2015 VJC JC2 Prelim Paper 1 Solutions

Q1) Let P_n be the statement:

$$w_n = an + (n-1),\ n \in \mathbb{Z}^+.$$

LHS of $P_1 = w_1 = a$ (given)

RHS of $P_1 = a(1) + (1-1) = a$

$\therefore P_1$ is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$ i.e. $w_k = ak + (k-1)$

We want to show P_{k+1} is true i.e. $w_{k+1} = a(k+1) + k$

LHS of $P_{k+1} = w_{k+1}$

$$= \frac{1}{k}[(k+1)w_k + 1]$$

$$= \frac{1}{k}\{(k+1)[ak + (k-1)] + 1\}$$

$$= a(k+1) + \frac{(k+1)(k-1)+1}{k}$$

$$= a(k+1) + \frac{k^2}{k}$$

$$= a(k+1) + k$$

= RHS of P_{k+1}

$\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true

Since we have shown that

(1) P_1 is true and

(2) P_k is true $\Rightarrow P_{k+1}$ is true.

$\therefore$ By mathematical induction, P_n is true for all positive integers n.

Q2)

Sub $(1,1)$ and $(2,2)$ into $y = \mathrm{h}(x)$.

$a + b + c + d = 1$ ----- (1)

$8a + 4b + 2c + d = 2$ -----(2)

Since (2,2) is also the stationary point, $\mathrm{h}'(2) = 0$. i.e.

$12a + 4b + c = 0$ ----- (3)

Using the GC,

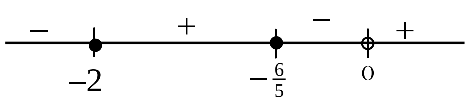
$$a = -\frac{1}{2} - \frac{1}{4}d$$

$$b = \frac{3}{2} + \frac{5}{4}d$$

$$c = -2d$$

$$\frac{ab}{c} \le 0$$

$$\frac{\left(-\frac{1}{2} - \frac{1}{4}d\right)\left(\frac{3}{2} + \frac{5}{4}d\right)}{-2d} \le 0$$



$$\{d \in \mathbb{R} : d \le -2 \text{ or } -\frac{6}{5} \le d < 0\}$$

Q3) $y=\dfrac{ax^2+bx+d}{x-2}=2x+3+\dfrac{k}{x-2}$

By observation, $a=2$

$\Rightarrow 2x^2+bx+d=(2x+3)(x-2)+k$

Compare coefficents of x: $b=3-4\Rightarrow b=-1$

$y=\dfrac{2x^2-x+d}{x-2}$

Given: $d<-6$

Asymptotes: $y=2x+3$, $x=2$

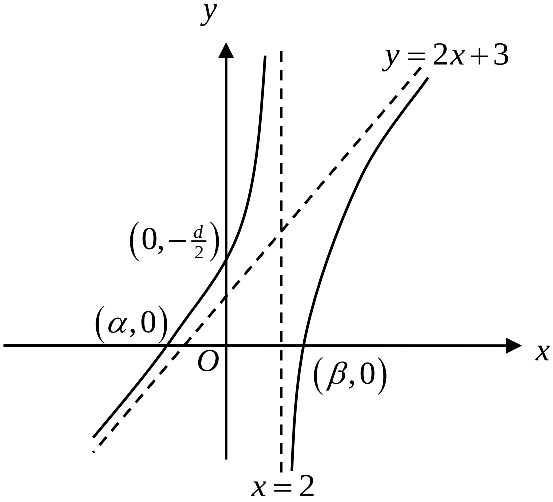
Axial intercepts: when $x=0$, $y=-\dfrac{d}{2}$

When $y=0$, $2x^2-x+d=0$

$$x=\frac{1\pm\sqrt{1-4(2)(d)}}{4}=\frac{1\pm\sqrt{1-8d}}{4}$$

The coordinates are $\left(0,\dfrac{d}{2}\right)$, $\left(\dfrac{1-\sqrt{1-8d}}{4},0\right)$, $\left(\dfrac{1+\sqrt{1-8d}}{4},0\right)$.

Let $\alpha=\dfrac{1-\sqrt{1-8d}}{4}$ and $\beta=\dfrac{1+\sqrt{1-8d}}{4}$



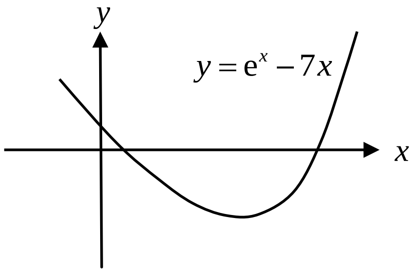
Q4(i)

Let $y=\mathrm{e}^x-7x$. So, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x-7$.

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow \quad \mathrm{e}^x-7=0$

$x=\ln 7$

$\therefore \min \lambda=\ln 7$



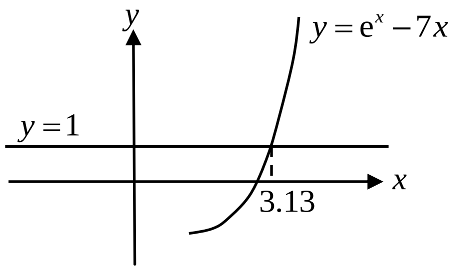
Q4(ii) Let $x=\mathrm{g}^{-1}(1)$

$\Rightarrow \mathrm{g}(x)=1$

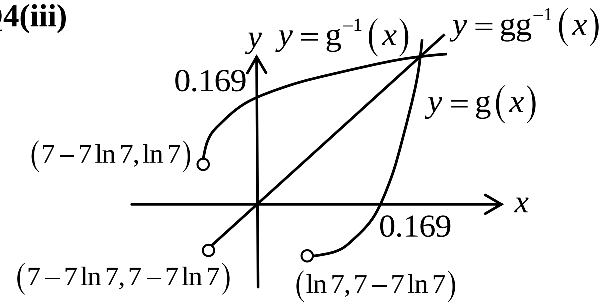
$\Rightarrow \mathrm{e}^x-7x=1$

From the GC,

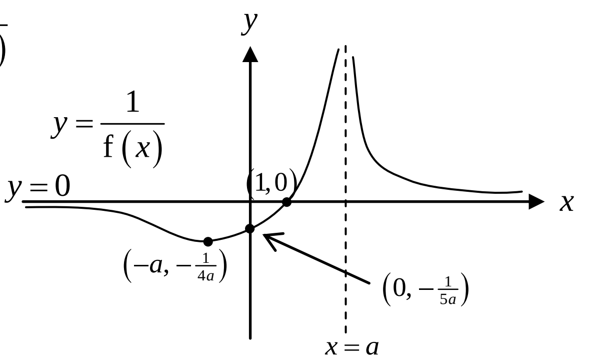
$x=3.13$



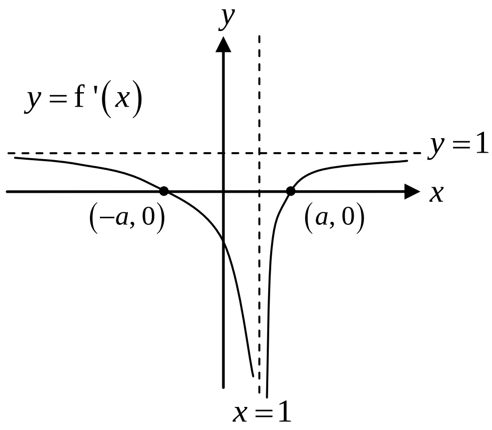
Q4(iii)



Q5(i) $y = \dfrac{1}{\mathrm{f}(x)}$



Q5(ii)



$$\int_{-a}^{0} \left[2 - \mathrm{f}'(x)\right] \mathrm{d}x = \left[2x - \mathrm{f}(x)\right]_{-a}^{0}$$
$$= \left[2(0) - \mathrm{f}(0)\right] - \left[2(-a) - \mathrm{f}(-a)\right]$$
$$= -(-5a) + 2a + (-4a)$$
$$= 3a$$

Q6(i)

n	Amount at end of year n
1	1.08(1000)
2	$1.08\left[1000 + 1.08(1000)\right] = 1000\left(1.08 + 1.08^2\right)$
3 :	$1.08\left[1000 + 1000(1.08) + 1000(1.08)^2\right]$ $= 1000\left(1.08 + 1.08^2 + 1.08^3\right)$
n	$1000\left(1.08 + 1.08^2 + \ldots + 1.08^n\right)$

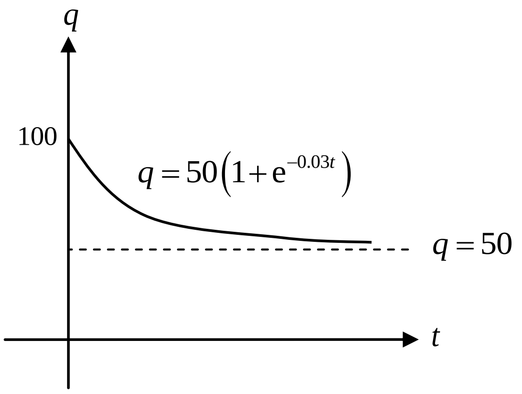
Amount at the end of year 2040 $=1000\left[(1.08)+(1.08)^2+...+(1.08)^{26}\right]$ $=1000\left\{\frac{1.08\left[1-(1.08)^{26}\right]}{1-1.08}\right\}$ $=86351$ (to nearest dollar) **Q6(ii)** $S_n=\underbrace{1000+1080+1160+...}_{n\text{ terms}}>86351$ $S_n=\frac{n}{2}\left[2(1000)+(n-1)(80)\right]>86351$ $\Rightarrow 40n^2+960n-86351>0$ $\Rightarrow n<-59.987$ (N.A.) or $n>35.987$ $\therefore$ Least number of years that he still needs to save = 36 The year at which Mr Woo's savings in this savings plan will first exceed \$86351 $=2015+36-1=2050$	
Q7 Rate of salt flowing into tank per minute is $12\times(0.125)=1.5$ kg Rate of salt flowing out per minute is $\frac{12}{400}\times q=0.03q$ Therefore, $\frac{dq}{dt}=1.5-0.03q$. $\frac{dq}{dt}=1.5-0.03q$ $\int\frac{1}{1.5-0.03q}\,dq=\int 1\,dt$ $-\frac{1}{0.03}\ln\left\|1.5-0.03q\right\|=t+C$ $\left\|1.5-0.03q\right\|=Ae^{-0.03t}$ $1.5-0.03q=Be^{-0.03t}$ When $t=0$, $q=100$, $1.5-0.03(100)=B$ $B=-1.5$ $1.5-0.03q=-1.5e^{-0.03t}$ 1.6 kg per litre = 0.16×400 = 64 kg of salt in the tank Thus $1.5-0.03(64)=-1.5e^{-0.03t}$ $t=42.4$ min **(3 s.f)**	

$0.03q = 1.5\left(1+\mathrm{e}^{-0.03t}\right)$

$q = 50\left(1+\mathrm{e}^{-0.03t}\right)$

When t is large, $\mathrm{e}^{-0.03t} \to 0$

Thus, the amount of salt in the tank decreases to 50kg.



Q8(i)

$\text{LHS} = \left((x+\mathrm{i}\,y)^2\right)^*$

$= \left(x^2 + (\mathrm{i}\,y)^2 + 2xy\,\mathrm{i}\right)^*$

$= \left(x^2 - y^2 + 2xy\,\mathrm{i}\right)^*$

$= x^2 - y^2 - 2xy\,\mathrm{i}$

$\text{RHS} = \left((x+\mathrm{i}\,y)^*\right)^2$

$= \left(x - \mathrm{i}\,y\right)^2$

$= x^2 + \left(\mathrm{i}\,y\right)^2 - 2xy\,\mathrm{i}$

$= x^2 - y^2 - 2xy\,\mathrm{i}$

Q8(ii) Let $z = x + \mathrm{i}\,y$, where $x, y \in \mathbb{R}$

$(x+\mathrm{i}\,y)^2 = 1 - 4\sqrt{3}\,\mathrm{i}$

$$\Rightarrow \begin{cases} x^2 - y^2 = 1 & -(1) \\ 2xy = -4\sqrt{3} & -\ (2) \end{cases}$$

$(2) \Rightarrow y = \dfrac{-2\sqrt{3}}{x}$

$(1) \Rightarrow x^2 - \dfrac{12}{x^2} = 1 \Rightarrow x^4 - x^2 - 12 = 0 \Rightarrow (x^2-4)(x^2+3) = 0$

$x \in \mathbb{R} \Rightarrow x^2 \geq 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2, \quad y = \mp\sqrt{3}$

$\therefore z = 2 - \sqrt{3}\,\mathrm{i} \quad \text{or} \quad -2 + \sqrt{3}\,\mathrm{i}$

Q8(iii) $w^2 = 4 + 16\sqrt{3}\,\mathrm{i} = 4(1 + 4\sqrt{3}\,\mathrm{i})$

* both sides: $\left(w^2\right)^* = 4(1 - 4\sqrt{3}\,\mathrm{i})$

using (i) : $\left(w^*\right)^2 = 4(1 - 4\sqrt{3}\,\mathrm{i})$

using (ii) : $w^* = \sqrt{4}\ (2 - \sqrt{3}\,\mathrm{i}) \quad \text{or} \quad \sqrt{4}\ (-2 + \sqrt{3}\,\mathrm{i})$

$= \ 4 - 2\sqrt{3}\,\mathrm{i} \quad \text{or} \quad -4 + 2\sqrt{3}\,\mathrm{i}$

$\therefore w = \ 4 + 2\sqrt{3}\,\mathrm{i} \quad \text{or} \quad -4 - 2\sqrt{3}\,\mathrm{i}$

Q8(iv) $z_1 = 2 - \text{Ö}\,3\,\mathrm{i}$ and $z_2 = -2 + \text{Ö}\,3\,\mathrm{i}$

Given: $\arg\left(z^2\right) = \theta$.

$$\begin{aligned}
\arg\left(z_1 z_2\right) &= \arg\left[\left(-2+\sqrt{3}\mathrm{i}\right)\left(2-\sqrt{3}\mathrm{i}\right)\right] \\
&= \arg\left(-4+4\sqrt{3}\mathrm{i}+3\right) \\
&= \arg\left(-1+4\sqrt{3}\mathrm{i}\right) \\
&= \arg\left(-z^2\right) \\
&= \arg\left(-1\right)+\arg\left(-z^2\right) \\
&= \theta+\pi
\end{aligned}$$

<u>Alternative</u>

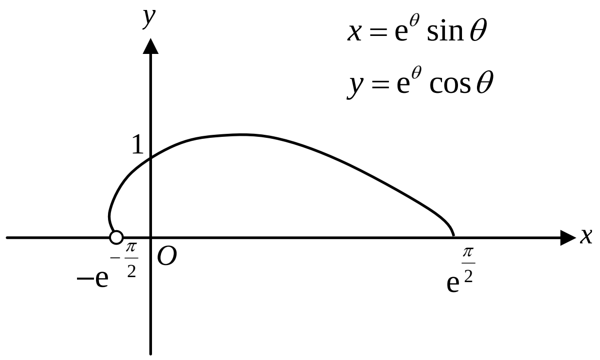
Given: $\arg\left(z^2\right) = \theta$ (where $\theta < 0$)

$$\begin{aligned}
\arg\left(z_1 z_2\right) &= \arg\left(z_1\right)+\arg\left(z_2\right) \\
&= \frac{\theta}{2}+\left(\frac{\theta}{2}+\pi\right) \\
&= \theta+\pi
\end{aligned}$$

Q9(i)

$x = \mathrm{e}^{\theta}\sin\theta$

$y = \mathrm{e}^{\theta}\cos\theta$



Q9(ii) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0.1,\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = ?$ at $x = \dfrac{1}{2}\mathrm{e}^{\frac{\pi}{6}}$

$$\mathrm{e}^{\theta}\sin\theta = \frac{1}{2}\mathrm{e}^{\frac{\pi}{6}} \Rightarrow \theta = \frac{\pi}{6}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{\theta}\cos\theta - \mathrm{e}^{\theta}\sin\theta}{\mathrm{e}^{\theta}\cos\theta + \mathrm{e}^{\theta}\sin\theta} = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\cos\left(\frac{\pi}{6}\right)-\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{6}\right)}\times\left(0.1\right) \\
&= \frac{\sqrt{3}/2 - 1/2}{\sqrt{3}/2 + 1/2}\left(0.1\right) \\
&= \frac{\sqrt{3}-1}{10\left(\sqrt{3}+1\right)}
\end{aligned}$$

Q9(iii) $\dfrac{dy}{dx}=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

At point P , tangent // y-axis

$\Rightarrow \dfrac{dy}{dx}$ is undefined

$\Rightarrow \cos\theta+\sin\theta=0$

$\Rightarrow \tan\theta=-1$

$\Rightarrow \theta=-\dfrac{\pi}{4} \quad \left(\because -\dfrac{\pi}{2}<\theta \leq \dfrac{\pi}{2}\right)$

$x=e^{-\frac{\pi}{4}}\sin\left(-\dfrac{\pi}{4}\right)=-\dfrac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}$

Equation of tangent at point P is

$x=-\dfrac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}$

Q9(iv)

Coordinates of P: $\left(-\dfrac{1}{\sqrt{2}}e^{-\frac{\pi}{4}},\ \dfrac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}\right)$

$OP=\sqrt{\dfrac{1}{2}e^{-\frac{\pi}{2}}+\dfrac{1}{2}e^{-\frac{\pi}{2}}}=e^{-\frac{\pi}{4}}$

Note that P lies on the line $y=-x$ and $OP\perp OQ$,
then Q lies on the line of $y=x$.

$e^{\theta}\sin\theta=e^{\theta}\cos\theta$

$\tan\theta=1$

$\theta=\dfrac{\pi}{4}$

$x=e^{\frac{\pi}{4}}\sin\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}e^{\frac{\pi}{4}}$

$y=e^{\frac{\pi}{4}}\cos\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}e^{\frac{\pi}{4}}$

$OQ=\sqrt{\dfrac{1}{2}e^{\frac{\pi}{2}}+\dfrac{1}{2}e^{\frac{\pi}{2}}}=e^{\frac{\pi}{4}}$

Area of $\Delta POQ=\dfrac{1}{2}(OP)(OQ)=\dfrac{1}{2}\left(e^{-\frac{\pi}{4}}\right)\left(e^{\frac{\pi}{4}}\right)=\dfrac{1}{2}$ units^2

Q10(i) $v_n = u_1 + u_2 + u_3 + ... + u_n$

$$= \frac{A}{1!} - \frac{2A^2}{2!} + \cancel{\frac{A^3}{3!}}$$
$$+ \frac{A^2}{2!} - \cancel{\frac{2A^3}{3!}} + \cancel{\frac{A^4}{4!}}$$
$$+ \cancel{\frac{A^3}{3!}} - \cancel{\frac{2A^4}{4!}} + \cancel{\frac{A^5}{5!}}$$
$$+$$
$$.$$
$$.$$
$$+ \cancel{\frac{A^{n-2}}{(n-2)!}} - \cancel{\frac{2A^{n-1}}{(n-1)!}} + \cancel{\frac{A^n}{n!}}$$
$$+ \cancel{\frac{A^{n-1}}{(n-1)!}} - \cancel{\frac{2A^n}{n!}} + \frac{A^{n+1}}{(n+1)!}$$
$$+ \cancel{\frac{A^n}{n!}} - \frac{2A^{n+1}}{(n+1)!} + \frac{A^{n+2}}{(n+2)!}$$
$$= A - \frac{2A^2}{2!} + \frac{A^2}{2!} + \frac{A^{n+1}}{(n+1)!} - \frac{2A^{n+1}}{(n+1)!} + \frac{A^{n+2}}{(n+2)!}$$
$$= A - \frac{A^2}{2} - \frac{A^{n+1}}{(n+1)!} + \frac{A^{n+2}}{(n+2)!} \quad \text{(shown)}$$

Q10(ii) $\sum_{n=2}^{N} \left\{ \frac{1}{N}\left(v_n + \frac{A^{n+1}}{(n+1)!} - \frac{A^{n+2}}{(n+2)!} \right) + 7^{n-N} \right\}$

$$= \sum_{n=2}^{N} \left\{ \frac{1}{N}\left(A - \frac{A^2}{2} \right) + 7^{n-N} \right\}$$
$$= \frac{1}{N}\left(A - \frac{A^2}{2} \right)(N-1) + \frac{1}{7^N}\sum_{n=2}^{N} 7^n$$
$$= \left(\frac{N-1}{N} \right)\left(A - \frac{A^2}{2} \right) + \frac{1}{7^N}\left[\frac{7^2(1-7^{N-1})}{1-7} \right]$$
$$= \left(\frac{N-1}{N} \right)\left(A - \frac{A^2}{2} \right) + \frac{1}{7^N} . \frac{7^2}{6} . \left(7^{N-1} - 1 \right)$$
$$= \left(\frac{N-1}{N} \right)\left(A - \frac{A^2}{2} \right) + \frac{49}{6}\left(\frac{1}{7} - \frac{1}{7^N} \right)$$

$$\frac{N-1}{N} = 1 - \frac{1}{N}$$

As $N \to \infty$, $\frac{1}{N} \to 0$, so $1 - \frac{1}{N} \to 1$. Also, $\frac{1}{7^N} \to 0$.

Hence as $N \to \infty$, Series $\to A - \frac{A^2}{2} + \frac{7}{6}$

$\sum_{n=2}^{N} \left\{ \frac{1}{N}\left(v_n + \frac{A^{n+1}}{(n+1)!} - \frac{A^{n+2}}{(n+2)!} \right) + 7^{n-N} \right\}$ converges.

Limit = $A - \frac{A^2}{2} + \frac{7}{6}$.

Q11) Let M be a point on plane π.

$$\overrightarrow{OM}=\begin{pmatrix}0\\0\\13\end{pmatrix},\overrightarrow{MP}=\begin{pmatrix}1\\2\\-1\end{pmatrix},$$

$$\frac{\left|\begin{pmatrix}1\\2\\-1\end{pmatrix}.\begin{pmatrix}a\\-2\\1\end{pmatrix}\right|}{\sqrt{a^2+5}}=1$$

$$\sqrt{a^2+5}=|a-4-1|$$

$$(a^2+5)=a^2-10a+25$$

$$a=2$$

Q11ii) $l_1:\underset{\sim}{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$**,** $\lambda\in\mathbb{R}$

$$l_2:\underset{\sim}{r}=\begin{pmatrix}4\\0\\5\end{pmatrix}+\mu\begin{pmatrix}1\\1\\0\end{pmatrix},\ \mu\in\mathbb{R}$$

$$\overrightarrow{OA}=\begin{pmatrix}2+4\lambda\\1+3\lambda\\3-2\lambda\end{pmatrix},\quad \overrightarrow{OB}=\begin{pmatrix}4+\mu\\\mu\\5\end{pmatrix}$$

$$\overrightarrow{AB}=\begin{pmatrix}2+\mu-4\lambda\\\mu-3\lambda-1\\2+2\lambda\end{pmatrix}$$

$$\overrightarrow{AB}.\begin{pmatrix}1\\1\\0\end{pmatrix}=0,\qquad \overrightarrow{AB}.\begin{pmatrix}4\\3\\-2\end{pmatrix}=0$$

$$1=-2\mu+7\lambda,\qquad 1=-7\mu+29\lambda$$

Solving,

$$\therefore\ \mu=-\frac{22}{9},\quad \lambda=-\frac{5}{9}$$

$$\overrightarrow{OA}=\begin{pmatrix}-\frac{2}{9}\\-\frac{2}{3}\\\frac{37}{9}\end{pmatrix},\quad \overrightarrow{OB}=\begin{pmatrix}\frac{14}{9}\\-\frac{22}{9}\\5\end{pmatrix}$$

Q11(ii)

$$\left[\begin{pmatrix}4\\0\\5\end{pmatrix}+\mu\begin{pmatrix}1\\1\\0\end{pmatrix}\right].\begin{pmatrix}2\\-2\\1\end{pmatrix}=8+2\mu-2\mu+5=13$$

Hence l_2 is in plane π.

Alternative

$l_2: \quad r = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 2 - 2 = 0$

$\Rightarrow l_2$ is perpendicular to $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

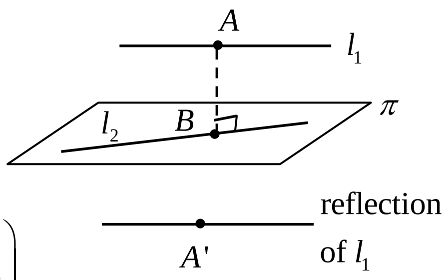
$\Rightarrow l_2$ is parallel to π.

$\begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 8 + 5 = 13$

$\Rightarrow$ a point in l_2 is also in π.

Hence l_2 is in plane π.

Q11(ii)



$\overrightarrow{OB} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$

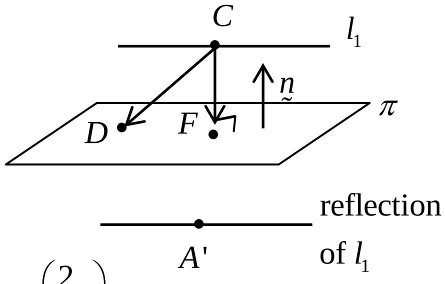
$\overrightarrow{OA'} = 2 \begin{pmatrix} \frac{14}{9} \\ -\frac{22}{9} \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{2}{9} \\ -\frac{2}{3} \\ \frac{37}{9} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 30 \\ -38 \\ 53 \end{pmatrix}$

Line of reflection of l_1 in π: $\quad r = \frac{1}{9} \begin{pmatrix} 30 \\ -38 \\ 53 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}, \beta \in \mathbb{R}$

Alternative

Let D be the point $(0,0,13)$ on π

and C be the point $(2,1,3)$ on l_1



$\overrightarrow{CD} = \begin{pmatrix} 0 \\ 0 \\ 13 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 10 \end{pmatrix}$

$\overrightarrow{CF} = \left[\begin{pmatrix} -2 \\ -1 \\ 10 \end{pmatrix} \cdot \frac{1}{\sqrt{9}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{9}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{8}{9} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\overrightarrow{OF} = \overrightarrow{CF} + \overrightarrow{OC} = \frac{8}{9} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 34 \\ -7 \\ 35 \end{pmatrix}$

$\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\overrightarrow{OA'}=2\begin{pmatrix}\frac{34}{9}\\-\frac{7}{9}\\\frac{35}{9}\end{pmatrix}-\begin{pmatrix}2\\1\\3\end{pmatrix}=\frac{1}{9}\begin{pmatrix}50\\-23\\43\end{pmatrix}$ Line of reflection of l_1 in π**:** $r=\frac{1}{9}\begin{pmatrix}50\\-23\\43\end{pmatrix}+\beta\begin{pmatrix}4\\3\\-2\end{pmatrix}, \beta\in\mathbb{R}$ **Q11(iii)** $n=\begin{pmatrix}2\\-2\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\1\\4\end{pmatrix}$ $r.\begin{pmatrix}-1\\1\\4\end{pmatrix}=\begin{pmatrix}4\\0\\5\end{pmatrix}.\begin{pmatrix}-1\\1\\4\end{pmatrix}$ Equation of plane p is $-x+y+4z=16$	