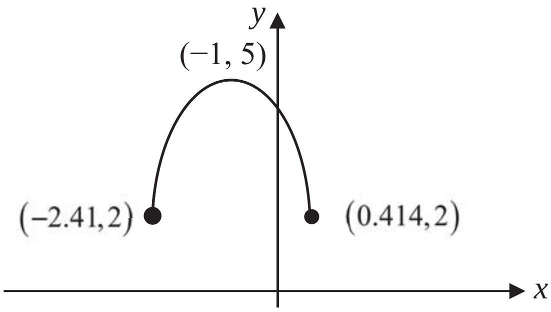
$\mathrm{fg} : x \mapsto \ln(x+a) + 2 + \dfrac{4}{\ln(x+a) - 1} \qquad$ for $x \in \mathbb{R}, x > 0$

(iv)	If $a > e^3$, $\ln a > 3$, $D_{fg} = D_g \to R_g = (\ln a, \infty) \to R_{fg} = \left(\ln a + 2 + \dfrac{4}{\ln a - 1}, \infty\right)$ 

Qn	Solution
6	**Transformation of Graphs**
(i)	
(ii)	
(iii)	

Qn	Solution
7	**Maclaurin Series**
	$y=\ln(1+\sin(x)), \Rightarrow e^y = 1+\sin(x)$ Differentiate with respect to x: $e^y \frac{dy}{dx} = \cos(x)$ Differentiate with respect to x: $\frac{d^2y}{dx^2}e^y + \frac{dy}{dx}e^y\frac{dy}{dx} = -\sin(x)$ $e^y\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] = -\sin(x)$ (shown)
(i)	Differentiate with respect to x: $e^y\left[\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)\right]+e^y\frac{dy}{dx}\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] = -\cos(x)$ $e^y\left[\frac{d^3y}{dx^3}+3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^3\right] = -\cos(x)$ When $x=0,\ y=0,\ \frac{dy}{dx}=1,\ \frac{d^2y}{dx^2}=-1,\ \frac{d^3y}{dx^3}=1$ Maclaurin series for y is $y \approx x - \frac{1}{2}x^2 + \frac{1}{6}x^3$
(ii)	Using GC,  The graphs of $y=g(x)=x-\frac{1}{2}x^2+\frac{1}{6}x^3$ and $y=f(x)=\ln(1+\sin x)$ **deviate/differ significantly at *x* = 2**, where $g(2)=\frac{4}{3}\approx 1.33$ and $f(2)=0.647$. Hence the approximation is not accurate. **<u>OR</u>** The value of $x=2$ is **not close to zero**, hence the approximation is not very good.

Qn	Solution
8	**Parametric Equations (Cross topical with Differentiation)**
(i)	$x=\sqrt{2}\sin\theta-1,\ \ y=2+3\cos\theta$ $\frac{dx}{d\theta}=\sqrt{2}\cos\theta,\ \ \frac{dy}{d\theta}=-3\sin\theta$ $\frac{dy}{dx}=\frac{-3\sin\theta}{\sqrt{2}\cos\theta}$ For tangent to be parallel to *y*-axis, $\sqrt{2}\cos\theta=0$ Since $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ \ \theta=-\frac{\pi}{2}$ or $\frac{\pi}{2}$ At $\theta=\frac{\pi}{2},\ x=\sqrt{2}\sin\left(\frac{\pi}{2}\right)-1=\sqrt{2}-1,\ \ y=2+3\cos\left(\frac{\pi}{2}\right)=2$ Coordinates of point is $(\sqrt{2}-1, 2)$ or (-2.41,2).

<table>
<tr><td></td><td>At $\theta=-\frac{\pi}{2}$, $x=\sqrt{2}\sin\left(-\frac{\pi}{2}\right)-1=-\sqrt{2}-1$, $y=2+3\cos\left(-\frac{\pi}{2}\right)=2$
Coordinates of point is $(-\sqrt{2}-1,2)$ or (0.414,2).</td></tr>
<tr><td>(ii)</td><td>
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<tr><td>(iii)</td><td>

$x=\sqrt{2}\sin\theta-1 \qquad y=2+3\cos\theta$

$\sin\theta=\frac{x+1}{\sqrt{2}} \qquad \cos\theta=\frac{y-2}{3}$

Since $\sin^2\theta+\cos^2\theta=1$,

$\left(\frac{x+1}{\sqrt{2}}\right)^2+\left(\frac{y-2}{3}\right)^2=1$

$\frac{(x+1)^2}{2}+\frac{(y-2)^2}{9}=1$

$(y-2)^2=9-\frac{9(x+1)^2}{2}$

$y=2\pm\sqrt{9-\frac{9(x+1)^2}{2}}$

Since $y\geq 2$, $\quad y=2+\sqrt{9-\frac{9(x+1)^2}{2}}$

</td></tr>
</table>

Qn	**Solution**
9	**Vectors**
(i)	$l_1:\mathbf{r}=\begin{pmatrix}-1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\2\end{pmatrix},\lambda\in\mathbb{R}$ $\mathbf{d}_2=\begin{pmatrix}1\\0\\3\end{pmatrix}-\begin{pmatrix}-1\\2\\2\end{pmatrix}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$ $\mathbf{n}=\begin{pmatrix}2\\-2\\1\end{pmatrix}\times\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-4\\-3\\2\end{pmatrix}$

	$\mathbf{r}.\begin{pmatrix}-4\\-3\\2\end{pmatrix}=\begin{pmatrix}1\\0\\3\end{pmatrix}.\begin{pmatrix}-4\\-3\\2\end{pmatrix}$ $\Rightarrow \pi_1:\ \mathbf{r}.\begin{pmatrix}-4\\-3\\2\end{pmatrix}=2$ $-4x-3y+2z=2$
(ii)	$l_2:\mathbf{r}=\begin{pmatrix}5\\2\\8\end{pmatrix}+\mu\begin{pmatrix}2\\0\\1\end{pmatrix},\mu\in\mathbb{R}$ Let X be the point of intersection. Since X is on l_2 , $\overrightarrow{OX}=\begin{pmatrix}5+2\mu\\2\\8+\mu\end{pmatrix}$ for some $\mu\in\mathbb{R}$. Since X is on π_1 , $\begin{pmatrix}5+2\mu\\2\\8+\mu\end{pmatrix}.\begin{pmatrix}-4\\-3\\2\end{pmatrix}=2$ $-20-8\mu-6+16+2\mu=2$ $\therefore \mu=-2$ $\overrightarrow{OX}=\begin{pmatrix}5+2(-2)\\2\\8+(-2)\end{pmatrix}=\begin{pmatrix}1\\2\\6\end{pmatrix}$
(iii)	Let F be the foot of perpendicular from B to the plane π_1. $l_{BF}:\mathbf{r}=\begin{pmatrix}5\\2\\8\end{pmatrix}+\alpha\begin{pmatrix}-4\\-3\\2\end{pmatrix},\alpha\in\mathbb{R}$ Sub l_{BF} into eqn of π_1 , $\begin{pmatrix}5-4\alpha\\2-3\alpha\\8+2\alpha\end{pmatrix}.\begin{pmatrix}-4\\-3\\2\end{pmatrix}=2$ $29\alpha=12$ $\therefore \alpha=\frac{12}{29}$ $\overrightarrow{OF}=\begin{pmatrix}5\\2\\8\end{pmatrix}+\frac{12}{29}\begin{pmatrix}-4\\-3\\2\end{pmatrix}$ $=\frac{1}{29}\begin{pmatrix}97\\22\\256\end{pmatrix}$
(iv)	**Method 1:**

$$\pi_1 : \mathbf{r}.\begin{pmatrix}-4\\-3\\2\end{pmatrix} = 2 \Rightarrow \mathbf{r}.\hat{\mathbf{n}} = \frac{2}{\sqrt{(-4)^2+(-3)^2+2^2}}$$

$$= \frac{2}{\sqrt{29}}$$

$$\pi_2 : \mathbf{r}.\begin{pmatrix}8\\6\\-4\end{pmatrix} = -62 \Rightarrow \mathbf{r}.\begin{pmatrix}-4\\-3\\2\end{pmatrix} = 31$$

$$\mathbf{r}.\hat{\mathbf{n}} = \frac{31}{\sqrt{29}}$$

$$\text{Dist between the two planes} = \left|\frac{31}{\sqrt{29}} - \frac{2}{\sqrt{29}}\right|$$

$$= \frac{29}{\sqrt{29}}$$

$$= \sqrt{29}$$

Method 2 (length of projection onto its normal):

Let $P(0,0,1)$ be a point on π_1 and $Q\left(0,0,\frac{31}{2}\right)$ be a point on π_2 .

$$\text{Dist between the two planes} = \left|\overrightarrow{PQ}.\hat{\mathbf{n}}\right|$$

$$= \frac{\left|\begin{pmatrix}0\\0\\\frac{31}{2}-1\end{pmatrix}.\begin{pmatrix}-4\\-3\\2\end{pmatrix}\right|}{\sqrt{29}}$$

$$= \frac{29}{\sqrt{29}}$$

$$= \sqrt{29}$$

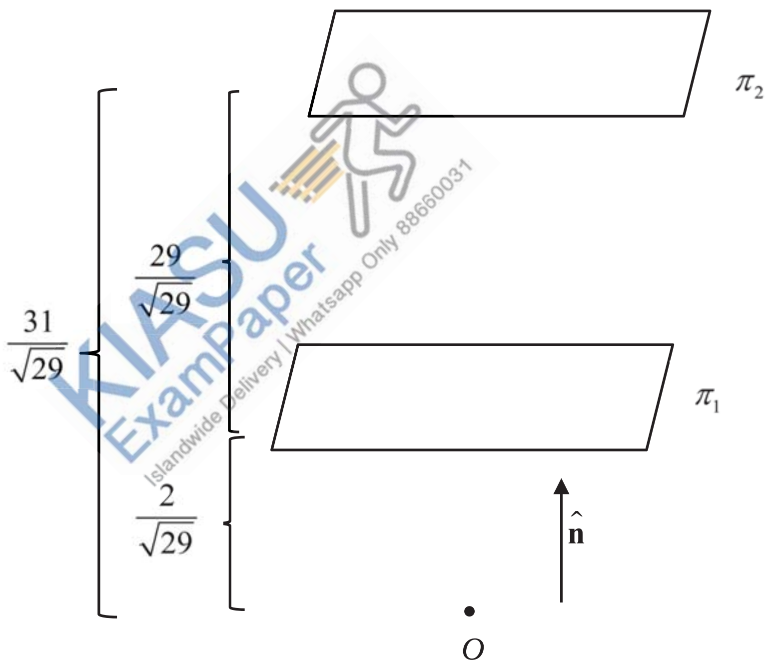
(v)

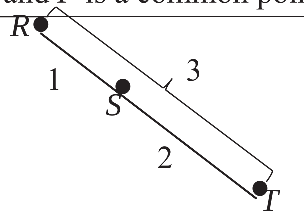
$$\pi_1 : \mathbf{r}.\hat{\mathbf{n}} = \frac{2}{\sqrt{29}} > 0$$

$$\pi_2 : \mathbf{r}.\hat{\mathbf{n}} = \frac{31}{\sqrt{29}} > 0$$

Since $\frac{2}{\sqrt{29}}$ and $\frac{31}{\sqrt{29}}$ are both positive, therefore origin is not between the two planes.

Diagram for understanding



Qn	Solution
10	**Vector Algebra**
(i)	$\alpha+\beta+1=0 \Rightarrow \alpha=-\beta-1$ Sub into $\alpha\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$, ← This is $\underset{\sim}{0}$ (a vector). Important to use the correct notation. $(-\beta-1)\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$ $\beta(\mathbf{q}-\mathbf{p})+(\mathbf{r}-\mathbf{p})=\mathbf{0}$ $\beta\overrightarrow{PQ}+\overrightarrow{PR}=\mathbf{0}$ $\overrightarrow{PQ}=k\overrightarrow{PR},\quad k=-\frac{1}{\beta},\ \beta\neq 0$ Since $\overrightarrow{PQ}$ is parallel to $\overrightarrow{PR}$, and P is a common point, P, Q and R are collinear.
(ii)	$\overrightarrow{OS}=\frac{\overrightarrow{OT}+2\overrightarrow{OR}}{3}$ $\mathbf{s}=\frac{\overrightarrow{OT}+2\mathbf{r}}{3}$ $\overrightarrow{OT}=3\mathbf{s}-2\mathbf{r}$ 
(iii)	$\frac{\lvert(\mathbf{p}-\mathbf{r})\cdot(\mathbf{s}-\mathbf{r})\rvert}{\lvert\mathbf{s}-\mathbf{r}\rvert}=\left\lvert(\mathbf{p}-\mathbf{r})\cdot\frac{(\mathbf{s}-\mathbf{r})}{\lvert\mathbf{s}-\mathbf{r}\rvert}\right\rvert$ is the length of projection of $\overrightarrow{RP}$ on $\overrightarrow{RS}$.
(iv)	Area of triangle $PRT=\frac{1}{2}\left\lvert\overrightarrow{RP}\times\overrightarrow{RT}\right\rvert$ $=\frac{1}{2}\lvert(\mathbf{p}-\mathbf{r})\times(3\mathbf{s}-2\mathbf{r}-\mathbf{r})\rvert$ $=\frac{1}{2}\lvert(\mathbf{p}-\mathbf{r})\times(3\mathbf{s}-3\mathbf{r})\rvert$ $=\frac{3}{2}\lvert\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}+\mathbf{r}\times\mathbf{r}\rvert$ $=\frac{3}{2}\lvert\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}\rvert\quad(\because \mathbf{r}\times\mathbf{r}=\mathbf{0})$ $\therefore \gamma=\frac{3}{2}$ ← To be stated explicitly

Qn	Solution
11	**Applications of Differentiation**
(i)	Volume from external wall $=\frac{1}{3}\pi(r+3)^2(h+5)+\pi(r+3)^2(4h)$ $=\frac{13}{3}\pi(r+3)^2h+\frac{5}{3}\pi(r+3)^2$ Volume from internal wall $=\frac{1}{3}\pi r^2h+\pi r^2(4h)=k$ $\Rightarrow h=\frac{3k}{13\pi r^2}$ $V=\frac{13}{3}\pi(r+3)^2h+\frac{5}{3}\pi(r+3)^2-k$ $V=\frac{13}{3}\pi(r+3)^2\left(\frac{3k}{13\pi r^2}\right)+\frac{5}{3}\pi(r+3)^2-k$ $=(r+3)^2\left(\frac{k}{r^2}\right)+\frac{5}{3}\pi(r+3)^2-k\quad$ (Shown)

(ii)

$$V=(r+3)^2\left(\frac{k}{r^2}\right)+\frac{5}{3}\pi(r+3)^2-k$$

$$\frac{dV}{dr}=2(r+3)\left(\frac{k}{r^2}\right)+(r+3)^2\left(\frac{-2k}{r^3}\right)+\frac{10}{3}\pi(r+3)$$

To find stationary point, $\frac{dV}{dr}=0$

$$2(r+3)\left(\frac{k}{r^2}\right)+(r+3)^2\left(\frac{-2k}{r^3}\right)+\frac{10}{3}\pi(r+3)=0$$

$$(r+3)\left[\frac{2k}{r^2}-\frac{2k}{r^3}(r+3)+\frac{10}{3}\pi\right]=0$$

$$(r+3)\left[\frac{2k}{r^2}-\frac{2k}{r^2}-\frac{6k}{r^3}+\frac{10}{3}\pi\right]=0$$

$$(r+3)\left[-\frac{6k}{r^3}+\frac{10}{3}\pi\right]=0$$

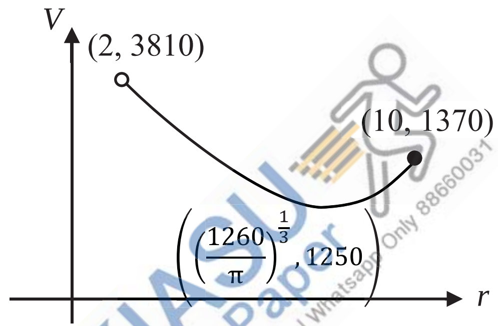
$-\frac{6k}{r^3}+\frac{10}{3}\pi=0$ or $r=-3$ (Reject, $\because r>0$)

$$r^3=\frac{9k}{5\pi}$$

$$r=\left(\frac{9k}{5\pi}\right)^{\frac{1}{3}}$$

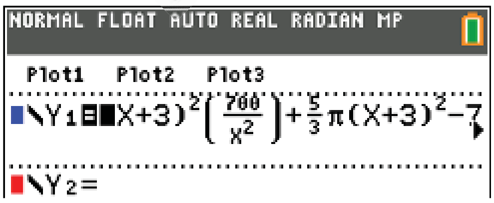
(iii)

When $k = 700$, $r=\left(\frac{9(700)}{5\pi}\right)^{\frac{1}{3}}=7.3746$

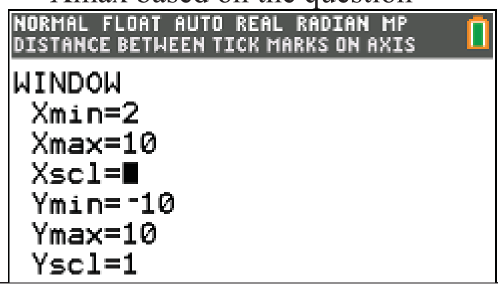


GC Skill:

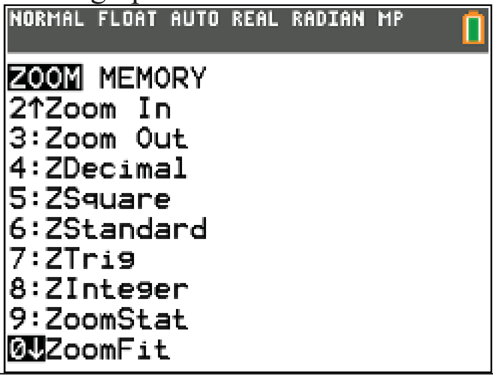
This part can be done using the GC



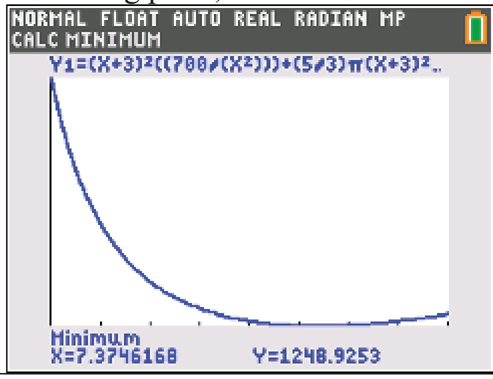
2) Press WINDOW and set the Xmin and Xmax based on the question



3) Press [ZOOM] and select **0: ZoomFit** to see the graph



4) From the graph, you can clearly see a turning point, use GC to find it



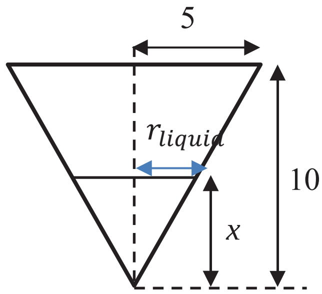
(iv)

Volume of cone section = $\frac{1}{3}\pi\left(5^2\right)(10)=261.80$

Hence when volume of liquid is 200 m^3 it is still in the cone section

By similar triangles,

$\frac{r_{liquid}}{x}=\frac{5}{10} \quad \Rightarrow r_{liquid}=\frac{1}{2}x$



Let volume of the liquid at time t minutes be W

$$W=\frac{1}{3}\pi\left(r_{liquid}\right)^2 x$$

$$=\frac{1}{3}\pi\left(\frac{1}{2}x\right)^2 x=\frac{1}{12}\pi x^3$$

$$\frac{dW}{dx}=\frac{1}{4}\pi x^2$$

When $W=200$, $200=\frac{1}{12}\pi x^3 \Rightarrow x=\left(\frac{2400}{\pi}\right)^{\frac{1}{3}}$

$$\frac{dW}{dt}=\frac{dW}{dx}\left(\frac{dx}{dt}\right)$$

When $x=\left(\frac{2400}{\pi}\right)^{\frac{1}{3}}$, $\frac{dx}{dt}=\frac{0.1}{\left(\frac{1}{4}\pi\left(\frac{2400}{\pi}\right)^{\frac{2}{3}}\right)}=0.00152 \quad$ (3 s.f.)

The rate of change of depth of the acid in the storage well is 0.00152 m/min

1 **(i)** Show that $\dfrac{1}{r(r+1)}-\dfrac{1}{(r+1)(r+2)}=\dfrac{A}{r(r+1)(r+2)}$, where A is a constant to be found. [1]

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$. (There is no need to express your answer as a single algebraic fraction.) [3]

(iii) Given a reason why the series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ converges, and write down its value. [2]

[Turn Over

2 **(i)** Given that $y = \sin^{-1}\left(x^2\right)$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. [1]

(ii) Evaluate $\int_2^3 \ln\left[\tan^{-1} x\right] \mathrm{d}x$, giving your answer correct to 3 decimal places. [1]

(iii) Given that $k > 0$ and $\int_0^k x^2 \sin^{-1}\left(x^2\right) \mathrm{d}x = \int_2^3 \ln\left[\tan^{-1} x\right] \mathrm{d}x$, show that

$$\int_0^k \frac{x^4}{\sqrt{1-x^4}} \mathrm{d}x = ak^3 \sin^{-1} k^2 + b,$$

where a and b are constants to be determined. [4]

3 (a) Do not use a calculator in answering this question.

The equation $z^2 + (-5+2\mathrm{i})z + (21-\mathrm{i}) = 0$ has a root $z = 3 + \mathrm{i}\,c$, where c is a real constant. Find the value of c and hence find the second root of the equation in Cartesian form, $a + \mathrm{i}b$, showing your working. [5]

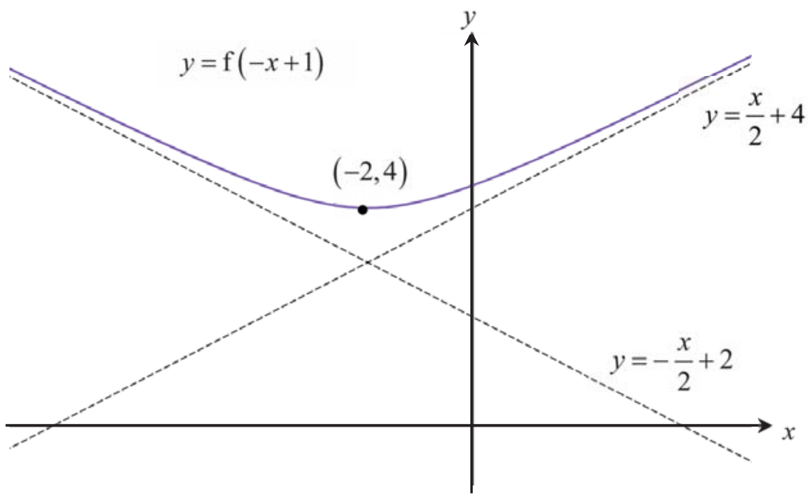
[Turn Over

(b) The complex number z is such that $8z^3 + 125 = 0$.

(i) Given that one possible value of z is $-\frac{5}{2}$, use a **non-calculator method** to find the other possible values of z. Give your answers in the form $a + \mathrm{i}b$, where a and b are exact values. [3]

(ii) Write these values of z in modulus argument form and represent them on an Argand diagram. [4]

4 The diagram below shows the graph of $y = \mathrm{f}(-x+1)$ which has a minimum point at $(-2, 4)$ and has lines $y = \frac{x}{2} + 4$ and $y = -\frac{x}{2} + 2$ as asymptotes.



[Turn Over

(i) The diagram above shows a part of the curve with equation $\frac{(y+a)^2}{m}-\frac{(x+b)^2}{k}=1$. Find the values of *a*, *b* and *k*. [4]

(ii) Sketch the graph of $y=\mathrm{f}(x)$. [2]

5 Given that $f(\theta) = \tan\left(\frac{1}{2}\theta\right)$, show that $f'(\theta) = \frac{1}{2}\left\{1 + \left[f(\theta)\right]^2\right\}$. [2]

By repeated differentiation, find the Maclaurin series for $f(\theta)$, up to and including the term in θ^3. [4]

Explain why a Maclaurin series for $g(\theta) = \cot\left(\frac{1}{2}\theta\right)$ cannot be found. [1]

[Turn Over

The equation $\cot\left(\frac{\pi}{2}-\frac{1}{2}\theta\right)-\frac{5\sqrt{35}}{59}\theta=0$ has a positive root close to zero. Use the expansion above to obtain the approximate value of the root, correct to 4 decimal places. [3]

6 **(a)** Sketch the graph of $y=\frac{x+3}{(2x-1)(x+2)}$. Give the equations of the asymptotes, the coordinates of the point(s) where the curve crosses either axis and the coordinates of the two stationary points. [4]

(b) The curve C has equation $\frac{(3y-3)^2}{a^2}+\frac{(4-2x)^2}{b^2}=1$, where a and b are positive constants. Given that C is a circle and the x-axis is a tangent to C, state the values of a and b. [2]

Describe a sequence of transformations that transforms the graph of C to $\frac{(y-3)^2}{a^2}+\frac{(5-2x)^2}{b^2}=1$. [2]

[Turn Over

7 A curve is such that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}$ and P is a point that moves along the curve. The y-coordinate of P is decreasing at 0.3 units per second at $(1,0)$.

(i) Find the rate of decrease of the x-coordinate of P at $(1,0)$. [2]

(ii) Find the equation of the curve, leaving your answer in the form $y=\mathrm{f}(x)$. [3]

Mr Tan owns a company that manufactures and sells chipsets for smartphones. He models the average profit from one chipset, \$$y$, $y\geqslant 0$, when the selling price of each chipset is \$$x$, with the equation $y=\mathrm{f}(x)$.

(iii) Find algebraically, the selling price of each chipset which gives the maximum average profit. [3]

(iv) Sketch the graph showing the average profit from one chipset as the selling price of each chip set varies. State a suitable range of values that Mr Tan should set for the selling price of a chipset. [2]

8 **(i)** Show that $\frac{2x^2+5}{\left(x^2+1\right)^2}=\frac{A}{x^2+1}+\frac{B}{\left(x^2+1\right)^2}$, where A and B are constants to be determined. [1]

[Turn Over

(ii) Using the substitution $x = \tan u$, show that $\int \frac{1}{\left(x^2+1\right)^2}\ \mathrm{d}x = \frac{ax}{x^2+1} + b\tan^{-1} x + C$, where a and b are constants to be found. [5]

(iii) Hence, find $\int \frac{2x^2+7x+5}{\left(x^2+1\right)^2}\ \mathrm{d}x$. [4]

9 A curve C has parametric equations

$$x = \frac{1}{t} + t \quad , \qquad y = \frac{1}{t} - t \, , \qquad \text{where } t \neq 0.$$

(i) Show that the gradient of the normal to the curve at $P\left(\frac{1}{p} + p, \frac{1}{p} - p\right)$ is given by $\frac{p^2 - 1}{p^2 + 1}$.

[2]

(ii) Given that $p = \frac{1}{\sqrt{3}}$, determine the acute angle between the normal and the line $y = x + 3$.

[3]

[Turn Over

(iii) Point Q on C has parameter q. Show that the gradient of PQ is $\frac{1+pq}{1-pq}$. [2]

(iv) The normal at P cuts the curve again at point R with parameter r. Show that $r=-\frac{1}{p^3}$. [3]

10 Crypto currency Miners are rewarded certain amounts of the currency at varying time intervals when they carry out validation tasks of transactions using their personal computers.

A crypto currency, Shockcoin (SHC), has a reward payment schedule such that the first reward payout is 20 days after a Miner commences mining and the second reward payout is 28 days after the first. The duration of each subsequent payout is 8 days more than the duration between the two preceding payouts. For example, if the duration between the last and current payout is 100 days, then it would take 108 days to the next reward payout. Albert plans to commence mining on 1 January 2022.

(i) After receiving his first reward payout on 21 January 2022, on what day and month will he receive his second reward payout? [1]

(ii) Show that from the start of mining, the number of days Albert takes to receive his n^{th} reward payout is given by $4n^2 + 16n$. [2]

[You may assume that there are 365 days in a year]

(iii) Find the number of reward payouts Albert have received as at 31 December 2023. [2]

(iv) Find the amount, in SHC Albert is expected to receive in his last payout in 2023. [2]

(v) Write an expression for the total amount, in SHC, Albert would have received after the n^{th} reward payout. [2]

It is estimated that it would cost Albert 25 SHC per day for the electricity to keep the computer on to mine SHC.

(vi) Show that the total net gain for Albert immediately after receiving n reward payouts is given by $a\left(8000\left(1-0.85^n\right)-bn^2-cn\right)$ where a, b and c are constants to be found. [2]

[Turn Over

(v) In order to maximise his total net gain, Albert should stop mining after the kth payout. Find the value of k and the maximum total net gain. [2]

11 **(i)** The complex number w can be expressed as $e^{i\theta}$.

(a) Find $w^n + \frac{1}{w^n}$ and $w^n - \frac{1}{w^n}$ in simplified trigonometric form. [3]

(b) By considering the binomial expansion of $\left[\left(w+\frac{1}{w}\right)\left(w-\frac{1}{w}\right)\right]^3$ and the results from **part (a)**, show that $\sin^3\theta\cos^3\theta = \frac{3}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$. [4]

(ii) Hence find the first two non-zero terms of the Maclaurin series for $\sin^3\theta\cos^3\theta$. [2]

[Turn Over

1 **(i)** Show that $\dfrac{1}{r(r+1)}-\dfrac{1}{(r+1)(r+2)}=\dfrac{A}{r(r+1)(r+2)}$, where A is a constant to be found. [1]

$\dfrac{1}{r(r+1)}-\dfrac{1}{(r+1)(r+2)}=\dfrac{r+2-r}{r(r+1)(r+2)}$ $=\dfrac{2}{r(r+1)(r+2)}$	

(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$. (There is no need to express your answer as a single algebraic fraction.) [3]

$\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$ $=\dfrac{1}{2}\Big[\dfrac{1}{1(2)}-\dfrac{1}{2(3)}$ $+\dfrac{1}{2(3)}-\dfrac{1}{3(4)}$ $+\ \ldots$ $+\dfrac{1}{(n-1)n}-\dfrac{1}{n(n+1)}$ $+\dfrac{1}{n(n+1)}-\dfrac{1}{(n+1)(n+2)}\Big]$ $=\dfrac{1}{2}\left[\dfrac{1}{2}-\dfrac{1}{(n+1)(n+2)}\right]=\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}$	

(iii) Given a reason why the series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ converges, and write down its value. [2]

$\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$ As $n\to\infty$, $\dfrac{1}{2(n+1)(n+2)}\to 0$. Hence $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}\to\frac{1}{4}$ Thus series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ converges. $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}$	Learn how to present your argument logically. A lot of presentation errors are seen. Wrong to say: As $r\to\infty$, $\dfrac{1}{r(r+1)(r+2)}\to 0$ $\dfrac{1}{r(r+1)(r+2)}$ is the term, not the sum. Pay attention to the presentation. **The following are correct** ✓ $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$ ✓ $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}$

[Turn Over

	✓ $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)} \to \frac{1}{4}$ as $n\to\infty$ Since the question asks for the value, you must state the value of sum to infinity. Thus it is insufficient to stop at line 3 of the solution $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)} \to \frac{1}{4}$ You must conclude and state that $\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)} = \frac{1}{4}$

2 **(i)** Given that $y=\sin^{-1}(x^2)$, find $\frac{dy}{dx}$ in terms of x. [1]

$\frac{d}{dx}(\sin^{-1}x^2)=\frac{1}{\sqrt{1-x^4}}\times\frac{d}{dx}(x^2)$ $=\frac{2x}{\sqrt{1-x^4}}$	Remember chain rule

(ii) Evaluate $\int_2^3 \ln[\tan^{-1}x]\ dx$, giving your answer correct to 3 decimal places. [1]

$\int_2^3 \ln(\tan^{-1}x)dx = 0.17022 = 0.170 \quad (3\text{ d.p.})$	Use GC since only 1 mark is given and question does not require exact value.

(iii) Given that $k>0$ and $\int_0^k x^2\sin^{-1}(x^2)\ dx = \int_2^3 \ln[\tan^{-1}x]\ dx$, show that

$$\int_0^k \frac{x^4}{\sqrt{1-x^4}}\ dx = ak^3\sin^{-1}k^2+b,$$

where a and b are constants to be determined. [4]

$\int_0^k x^2\sin^{-1}(x^2)\ dx$ $=\left[\frac{x^3}{3}\sin^{-1}(x^2)\right]_0^k-\int_0^k\frac{x^3}{3}\cdot\frac{2x}{\sqrt{1-x^4}}dx$ $=\frac{k^3}{3}\sin^{-1}k^2-\frac{2}{3}\int_0^k\frac{x^4}{\sqrt{1-x^4}}dx$ $\therefore \frac{k^3}{3}\sin^{-1}k^2-\frac{2}{3}\int_0^k\frac{x^4}{\sqrt{1-x^4}}dx = 0.17022$ Hence $\int_0^k\frac{x^4}{\sqrt{1-x^4}}dx=\frac{3}{2}\left[\frac{k^3}{3}\sin^{-1}k^2-0.17022\right]$ $=\frac{1}{2}k^3\sin^{-1}k^2-0.255$ **Alternative**	Let $u=\sin^{-1}(x^2)$ and $\frac{dv}{dx}=x^2$ $\frac{du}{dx}=\frac{2x}{\sqrt{1-x^4}}$ and $\frac{dv}{dx}=\frac{x^3}{3}$ You should substitute in the 5 d.p. answers from (ii) so as to avoid rounding off error if any. **Alternative**

$\int_0^k \frac{x^4}{\sqrt{1-x^4}}\,\mathrm{d}x$ $=\int_0^k \frac{x^3}{2}\left(\frac{2x}{\sqrt{1-x^4}}\right)\mathrm{d}x$ $=\left[\frac{x^3}{2}\sin^{-1}\left(x^2\right)\right]_0^k - \int_0^k \frac{3}{2}x^2\sin^{-1}\left(x^2\right)\,\mathrm{d}x$ $=\frac{k^3}{2}\sin^{-1}\left(k^2\right)-\frac{3}{2}\int_2^3 \ln\left[\tan^{-1}x\right]\,\mathrm{d}x$ $=\frac{k^3}{2}\sin^{-1}\left(k^2\right)-\frac{3}{2}(0.17022)$ $=\frac{k^3}{2}\sin^{-1}\left(k^2\right)-0.255$	Let $u=\frac{x^3}{2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{2x}{\sqrt{1-x^4}}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{3x^2}{2}$ and $v=\sin^{-1}\left(x^2\right)$

3 (a) Do not use a calculator in answering this question.

The equation $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=0$ has a root $z=3+\mathrm{i}\,c$, where c is a real constant.

Find the value of c and hence find the second root of the equation in Cartesian form, $a+ib$, showing your working. [5]

Since $3+\mathrm{i}\,c$ is a root, $(3+\mathrm{i}\,c)^2+(-5+2\mathrm{i})(3+\mathrm{i}\,c)+(21-\mathrm{i})=0$ $(9+6c\mathrm{i}-c^2)+(-15+6\mathrm{i}-5c\mathrm{i}-2c)+(21-\mathrm{i})=0$ $(15-2c-c^2)+\mathrm{i}(c+5)=0$ Comparing real and imaginary parts: $15-2c-c^2=0$ and $c+5=0$ $(c-3)(c+5)=0$ and $c+5=0$ $\therefore c=-5$ or $c=3$ and $c=-5$ Hence $c=-5$ Let $z=k$ be the second root. $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=[z-(3-5\mathrm{i})][z-k]$ Comparing coefficients: z^0 : $21-\mathrm{i}=(3-5\mathrm{i})k$ $k=\frac{21-\mathrm{i}}{3-5\mathrm{i}}\times\frac{3+5\mathrm{i}}{3+5\mathrm{i}}$ $=\frac{63+105\mathrm{i}-3\mathrm{i}+5}{9+25}$ $=2+3\mathrm{i}$ Hence the second root is $2+3\mathrm{i}$	Coefficients of the equation $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=0$ are not all real. Hence, the conjugate $3-\mathrm{i}c$ is not a root. c has to satisfy both equations. Hence $c=3$ is rejected.

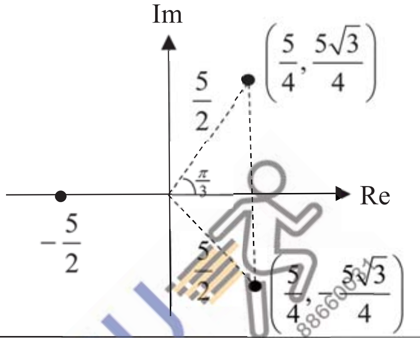
[Turn Over

(b) The complex number z is such that $8z^3+125=0$.

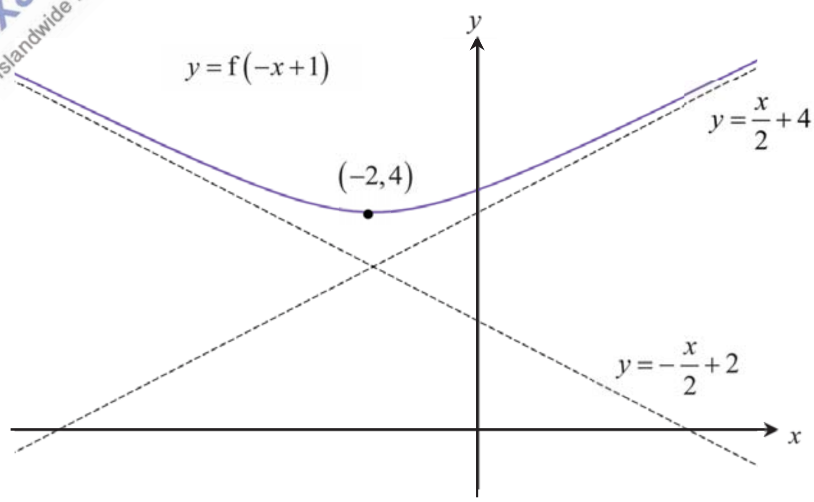
(i) Given that one possible value of z is $-\frac{5}{2}$, use a **non-calculator method** to find the other possible values of z. Give your answers in the form $a+\mathrm{i}b$, where a and b are exact values. [3]

$8z^3+125=0$ Since $z=-\frac{5}{2}$, then $2z+5$ is a factor. $8z^3+125=(2z+5)(Az^2+Bz+C)$ $=(2z+5)(4z^2+Bz+25)$ Comparing coefficient of z^2: $0=2B+20$ $\Rightarrow B=-10$ $\therefore (2z+5)(4z^2-10z+25)=0$ $z=-\frac{5}{2}$ or $z=\frac{10\pm\sqrt{100-4(4)(25)}}{8}$ $=\frac{10\pm\sqrt{-300}}{8}$ $=\frac{10\pm\sqrt{300}\,\mathrm{i}}{8}$ $=\frac{5\pm5\sqrt{3}\,\mathrm{i}}{4}$	By Fundamental Theorem of Algebra, $8z^3+125=0$ has 3 roots. To find all roots, we need to 1. Obtain the first root (in this case given by question), then write down the equivalent linear factor 2. The quadratic factor has to be found either through long division or comparing of coefficients. 3. Solve the quadratic equation $az^2+bz+c=0$ using quadratic formula i.e $z=\frac{-b\pm\sqrt{b^2-4(a)(c)}}{2a}$ **<u>GC CANNOT be used!</u>** There are many different methods to solve this question. Learn the most efficient method.

(ii) Write these values of z in modulus argument form and represent them on an Argand diagram. [4]

$z=-\frac{5}{2},\ \frac{5}{4}\pm\frac{5\sqrt{3}}{4}i$ $\sqrt{\left(\frac{5}{2}\right)^2+\left(\pm\frac{5\sqrt{3}}{4}\right)^2}=\frac{5}{2}$ $\tan\alpha=\sqrt{3}$ $\Rightarrow\alpha=\frac{\pi}{3}$ $\therefore$ The three roots are $z=\frac{5}{2}e^{i\pi},\ \frac{5}{2}e^{i\frac{\pi}{3}}$ and $\frac{5}{2}e^{-i\frac{\pi}{3}}$. Or $z=\frac{5}{2}(\cos\pi+i\sin\pi)$, $\frac{5}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ and $\frac{5}{2}\left[\cos\left(-\frac{\pi}{3}\right)++i\sin\left(-\frac{\pi}{3}\right)\right]$ 	All the values of z need to be represented in one single Argand diagram. You need to give the values of z in modulus argument form.

4 The diagram below shows the graph of $y=f(-x+1)$ which has a minimum point at $(-2,4)$ and has lines $y=\frac{x}{2}+4$ and $y=-\frac{x}{2}+2$ as asymptotes.



[Turn Over

(i) The diagram above shows a part of the curve with equation $\frac{(y+a)^2}{m}-\frac{(x+b)^2}{k}=1$. Find the values of a, b and k. [4]

$y=\frac{x}{2}+4 \quad ---(1)$

$y=-\frac{x}{2}+2 \quad ---(2)$

Solving (1) and (2): $x=-2$, $y=3$

Hence centre of the hyperbola is $(-2,3)$

$\therefore a=-3$ and $b=2$

$$\frac{(y-3)^2}{m}-\frac{(x+2)^2}{k}=1$$

Since $(-2,4)$ is a vertex on the hyperbola,

$3+\sqrt{m}=4$

$\Rightarrow m=1$

Asymptotes: $\frac{(y-3)^2}{1}=\frac{(x+2)^2}{k}$

$$(y-3)=\pm\left(\frac{1}{\sqrt{k}}\right)(x+2)$$

Comparing gradient: $\frac{1}{\sqrt{k}}=\frac{1}{2} \quad \Rightarrow k=4$

Given $\frac{(y+a)^2}{m}-\frac{(x+b)^2}{k}=1$

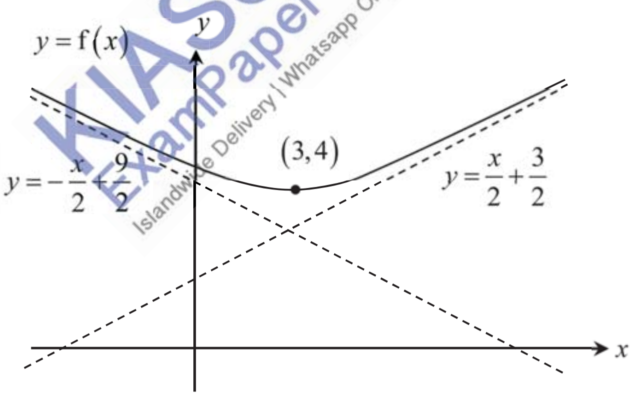
(1) Centre: $(-b,-a)$

(2) Vertices: $\left(-b,-a+\sqrt{m}\right)$ and $\left(-b,-a-\sqrt{m}\right)$

(3) Equations of asymptotes are

$$y=-a\pm\sqrt{\frac{m}{k}}(x+b)$$

(ii) Sketch the graph of $y=\mathrm{f}(x)$. [2]



$y=\mathrm{f}(-x+1)\xrightarrow{\text{replace } x \text{ with } -x} y=\mathrm{f}(x+1)$

Reflect the curve in the y-axis.

$y=\mathrm{f}(x+1)\xrightarrow{\text{replace } x \text{ with } x-1} y=\mathrm{f}(x)$

Translate 1 unit in the positive x-direction

Note that asymptotes will also undergo the same transformations as above. Hence need to find the equations of the new asymptotes.

E.g:

$y=\frac{x}{2}+4\rightarrow y=\frac{-x}{2}+4\rightarrow y=\frac{-(x-1)}{2}+4$

i.e $y=\frac{-x}{2}+\frac{9}{2}$

which is a line with negative gradient

5 Given that $f(\theta)=\tan\left(\frac{1}{2}\theta\right)$, show that $f'(\theta)=\frac{1}{2}\left\{1+\left[f(\theta)\right]^2\right\}$. [2]

$f(\theta)=\tan\left(\frac{1}{2}\theta\right)$ $f'(\theta)=\frac{1}{2}\sec^2\left(\frac{1}{2}\theta\right)$ $=\frac{1}{2}\left\{1+\tan^2\left(\frac{1}{2}\theta\right)\right\}$ $=\frac{1}{2}\left\{1+\left[f(\theta)\right]^2\right\}$	

By repeated differentiation, find the Maclaurin series for $f(\theta)$, up to and including the term in θ^3. [4]

$f''(\theta)=\frac{1}{2}\left[2f(\theta)\times\frac{d}{d\theta}f(\theta)\right]$ $=f(\theta)f'(\theta)$	Remember chain rule.
$f'''(\theta)=f(\theta)\times\frac{d}{d\theta}f'(\theta)+f'(\theta)\times\frac{d}{d\theta}f(\theta)$ $=f(\theta)f''(\theta)+\left[f'(\theta)\right]^2$ When $\theta=0$, $f(0)=\tan\left(\frac{1}{2}\times0\right)=0$ $f'(0)=\frac{1}{2}\left(1+0^2\right)=\frac{1}{2}$ $f''(0)=(0)\left(\frac{1}{2}\right)=0$ $f'''(0)=\left(\frac{1}{2}\right)^2+0=\frac{1}{4}$ The Maclaurin series for $f(\theta)$ is $f(\theta)=0+\frac{1}{2}(\theta)+0\left(\frac{\theta^2}{2!}\right)+\frac{1}{4}\left(\frac{\theta^3}{3!}\right)+...$ $=\frac{1}{2}\theta+\frac{1}{24}\theta^3+...$	Easier to carry out implicit differentiation.

Explain why a Maclaurin series for $g(\theta)=\cot\left(\frac{1}{2}\theta\right)$ cannot be found. [1]

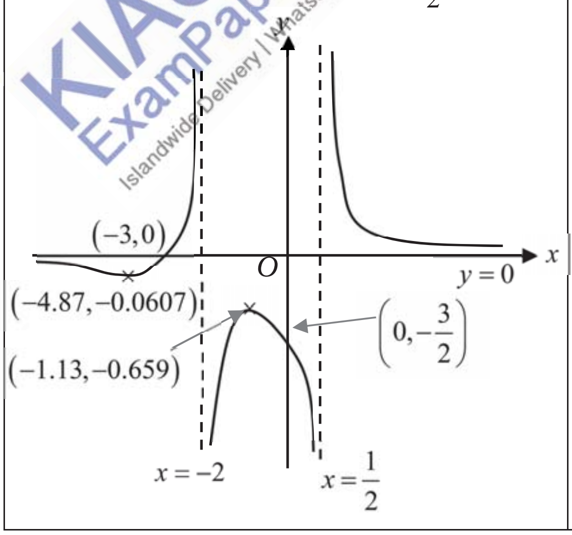
$g(0)=\cot(0)=\frac{1}{\tan(0)}$ is undefined. So a Maclaurin series for $g(\theta)$ cannot be found.	

[Turn Over

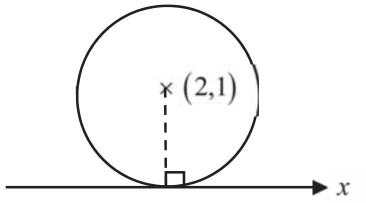
The equation $\cot\left(\frac{\pi}{2}-\frac{1}{2}\theta\right)-\frac{5\sqrt{35}}{59}\theta=0$ has a positive root close to zero. Use the expansion above to obtain the approximate value of the root, correct to 4 decimal places. [3]

$\cot\left(\frac{\pi}{2}-\frac{1}{2}\theta\right)-\frac{5\sqrt{35}}{59}\theta=0$ $\Rightarrow \tan\left(\frac{1}{2}\theta\right)-\frac{5\sqrt{35}}{59}\theta=0$ $\Rightarrow \frac{1}{2}\theta+\frac{1}{24}\theta^3-\frac{5\sqrt{35}}{59}\theta=0$ $\Rightarrow \frac{1}{24}\theta^3+\left(\frac{1}{2}-\frac{5\sqrt{35}}{59}\right)\theta=0$ From the GC, $\theta=0.1808$ or -0.1808 or 0 For the positive root close to zero, $\theta=0.1808$ (to 4 d.p)	Since we are prompted to use expansion in previous part (which is expanded up to θ^3) hence small angle approximation cannot be used here. Recall the following identity: $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ $\cos\left(\frac{\pi}{2}-x\right)=\sin x$ $\therefore \cot\left(\frac{\pi}{2}-x\right)=\tan x$

6 **(a)** Sketch the graph of $y=\frac{x+3}{(2x-1)(x+2)}$. Give the equations of the asymptotes, the coordinates of the point(s) where the curve crosses either axis and the coordinates of the two stationary points. [4]

$y=\frac{x+3}{(2x-1)(x+2)}$ Intercepts: $(-3,0)$ and $\left(0,-\frac{3}{2}\right)$ Asymptotes: $y=0$, $x=-2$ and $x=\frac{1}{2}$  	When drawing graphs with asymptotes, we need to make sure that the graph tends gradually to the asymptotes. Question asks for: (a) Equations of asymptotes (b) Coordinates of stationary points (c) Coordinates of axial intercepts Hence (a), (b) and (c) must be clearly labelled in our graph.

(b) The curve C has equation $\frac{(3y-3)^2}{a^2}+\frac{(4-2x)^2}{b^2}=1$, where a and b are positive constants. Given that C is a circle and the x-axis is a tangent to C, state the values of a and b. [2]

$\frac{(3y-3)^2}{a^2}+\frac{(4-2x)^2}{b^2}=1$ $\frac{3^2(y-1)^2}{a^2}+\frac{(-2)^2(x-2)^2}{b^2}=1$ Given C is a circle with centre at $(2,1)$. Since x-axis is a tangent to circle, radius = 1 Hence $a=3$, $b=2$	Recall circle property: tangent is perpendicular to radius 

Describe a sequence of transformations that transforms the graph of C to $\frac{(y-3)^2}{a^2}+\frac{(5-2x)^2}{b^2}=1$. [2]

$\frac{(3y-3)^2}{a^2}+\frac{(4-2x)^2}{b^2}=1$ Replace y with $\frac{1}{3}y$: $\frac{(y-3)^2}{a^2}+\frac{(4-2x)^2}{b^2}=1$ Replace x with $\left(x-\frac{1}{2}\right)$: $\frac{(y-3)^2}{a^2}+\frac{\left(4-2\left(x-\frac{1}{2}\right)\right)^2}{b^2}=1$ $\Rightarrow \frac{(y-3)^2}{a^2}+\frac{(5-2x)^2}{b^2}=1$ <u>Description:</u> (1) Stretch C parallel to y-axis by factor 3, with x-axis invariant, <u>**then**</u> (2) Translate <u>resultant curve</u> by $\frac{1}{2}$ units in the positive x direction.	

[Turn Over

7 A curve is such that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}$ and P is a point that moves along the curve. The y-coordinate of P is decreasing at 0.3 units per second at $(1,0)$.

(i) Find the rate of decrease of the x-coordinate of P at $(1,0)$. [2]

At $(1,0)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1}-\frac{1}{2\sqrt{1+8}}=\frac{5}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}t}=-0.3$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}t}$ $=\frac{6}{5}\times(-0.3)$ $=-0.36$ $\therefore$ rate of decrease of the x-coordinate at P is 0.36 units per second.	"y-coordinate of P is **decreasing** at 0.3 units per second" means $\frac{\mathrm{d}y}{\mathrm{d}t}=-0.3$. Remember to include –ve sign.

(ii) Find the equation of the curve, leaving your answer in the form $y=\mathrm{f}(x)$. [3]

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}$ $y=\int\frac{1}{x}-\frac{1}{2\sqrt{x+8}}\,\mathrm{d}x$ $=\ln\lvert x\rvert-\sqrt{x+8}+c$ (1, 0) is a point on the curve $\Rightarrow 0=\ln 1-3+c$ $\Rightarrow c=3$ $\therefore y=\ln\lvert x\rvert-\sqrt{x+8}+3$	$\int\frac{1}{x}\,\mathrm{d}x=\ln\lvert x\rvert+C$. You cannot drop the modulus sign. Since (1, 0) is on the curve, it satisfies the equation of curve, You can use it to find C.

Mr Tan owns a company that manufactures and sells chipsets for smartphones. He models the average profit from one chipset, \$$y$, $y\geqslant 0$, when the selling price of each chipset is \$$x$, with the equation $y=\mathrm{f}(x)$.

(iii) Find algebraically, the selling price of each chipset which gives the maximum average profit. [3]

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}=0$ $\Rightarrow\frac{2\sqrt{x+8}-x}{2x\sqrt{x+8}}=0$ $\Rightarrow 2\sqrt{x+8}=x$ ----(*) $\Rightarrow 4x+32=x^2$ $\Rightarrow x^2-4x-32=0$ $\Rightarrow(x+4)(x-8)=0$ $\therefore x=8,\ -4$ (reject since selling price is positive)	You are expected to solve this question "algebraically". You should either use factorisation or quadratic

$$\frac{dy}{dx}=\frac{1}{x}-\frac{1}{2\sqrt{x+8}}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{x^2}+\frac{1}{4(x+8)^{\frac{3}{2}}}$$

When $x = 8$, $\frac{d^2y}{dx^2}=-0.0117$ (or $-\frac{3}{256}$) <0

$\therefore$ Selling price which gives maximum profit = $8.

formula to solve for x. You should reject x = -4.

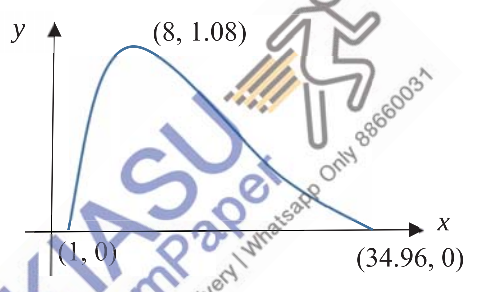
OR

x	7.99	8	8.01
$\frac{dy}{dx}$	1.1736×10^{-4}	0	-1.1701×10^{-4}
Sign of $\frac{dy}{dx}$	+		-

$\therefore$ Selling price which gives maximum profit = $8.

When choosing 2 values to do 1st derivative test, you must choose values which are close to 8. 7 and 9 are not considered close.

(iv) Sketch the graph showing the average profit from one chipset as the selling price of each chip set varies. State a suitable range of values that Mr Tan should set for the selling price of a chipset. [2]



$1\le x\le 34.96$

Note that $y\geqslant 0$. Do not include negative y portion in your sketch. You should label the maximum point (found in part (iii)) and the endpoints.

Coordinates of these points can be found using GC. y value should be given in 2 decimal places.

8 **(i)** Show that $\frac{2x^2+5}{(x^2+1)^2}=\frac{A}{x^2+1}+\frac{B}{(x^2+1)^2}$, where A and B are constants to be determined. [1]

$$\frac{2x^2+5}{(x^2+1)^2}=\frac{A}{x^2+1}+\frac{B}{(x^2+1)^2}$$

$$2x^2+5=A(x^2+1)+B$$

Comparing coefficients

$A=2$ and $B=3$

B1: A=2, B=3

[Turn Over

(ii) Using the substitution $x=\tan u$, show that $\int \frac{1}{(x^2+1)^2}\ \mathrm{d}x = \frac{ax}{x^2+1}+b\tan^{-1}x+C$, where a and b are constants to be found. [5]

$x=\tan u \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=\sec^2 u$ $\int \frac{1}{(x^2+1)^2}\ \mathrm{d}x = \int \frac{1}{(\tan^2 u+1)^2}\ \sec^2 u\ \mathrm{d}u$ $=\int \frac{1}{\sec^2 u}\ \mathrm{d}u$ $=\int \cos^2 u\ \mathrm{d}u$ $=\int \frac{\cos 2u+1}{2}\ \mathrm{d}u$ $=\frac{1}{2}\left(\frac{\sin 2u}{2}+u\right)+C$ $=\frac{1}{2}\sin u\cos u+\frac{1}{2}u+C$ $=\frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)+\frac{1}{2}\tan^{-1}x+C$ $=\frac{x}{2(x^2+1)}+\frac{1}{2}\tan^{-1}x+C$	Since $\tan u=\frac{x}{1}=\frac{opp}{adj}$, $\sin u=\frac{x}{\sqrt{x^2+1}}$ and $\cos u=\frac{1}{\sqrt{x^2+1}}$

(iii) Hence, find $\int \frac{2x^2+7x+5}{(x^2+1)^2}\ \mathrm{d}x$. [4]

$\int \frac{2x^2+7x+5}{(x^2+1)^2}\ \mathrm{d}x$ $=\int \frac{2x^2+5}{(x^2+1)^2}\mathrm{d}x+\int \frac{7x}{(x^2+1)^2}\ \mathrm{d}x$ $=\int\left(\frac{2}{x^2+1}+\frac{3}{(x^2+1)^2}\right)\ \mathrm{d}x+\frac{7}{2}\int 2x(x^2+1)^{-2}\ \mathrm{d}x$ $=2\tan^{-1}x+3\left(\frac{x}{2(x^2+1)}+\frac{1}{2}\tan^{-1}x\right)-\frac{7}{2(x^2+1)}+C$ $=\frac{7}{2}\tan^{-1}x+\frac{3x-7}{2(x^2+1)}+C$	Use standard form: $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n\ \mathrm{d}x=\frac{1}{n+1}[\mathrm{f}(x)]^{n+1}+C$

9 A curve C has parametric equations

$$x=\frac{1}{t}+t \quad , \qquad y=\frac{1}{t}-t\,, \qquad \text{where } t\neq 0.$$

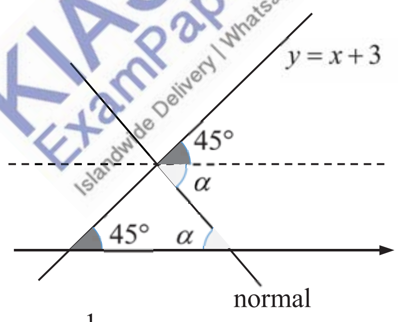
(i) Show that the gradient of the normal to the curve at $P\left(\frac{1}{p}+p,\frac{1}{p}-p\right)$ is given by $\frac{p^2-1}{p^2+1}$.

[2]

$x=\frac{1}{t}+t \quad , \quad y=\frac{1}{t}-t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=1-\frac{1}{t^2}=\frac{t^2-1}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-1-\frac{1}{t^2}=\frac{-t^2-1}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-t^2-1}{t^2}\times\frac{t^2}{t^2-1}=\frac{t^2+1}{1-t^2}$ Hence at $P\left(\frac{1}{p}+p,\frac{1}{p}-p\right)$, $t=p$ Gradient of normal $=-\frac{1}{\frac{p^2+1}{1-p^2}}=\frac{p^2-1}{p^2+1}$	This is a "show" question. You are expected to show working clearly.

(ii) Given that $p=\frac{1}{\sqrt{3}}$, determine the acute angle between the normal and the line $y=x+3$.

[3]

When $p=\frac{1}{\sqrt{3}}$, gradient of normal $=\frac{\frac{1}{3}-1}{\frac{1}{3}+1}=-\frac{1}{2}$  $\alpha=\tan^{-1}\frac{1}{2}$ $=26.565°$ Thus, acute angle between the 2 lines is $45°+26.565°=71.6°$ (1 d.p)	Note you are required to give the **acute** angle between the 2 lines.

[Turn Over

(iii) Point Q on C has parameter q. Show that the gradient of PQ is $\frac{1+pq}{1-pq}$. [2]

$x=\frac{1}{t}+t$, $y=\frac{1}{t}-t$ $P\left(\frac{1}{p}+p, \frac{1}{p}-p\right)$ $Q\left(\frac{1}{q}+q, \frac{1}{q}-q\right)$ Gradient of PQ $=\left(\frac{1}{p}-p-\frac{1}{q}+q\right)\div\left(\frac{1}{p}+p-\frac{1}{q}-q\right)$ $=\frac{q-p^2q-p+pq^2}{pq}\times\frac{pq}{q+p^2q-p-pq^2}$ $=\frac{(q-p)+pq(q-p)}{(q-p)-pq(q-p)}$ $=\frac{1+pq}{1-pq}$	This is a "show" question. You are expected to show working clearly.

(iv) The normal at P cuts the curve again at point R with parameter r. Show that $r=-\frac{1}{p^3}$. [3]

The normal at P cuts the curve again at point R, Hence gradient of normal at P = gradient of PR $\frac{p^2-1}{p^2+1}=\frac{1+pr}{1-pr}$ $(p^2-1)(1-pr)=(p^2+1)(1+pr)$ $p^2-1-p^3r+pr=p^2+1+p^3r+pr$ $2p^3r=-2$ $r=-\frac{1}{p^3}$	Note that $p\neq\frac{1}{\sqrt{3}}$ here.

10 Crypto currency Miners are rewarded certain amounts of the currency at varying time intervals when they carry out validation tasks of transactions using their personal computers.

A crypto currency, Shockcoin (SHC), has a reward payment schedule such that the first reward payout is 20 days after a Miner commences mining and the second reward payout is 28 days after the first. The duration of each subsequent payout is 8 days more than the duration between the two preceding payouts. For example, if the duration between the last and current payout is 100 days, then it would take 108 days to the next reward payout. Albert plans to commence mining on 1 January 2022.

(i) After receiving his first reward payout on 21 January 2022, on what day and month will he receive his second reward payout? [1]

18 February 2022	There are 31 days in the month of January.

(ii) Show that from the start of mining, the number of days Albert takes to receive his n^{th} reward payout is given by $4n^2+16n$. [2]

Number of days to n^{th} payout $=\frac{n}{2}\left[2(20)+(n-1)(8)\right]=4n^2+16n$	

[You may assume that there are 365 days in a year]

(iii) Find the number of reward payouts Albert have received as at 31 December 2023. [2]

$4n^2+16n=730$ $n=-15.66$ (reject) or $n=11.66$ Albert would have received 11 reward payouts by 31 December 2023.	From 1 January 2022 (i.e. start of mining) to 31 December 2023 is $365\times 2=730$ days.

The rewards for Miners are paid in Shockcoin (SHC).
The first reward for a Miner is SHC 120 000. For all subsequent reward payouts, the reward amount is 85% of the previous payout.
[Give non-exact numerical answers correct to 2 decimal places.]

(iv) Find the amount, in SHC Albert is expected to receive in his last payout in 2023. [2]

Reward for last payment in 2023 $=(120000)\left(0.85^{10}\right)$ $=23624.93$	Give non-exact numerical answers correct to 2 decimal places.

(v) Write an expression for the total amount, in SHC, Albert would have received after the n^{th} reward payout. [2]

Total reward received after the n^{th} payout $=\frac{120000(1-0.85^n)}{1-0.85}$ $=800000(1-0.85^n)$	

It is estimated that it would cost Albert 25 SHC per day for the electricity to keep the computer on to mine SHC.

(vi) Show that the total net gain for Albert immediately after receiving n reward payouts is given by $a\left(8000\left(1-0.85^n\right)-bn^2-cn\right)$ where a, b and c are constants to be found. [2]

[Turn Over

Total net gain after receiving n reward payout $=800000\left(1-0.85^n\right)-25(4n^2+16n)$ $=100\left(8000\left(1-0.85^n\right)-n^2-4n\right)$	

(v) In order to maximise his total net gain, Albert should stop mining after the kth payout. Find the value of k and the maximum total net gain. [2]

(table below) Quit mining after collecting the 21st reward payout so as maximise his total net gain. max net gain = SHC 721143.52	Give non-exact numerical answers correct to 2 decimal places.

n	$100\left(8000\left(1-0.85^n\right)-n^2-4n\right)$
20	720992.38
21	721143.52
22	720396.99

11 **(i)** The complex number w can be expressed as $\mathrm{e}^{\,\mathrm{i}\theta}$.

(a) Find $w^n+\dfrac{1}{w^n}$ and $w^n-\dfrac{1}{w^n}$ in simplified trigonometric form. [3]

$$w=\mathrm{e}^{i\theta}\Rightarrow w^n=\mathrm{e}^{in\theta},\ \frac{1}{w^n}=\mathrm{e}^{-in\theta}$$

$$\begin{aligned}w^n+\frac{1}{w^n}&=\mathrm{e}^{in\theta}+\mathrm{e}^{-in\theta}\\&=\cos n\theta+i\sin n\theta+\cos(-n\theta)+i\sin(-n\theta)\\&=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta\\&=2\cos n\theta\end{aligned}$$

$$\begin{aligned}w^n-\frac{1}{w^n}&=\mathrm{e}^{in\theta}-\mathrm{e}^{-in\theta}\\&=\cos n\theta+i\sin n\theta-\cos(-n\theta)-i\sin(-n\theta)\\&=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta\\&=2i\sin n\theta\end{aligned}$$

(b) By considering the binomial expansion of $\left[\left(w+\frac{1}{w}\right)\left(w-\frac{1}{w}\right)\right]^3$ and the results from **part (a)**, show that $\sin^3\theta\cos^3\theta=\frac{3}{32}\sin 2\theta-\frac{1}{32}\sin 6\theta$. [4]

$\left[\left(w+\frac{1}{w}\right)\left(w-\frac{1}{w}\right)\right]^3$ $=\left(w^2-\frac{1}{w^2}\right)^3$ $=\left(w^2\right)^3+3\left(w^2\right)^2\left(-\frac{1}{w^2}\right)+3\left(w^2\right)\left(-\frac{1}{w^2}\right)^2+\left(-\frac{1}{w^2}\right)^3$ $=w^6-\frac{1}{w^6}-3\left(w^2-\frac{1}{w^2}\right)$ $=2i\sin 6\theta-6i\sin 2\theta$ $\left[\left(w+\frac{1}{w}\right)\left(w-\frac{1}{w}\right)\right]^3=\left[\left(2\cos\theta\right)\left(2i\sin\theta\right)\right]^3$ $=-64i\sin^3\theta\cos^3\theta$ $\therefore 2i\sin 6\theta-6i\sin 2\theta=-64i\sin^3\theta\cos^3\theta$ $\Rightarrow \sin^3\theta\cos^3\theta=\frac{2i}{-64i}\sin 6\theta-\frac{6i}{-64i}\sin 2\theta$ $\Rightarrow \sin^3\theta\cos^3\theta=\frac{3}{32}\sin 2\theta-\frac{1}{32}\sin 6\theta$ (shown)	It is easier to expand $\left(w^2-\frac{1}{w^2}\right)^3$ than $\left(w+\frac{1}{w}\right)^3\times\left(w-\frac{1}{w}\right)^3$.

(ii) Hence find the first two non-zero terms of the Maclaurin series for $\sin^3\theta\cos^3\theta$. [2]

$\sin^3\theta\cos^3\theta$ $=\frac{3}{32}\sin 2\theta-\frac{1}{32}\sin 6\theta$ $=\frac{3}{32}\left(2\theta-\frac{(2\theta)^3}{3!}+\frac{(2\theta)^5}{5!}+\ldots\right)-\frac{1}{32}\left(6\theta-\frac{(6\theta)^3}{3!}+\frac{(6\theta)^5}{5!}+\ldots\right)$ $=-\frac{1}{8}\theta^3+\frac{9}{8}\theta^3+\frac{1}{40}\theta^5-\frac{81}{40}\theta^5+\ldots$ $=\theta^3-2\theta^5+\ldots$	It is much more efficient to use standard series of $\sin\theta$ than maclaurin series formula with repeated differentiation.

[Turn Over