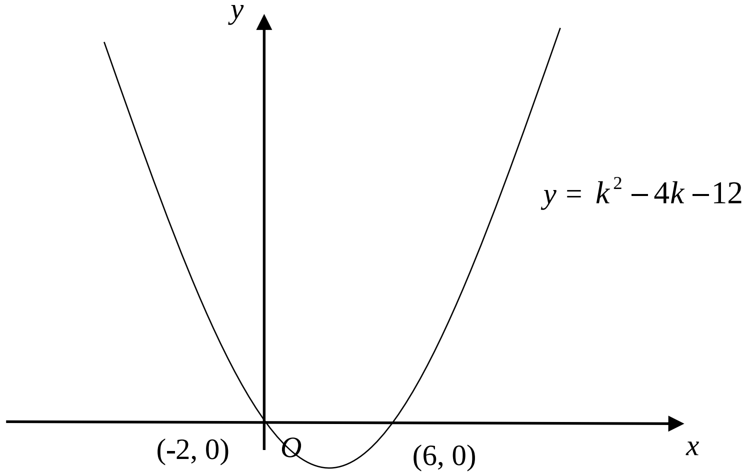
2017 H1 A Level Mathematics Solution

Section A

<table>
<tr><td>Q1.</td><td>For $x^2+(k-4)x-(k-7)>0$ for all real values of x,
Discriminant :
$b^2-4ac<0$
$(k-4)^2-4\left(-(k-7)\right)<0$
$(k^2-8k+16)+4k-28<0$
$k^2-4k-12<0$
$(k+2)(k-6)<0$


The set of values of k is $\{k : k\in\mathbb{R}, -2<k<6\}$.</td></tr>
<tr><td>Q2.</td><td>(i) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{5x-2}}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left((5x-2)^{-\frac{1}{2}}\right)=\left(-\frac{1}{2}\right)(5x-2)^{-\frac{3}{2}}(5)=-\frac{5}{2(5x-2)^{\frac{3}{2}}}$

(ii) $\int\frac{(2x^2-1)^2}{x^2}\,\mathrm{d}x=\int\frac{(4x^4-4x^2+1)}{x^2}\,\mathrm{d}x$
$=\int(4x^2-4+x^{-2})\,\mathrm{d}x$
$=\frac{4}{3}x^3-4x-\frac{1}{x}+c$ where c is an arbitrary constant</td></tr>
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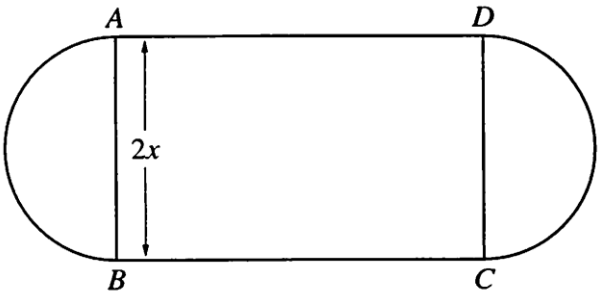
Q3.	

(i) Area of the sign

= Area of 2 half circles with diameters *AB* and *DC* + Area of rectangle *ABCD*
$= \pi x^2 + 2x(\text{AD})$
To find the length of AD,
Perimeter of the sign = $2\pi x + 2(\text{AD})$ = 10 (Given)

AD = $\frac{1}{2}(10 - 2\pi x) = 5 - \pi x$

Area of the sign = $\pi x^2 + 2x(5 - \pi x) = 10x - \pi x^2 = x(10 - \pi x)\,\text{cm}^2$ (Shown)

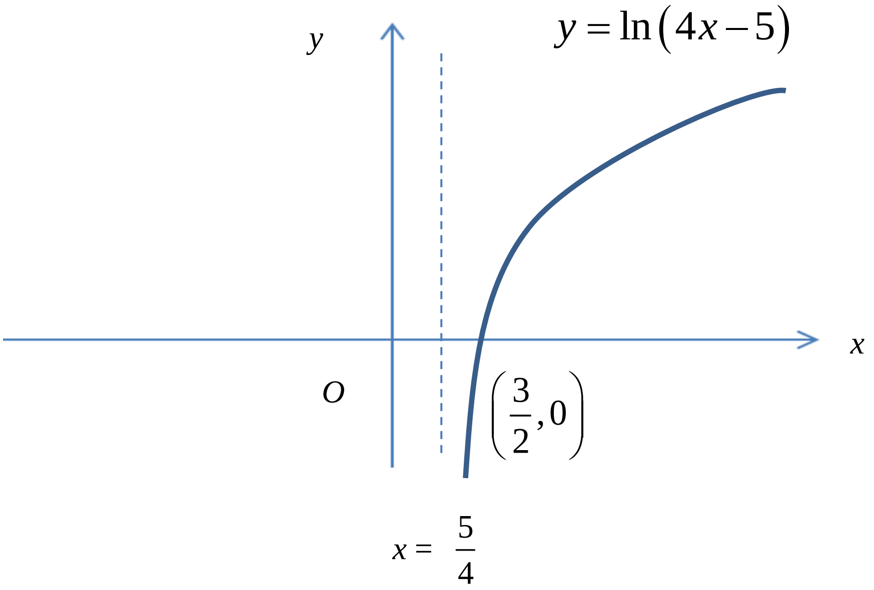
(ii) For a non-calculator method,
$y = 10x - \pi x^2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 10 - 2\pi x$
For stationary points,
$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
$10 - 2\pi x = 0$
$x = \frac{10}{2\pi} = \frac{5}{\pi}$

For $x = \frac{5}{\pi}$

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2\pi$ (< 0)

Hence the stationary value at $x = \frac{5}{\pi}$ gives a maximum area.

	Required Area $=10x-\pi x^2$ $=10\left(\frac{5}{\pi}\right)-\pi\left(\frac{5}{\pi}\right)^2$ $=\frac{50}{\pi}-\frac{25}{\pi}$ $=\frac{25}{\pi}\text{ units}^2$ The maximum value of this area is $\frac{25}{\pi}$ units^2.
Q4.	(i)  At x axis intercept (y = 0) $\ln(4x-5)=0$ $4x-5=e^0$ $4x-5=1$ $x=\frac{3}{2}$ The coordinates of point of intersection with the axes are $\left(\frac{3}{2},0\right)$. Asymptote : $x=\frac{5}{4}$.[Found by setting $4x-5=0$]

(ii) When $x = 2.5$, $y = \ln 5$.

$$y = \ln(4x-5)$$

$$\frac{dy}{dx} = \frac{1}{4x-5} \times 4 \qquad \left[\text{where } \frac{d}{dx}(4x-5) = 4\right]$$

$$\left.\frac{dy}{dx}\right|_{x=2.5} = \frac{4}{4(2.5)-5} = \frac{4}{5}$$

Equation of Tangent to the curve at the point $x = 2.5$,

$$y - (\ln 5) = \frac{4}{5}\left[x - \left(\frac{5}{2}\right)\right]$$

$$y = \frac{4}{5}x - 2 + \ln 5$$

$$5y = 4x - 10 + 5\ln 5$$

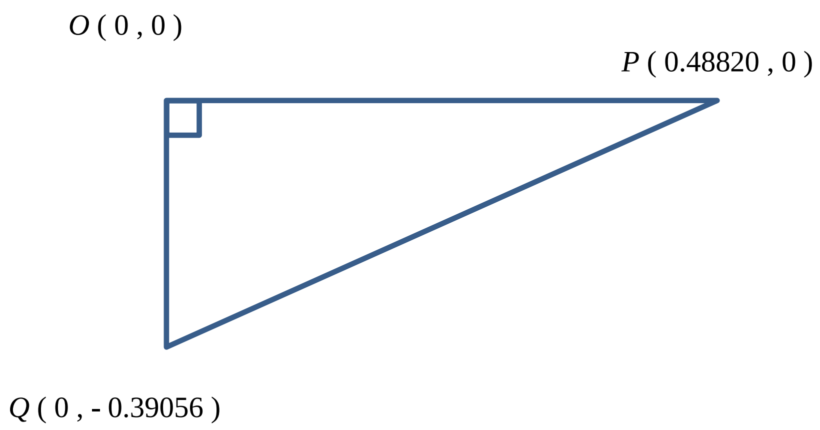
$$-4x + 5y = 5\ln 5 - 10$$

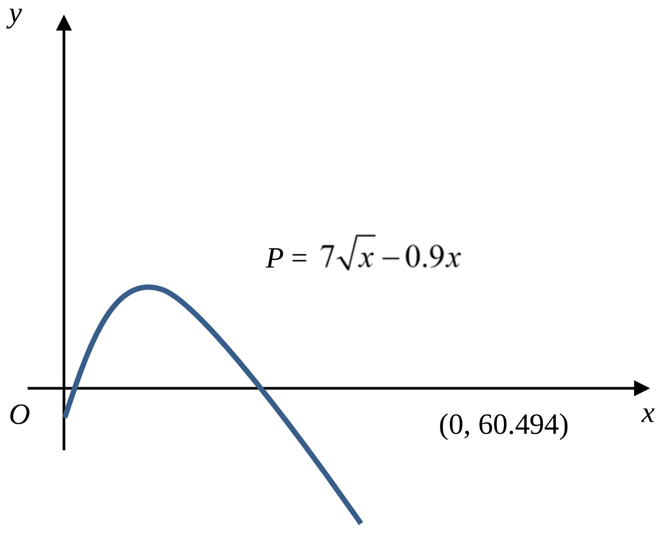
where $a = -4, b = 5, c = 5\ln 5 - 10$

(iii)

When $y = 0$, $x = \dfrac{10 - 5\ln 5}{4}$, point A is $\left(\dfrac{10-5\ln 5}{4}, 0\right)$ or $(0.48820, 0)$

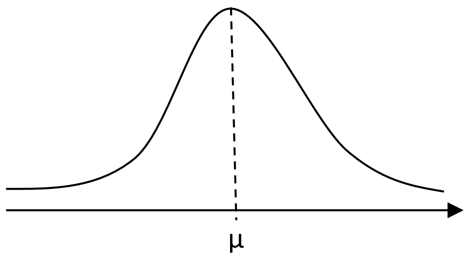
When $x = 0$, $y = \dfrac{5\ln 5 - 10}{5} = \ln 5 - 2$, point B is $(0, \ln 5 - 2)$ or $(0, -0.39056)$



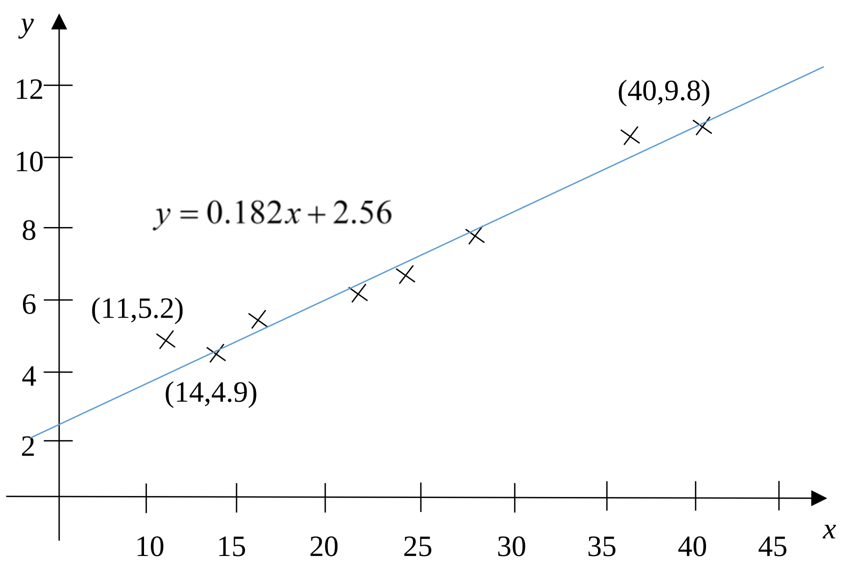
	By Pythagoras' Theorem, Length of PQ $= \sqrt{(0-\ 0.39056)^2 + (0.48820-0)^2}$ $= \sqrt{0.39088}$ $= 0.62520$ $= 0.625(3\text{ sig. fig})$
Q5.	(i) Let x, y, z be the manufacturing cost of a pair of Supers, Runners and Walkers respectively. $x = 2z$ ....(1) $10y - 7x = 50$ ....(2) $2x + 6y + 4z = 481$ ....(3) Rearranging $x - 2z = 0$ ....(1) $-7x + 10y = 50$ ....(2) $2x + 6y + 4z = 481$ ....(3) Using GC, $x = 55, \quad y = 43.5, \quad z = 27.5$ The manufacturing cost of a pair of Runners is \$43.50. (ii) Profit for 100 pairs of Supers = 100 [(80 – 55)] = \$2500 Profit for 100 pairs of Runners =100[(80 – 43.50)] = \$3650 Profit for 100 pairs of Walkers =100[(80 – 27.5)] = \$5250 (iii)  The coordinates of the points where the curve crosses the x-axis are (0, 0) and (0, 60.494).

	(iv) Using GC, The maximum point is (15.123, 13.611). Hence, the maximum value of *P* is \$13.61 when the value of *x* is \$15.12. (v) Using GC, when $x = 55$, $P = 2.4134$. Selling price is \$55 + \$2.41 (Profit) = \$57.41 The selling price of a pair of Extremes is \$57.41. (vi) When $x = 65$, $P = -2.064$ which indicates that the company will lose \$2.06 for every pair of Extremes produced. Hence, we will not advise the company to produce Extremes when the cost of a pair of Extremes increased to \$65.

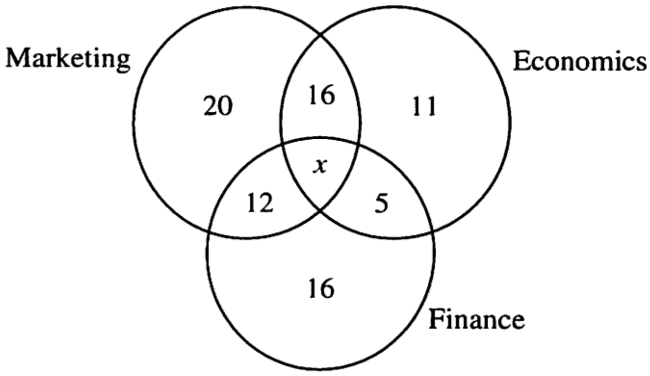
Section B

Q6.	Let X be the random variable of the height in meters of an adult male in a particular country. $$X \sim N(\mu, \sigma)$$ Given that $P(X > 1.75) = \frac{30}{100}$ and $P(X < 1.6) = \frac{20}{100}$  $P(X > 1.75) = \frac{30}{100}$ $P(X \le 1.75) = 1 - \frac{30}{100}$ $P\left(\frac{X-\mu}{\sigma} \le \frac{1.75-\mu}{\sigma}\right) = \frac{70}{100}$ $P\left(Z \le \frac{1.75-\mu}{\sigma}\right) = \frac{70}{100}$ $\frac{1.75-\mu}{\sigma} = 0.524405101$ $\mu + 0.524405101\sigma = 1.75$ (1) $P(X < 1.6) = \frac{20}{100}$ $P\left(\frac{X-\mu}{\sigma} < \frac{1.6-\mu}{\sigma}\right) = \frac{20}{100}$ $P\left(Z \le \frac{1.6-\mu}{\sigma}\right) = \frac{20}{100}$ $\frac{1.6-\mu}{\sigma} = -0.8416212335$ $\mu - 0.8416212335\sigma = 1.6$ (2) Solving equations (1) and (2), $\mu = 1.6924 \approx 1.69$ and $\sigma = 0.10981 \approx 0.110$ and $\sigma^2 = 0.012058 \approx 0.0121$. The mean of the distribution is 1.69 and the variance of the distribution is 0.0121.
Q7	Let X be the r.v. number of the ink cartridges that will last for one week or more, out of 8 cartridges. $X \sim B(8, 0.7)$ (i) $P(X = 5) = 0.25412184 = 0.254$ Let Y be the r.v. number of the ink cartridges that will less than one week or more, out of 8 cartridges. $Y \sim B(8, 0.3)$ (ii) Required probability = $P(Y \ge 4) = 1 - P(Y < 4) = 1 - P(Y \le 3) = 0.19410435 = 0.194$ Let W be the r.v. number of boxes of ink cartridges in stock out of 6 boxes of ink cartridges. $W \sim B(6, 0.19410435)$ (iii) Required probability = $P(W \le 2) = 0.90823 = 0.908$

Q8	(i) Number of codes that can be formed = ${}^6P_3\,{}^8P_3 = 40320$ (iia) P (contains the digit 5 exactly once and the letter H exactly twice) = $\dfrac{\left[{}^5C_1 \times \dfrac{3!}{2!} + {}^5C_2 \times 3!\right] \times \left[7 \times \dfrac{3!}{2!}\right]}{6^3 \times 8^3} = \dfrac{1575}{110592} = 0.0142$ Strategy for solving this part: Case 1: Number of ways where the code chosen at random where it contains the digit 5 exactly once with another two identical digits (eg. 511, 522, etc) = ${}^5C_1 \times \dfrac{3!}{2!}$ Case 2: Number of ways where the code chosen at random where it contains the digit 5 exactly once with another two different digits (eg. 512, 513, etc) = ${}^5C_2 \times 3!$ Case 3: Number of ways where the code chosen at random where it contains the letter H exactly twice with another one other letter (eg. HHA, HHB, etc) = $7 \times \dfrac{3!}{2!}$ **Alternative solution:** P (contains the digit 5 exactly once and the letter H exactly twice) = $\left[\dfrac{1}{6} \times \dfrac{5}{6} \times \dfrac{4}{6} \times 3 + \dfrac{1}{6} \times \dfrac{5}{6} \times \dfrac{1}{6} \times 3\right] \times \dfrac{1}{8} \times \dfrac{1}{8} \times \dfrac{7}{8} \times 3 = 0.0142$ *Explanation notes:* Case 1: Probability of the code chosen at random where it contains the digit 5 exactly once with another two identical digits (eg. 511, 522, etc) = $\dfrac{1}{6} \times \dfrac{5}{6} \times \dfrac{1}{6} \times 3$ Case 2: Probability of the code chosen at random where it contains the digit 5 exactly once with another two different digits (eg. 512, 513, etc) = $\dfrac{1}{6} \times \dfrac{5}{6} \times \dfrac{4}{6} \times 3$ Case 3: Probability of the code chosen at random where it contains the letter H exactly twice with another one other letter (eg. HHA, HHB, etc) = $\dfrac{1}{8} \times \dfrac{1}{8} \times \dfrac{7}{8} \times 3$

	(iib) P (has 2 as its first character) = $\frac{1}{6}$ P (has H as its sixth character) = $\frac{1}{8}$ P (has 2 as its first character and H as its sixth character) = $\frac{1}{6\times8}=\frac{1}{48}$ P (has 2 as its first character or H as its sixth character, but not both) $=\frac{1}{6}+\frac{1}{8}-2\left(\frac{1}{48}\right)=\frac{1}{4}$. **Alternative solution:** Case 1: 2 as its first character but not H as its sixth character P (has 2 as its first character but not H as its sixth character) = $\frac{1}{6}\times\frac{7}{8}=\frac{7}{48}$ Case 2: Not 2 as its first character but H as its sixth character P (not 2 as its first character but H as its sixth character) = $\frac{5}{6}\times\frac{1}{8}=\frac{5}{48}$ P (has 2 as its first character or H as its sixth character, but not both) $=\frac{7}{48}+\frac{5}{48}=\frac{12}{48}=\frac{1}{4}$.
Q9	 (ii) The product moment correlation coefficient, *r* is 0.978.

	The weekly earnings, *y* hundred dollars and the employees from the production line who have been with the company for *x* years has a strong positive linear relationship as *r* = 0.978 is close to 1. Moreover this is supported by the the scatter diagram which shows an increasing trend as the number of years at the company increases the weekly earnings in hundred dollars increases. (iii) The required equation is $y = 0.182x + 2.56$ where *a* = 0.182 and *b* = 2.56. (iv) When $x = 2$, $y = 2.9286 \approx 2.93$. Therefore the estimated of the weekly earnings for employees on the production line who have been with the company for 2 years is $293. (v) In the given data set, Employee C while having a shorter length of service with the company is earning more per week than Employee A. While this relationship may be true in general, but this may not be true for individual employees. The linear model is not appropriate here. Since it suggests as the number of years increases, the weekly earnings will increase proportionately, which is not realistic. The estimate is unreliable as the value of $x = 2$ years is out of the given data range, the number of years that the employees from the production line who have been with the company, i.e. the estimate is an extrapolation.
Q10	(i) Let *X* be the volume of a certain type of juice (in litres) in a bottle and μ be the population mean of the volumes of certain type of juice (in litres) in bottles. $H_0 : \mu = 0.6$ $H_1 : \mu < 0.6$ Under H_o, since *n* = 50 is large, by the Central Limit Theorem, $\overline{X} \sim N\left(0.6, \frac{0.01528}{50}\right)$ approximately. *z*-value $= -1.8305$ *p*-value $= 0.033586$ The probability that the sample mean volume of juice (in litres) in the bottles are measured is less than 0.568 litres, given that the population mean volume of juice (in litres) in the bottles is 0.6, is *p* = 0.033586 . There is a probability of 0.033586 of observing a result equal to or more extreme than that is actually observed when the mean volume is indeed 0.6. (ii) It is not necessary to assume a normal distribution for the test to be valid. Since *n* = 50 is large, by Central Limit Theorem, the mean volume in a bottle of juice is normally distributed approximately. (iii) Unbiased estimate of the population mean $\bar{y} = \frac{70.4}{110} = 0.64$

	Unbiased estimate of the population variance $= \frac{1}{109}\left(49.42 - \frac{70.4^2}{110}\right) = 0.0400366972 \approx 0.0400$ Let Y be the volume of certain type of juice (in litres) in a bottle and μ be the population mean of the volumes of certain type of juice (in litres) in bottles. $H_0 : \mu = 0.6$ $H_1 : \mu \neq 0.6$ Under H_0, since n = 110 is large, by the Central Limit Theorem, $\overline{Y} \sim N\left(0.6, \frac{0.0400366972}{110}\right)$ approximately, Use a two tailed z-test at 5 % level of significance, Test Statistic: $Z = \frac{\overline{X} - 0.6}{s/\sqrt{110}} \sim N(0,1)$ Using GC, the p-value = $0.036024 \leq 0.05$ Using GC, the z-value = $2.0967 > 1.96\,[invNorm(0.975)]$ Reject H_0 . There is sufficient evidence at 5% significance level to conclude that the population mean volume of juice differs from 0.6 litres.
Q11	 (i) $P(M) = \frac{20+16+x+12}{20+16+x+12+11+5+16} = \frac{48+x}{80+x}$ $P(E) = \frac{16+x+5+11}{20+16+x+12+11+5+16} = \frac{32+x}{80+x}$ (ii) M and E are independent $\Rightarrow P(M \cap E) = P(M) \times P(E)$ $\frac{16+x}{80+x} = \frac{48+x}{80+x} \times \frac{32+x}{80+x}$ $\frac{16+x}{80+x} = \frac{(48+x)(32+x)}{(80+x)^2}$

	$1=\frac{(48+x)(32+x)}{(16+x)(80+x)}$ Using GC, $x=16$ The value of x is 16. (iii) $\mathrm{P}(M\cup F')=\frac{20+16+11+12+16}{96}=\frac{75}{96}=\frac{25}{32}$ **Alternative solution:** $\mathrm{P}(M\cup F')=\frac{96-16-5}{96}=\frac{25}{32}$ (iv) If M is the event that the student studies Marketing and F is the event that the student studies Finance, where P(F) ≠ 0 and P(M) ≠ 0, In the Venn diagram representation, the possibility space for P($F \mid M$) is reduced to just M. That is, P($F \mid M$) is the probability of event F occurring (student studies in Finance) by considering event M as the sample space (student studies in Marketing). Hence, the meaning of P($F \mid M$) in the context of this question is by considering the sample of event that the student studies in Marketing, find the probability of the event that the students also studies in Finance. It is the probability that a student takes Finance given that he takes Marketing. $\mathrm{P}(F \mid M)=\frac{\mathrm{P}(F\cap M)}{\mathrm{P}(M)}=\frac{12+16}{20+16+16+12}=\frac{28}{64}=\frac{7}{16}$ (v) P(1 student studies exactly two of these three subjects) $=\frac{16+12+5}{96}=\frac{33}{96}$ P(3 students studies exactly two of these three subjects) $=\frac{33}{96}\times\frac{32}{95}\times\frac{31}{94}=\frac{341}{8930}$
Q12	Let Q be random variable of the journey time, in minutes, by bus between towns of Ayton and Beeton. $Q \sim N\left(45,4^2\right)$ Let Q be random variable of the journey time, in minutes, by train between towns of Ayton and Beeton. $R \sim N\left(42,3^2\right)$ (i) $P\left(Q<48\right)=0.77337\approx 0.773$ (ii) Required probability = $=\left[\mathrm{P}\left(Q>48\right)\right]^2=\left[1-\mathrm{P}\left(Q\le 48\right)\right]^2=\left(1-0.77337\right)^2=0.051360\approx 0.0514$

(iii) The cases found by (ii) is a subset of the cases found by *p*. Thus, *p* is greater.

Different Normal distribution, in which combinations of the total time for bus journeys is more than 96 minutes are included, not just 48 minutes twice.

The probability of two randomly chosen bus journeys more than 96 minutes include the case where a randomly chosen bus journey takes less than 48 minutes and the other randomly chosen bus journey takes more than 48 minutes and vice versa, but the total probabilities of both bus journeys are more than 96 minutes.

The total probability $= P(Q_1 < 48)P(Q_2 > 48) + P(Q_1 > 48)P(Q_2 < 48)$

$= 2(0.77337)(1 - 0.77337) = 0.35054$

The probability of two randomly chosen bus journeys each take more than 48 minutes, the total probability = $= \left[\mathrm{P}(Q > 48)\right]^2 = 0.051360$.

Hence, we conclude that of the total time for bus journeys is more than 96 minutes are included is more than just 48 minutes twice. The total time for two randomly chosen bus journeys is more than 96 minutes will be greater than the answer in part ii).

(iv)

Let $Y = Q_1 + Q_2 + Q_3 + R_1 + R_2 \sim N\left(3\times 45 + 2\times 42, 4^2 \times 3 + 3^2 \times 2\right)$

$Y \sim N(219, 66)$

$\mathrm{P}(Y > 210) = 0.86603 \approx 0.866$

(v)

	Mean	Variance
Cost of one journey from Ayton to Beeton by **Bus**	$0.12(45) = 5.4$	$0.12^2\left(4^2\right) = 0.2304$
Cost of one journey from Ayton to Beeton by **Train**	$0.15(42) = 6.3$	$0.15^2\left(3^2\right) = 0.2025$

Let *B* be random variable of the cost of one journey from Ayton and Beeton by bus.

$B \sim N(5.4, 0.2304)$

Let *T* be random variable of the cost of one journey from Ayton and Beeton by train.

$T \sim N(6.3, 0.2025)$

$\mathrm{E}(3B - 2T) = 3(5.4) - 2(6.3) = 3.6$

	$\text{Var}(3B-2T)=3^2(0.2304)+2^2(0.2025)=2.8836$ Let $C=3B-2T$. Let C be random variable of the total cost of 3 times the cost of randomly chosen one journey from Ayton and Beeton by bus exceed twice of the cost of randomly chosen one journey from Ayton to Beeton by train. $C \sim N(3.6, 2.8836)$ Required probability = $\text{P}(C<3) = \text{P}(3B-2T<3)=0.36192 \approx 0.362$ In the context of this question, $\text{P}(3B-2T<3)$ would represent the probability that the total cost of 3 times the cost of the randomly chosen bus trip from Ayton to Beeton exceed twice the cost of the randomly chosen train trip from Ayton to Beeton by \$3 is 0.362. It is the probability that 3 times the cost of one bus journey from Ayton to Beeton is less than 2 times the cost of one train journey from Ayton to Beeton by not more than \$3.