RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2018 Year 5

Chapter 10: Recurrence Relations

SYLLABUS INCLUDES:

H2 Further Mathematics

- Sequence generated by a simple recurrence relation, including the use of graphing calculator to generate the sequence defined by the recurrence relation
- Behavior of a sequence, such as the limiting behavior of a sequence
- Solution of
  (i) First order linear (homogeneous and non-homogeneous) recurrence relations with constant coefficients of the form $u_n = au_{n-1} + b, a, b \in \mathbb{R}, a \neq 0$
  (ii) Second order linear homogeneous recurrence relations with constant coefficients
- Modelling with recurrence relations of the forms above

CONTENT

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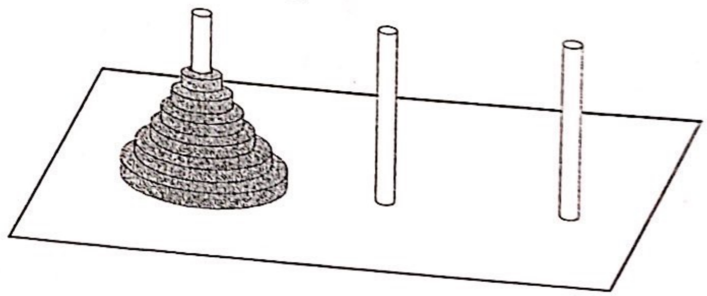
1 INTRODUCTION

The Tower of Hanoi is a popular puzzle invented by the French mathematician Édouard Lucas in 1883. The puzzle consists of three pegs mounted on a board together with disks of different sizes. Suppose initially there are r disks placed on the first peg in order of size, with the largest at the bottom (as shown). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The objective of the puzzle is to transfer all the disks in peg 1 to one of the other pegs in order of size, with the largest at the bottom.

How many moves are required? What is the relationship between the number of moves required for, say, 10 disks and for 9 disks?



In this chapter, we will learn how to solve such problems with the use of recurrence relations.

2 SEQUENCE GENERATED BY A RECURRENCE RELATION

Recall that a sequence is a set of numbers arranged in a defined order according to a certain rule. For example: the sequence that is in arithmetic progression 5, 8, 11, 14, …

A sequence can be generated by a **recurrence relation** of the form $u_{n+1} = \mathrm{f}(u_n)$, where $n \in \mathbb{Z}^+$ and the $(n+1)$th term, u_{n+1}, is linked to its previous term u_n, by a formula. If the initial condition (i.e. the value of u_1) is given, then we can determine $u_2, u_3, \ldots$ recursively by $u_2 = \mathrm{f}(u_1)$, $u_3 = \mathrm{f}(u_2)$ and so on.

For example, the sequence 5, 8, 11, 14, … can be defined by a recurrence relation of the form $u_n = u_{n-1} + 3$, with $u_1 = 5$.

Given a recurrence relation $u_{n+1} = \mathrm{f}(u_n)$, an explicit expression for the nth term of the sequence will depend not only on n but also its **initial value**.

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Example 1

A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = 3u_{n-1} + 2$, for $n = 2, 3, 4, \ldots$.
Given that $u_1 = 3$, find the values of u_4 and u_6.

Solution:

$u_1 = 3$

$u_2 = 3(3)+2 = 11$

$u_3 = 3(11)+2 = 35$

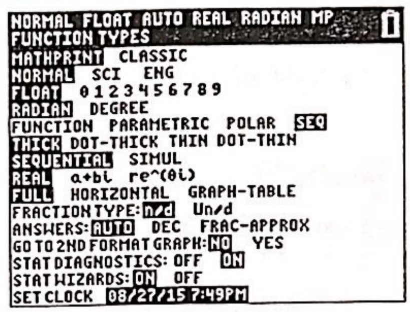
$u_4 = 3(35)+2 = 107$

$u_5 = 3(107)+2 = 323$

$u_6 = 3(323)+2 = 971$

Using the GC to obtain terms in a sequence generated by a recurrence relation

1. Press MODE and then scroll down to select **SEQ** mode.

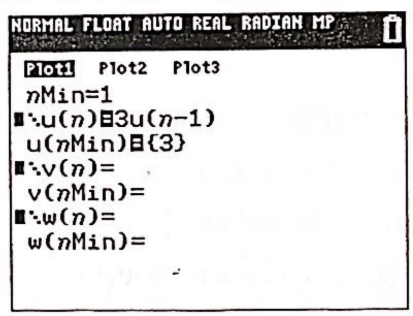


2. Press Y=.

a. Key in the starting value of n : $n\text{Min} = 1$

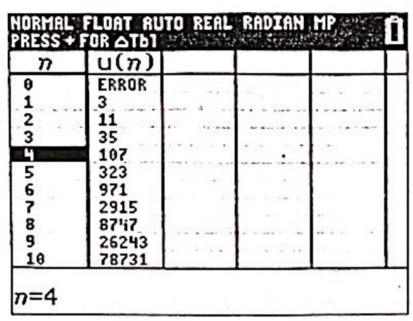
b. Key in the recurrence relation $u(n) = 3u(n-1) + 2$.
(to obtain the letter u, press 2nd 7;
to obtain the letter n, press X,T,Θ,n.)

c. Key in the initial condition: $u(n\text{Min}) = 3$



3. Press 2nd GRAPH to check the values of u_4 and u_6.

From the GC, $u_4 = 107$, $u_6 = 971$



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We can form a Mathematical model using a recurrence relation. Let's consider the following scenario.

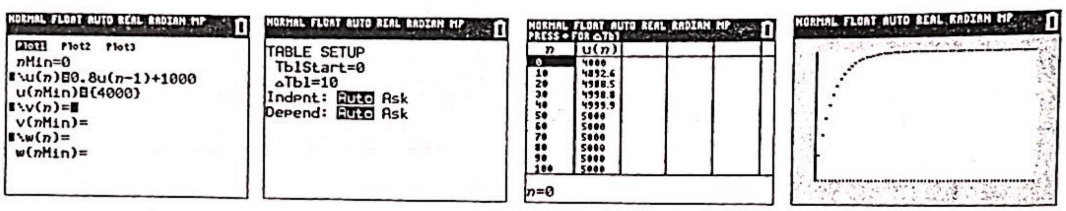
A small forest contains 4000 trees. Under a new forest management plan, 20% of the trees will be harvested each year and 1000 new trees will be planted; this pattern recurs year after year. We know that the number of trees will vary each year and is dependent on the number of trees in the previous year.

If we denote the number of trees at the end of the nth year as u_n, we see that

$$u_{n+1} = 0.8u_n + 1000.$$

Will the number of trees increase or decrease over time? Can we identify any long term behavior? Taking into account the constraint of land space, is this plan sustainable in the long run?

Since $u_{n+1} = 0.8u_n + 1000$ where u_n represents the number of trees at the end of the nth year and $u_0 = 4000$, we see from the GC that u_n forms an increasing sequence which converges to 5000. Hence, the plan is sustainable and there is no issue with the lack of land space over time.



Alternatively, we can form the explicit expression for the nth term of the sequence.

$$\begin{aligned}
u_n &= 0.8u_{n-1} + 1000 \\
&= 0.8(0.8u_{n-2} + 1000) + 1000 && [\text{by applying } u_{n-1} = 0.8u_{n-2} + 1000] \\
&= 0.8^2 u_{n-2} + 1000(1 + 0.8) && \text{simplify.} \\
&= 0.8^2(0.8u_{n-3} + 1000) + 1000(1 + 0.8) && [\text{by applying } u_{n-2} = 0.8u_{n-3} + 1000] \\
&= 0.8^3 u_{n-3} + 1000(1 + 0.8 + 0.8^2) \\
&\;\;\vdots
\end{aligned}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\begin{aligned}
&= 0.8^n u_o + 1000(1 + 0.8 + 0.8^2 + \ldots + 0.8^{n-1}) && \text{GP (apply } S_n \text{ formula).} \\
&= 0.8^n u_o + 1000\left(\frac{1-0.8^n}{1-0.8}\right) \\
&= \boxed{0.8^n u_o + 5000(1-0.8^n)} && \text{expressed in terms of initial condition \& no. of terms } (n).
\end{aligned}$$

In the above, we performed repeated substitution of the recurrence relation to solve the **1st order linear recurrence relation,** $u_n = 0.8u_{n-1} + 1000$.

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3 1st ORDER LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

3.1 Definitions

A recurrence relation is said to be **linear** if the expression is of the form no reciprocals.

$$u_n = a_{n-1}u_{n-1} + a_{n-2}u_{n-2} + \ldots + a_1u_1 + a_0,$$ no square root no power. $\boxed{u^1}$

where the a_i's are constants and not all are zeros. $u_n = 3u_{n-1} + 2u_{n-2}$

If $a_0 = 0$, we say that the recurrence relation is linear and **homogeneous.**

A recurrence relation is said to be **order** k if u_n is expressed in terms of some or all the previous k terms of the sequence, that is, u_{n-k} and no terms earlier than u_{n-k} appears in the expression. For example, $u_n = u_{n-1} + u_{n-2}$ with $u_1 = 1$ and $u_2 = 1$ is of order 2.

Example 2
For each of the recurrence relations, state the order and determine whether it is linear and if it is linear, whether it is homogeneous.

	Recurrence Relation	Order	Linear/Non-linear	Homogeneous/ Non-homogeneou
(i)	$u_n = 0.2u_{n-1} + 40$	1	linear.	non-homo.
(ii)	$u_n = u_{n-1} + n$	1	linear	non-homo
(iii)	$u_n = u_{n-2} + 1$	2	linear.	non-homo
(iv)	$x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right)$	1	non-linear.	homo
(v)	$I_n = \mathrm{e} + nI_{n-1}$	1	linear	non-homo

3.2 Solution of a 1st order linear recurrence relation

To solve a recurrence relation is to find a formula to express the general term u_n of the sequence. Knowing the solution to the recurrence relation is especially useful when we need to find the value of a certain term of the sequence efficiently; for instance the hundredth term u_{100} and where technology is not readily available. The solution also provides a better idea of the growth rate of the sequence. For example, $u_n = 2^n u_o$ as compared to $u_n = 2u_{n-1}$, where $u_0 = 200$, $n \geq 1$.

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In general, to solve for a 1st **order linear recurrence relation** of the form $\boxed{u_n = au_{n-1} + b}$ where $a \neq 0$ and $n \geq 1$,

we can perform repeated substitution of the recurrence relation as follows :

$$
\begin{aligned}
u_n &= au_{n-1} + b && \text{[by applying } u_{n-1} = au_{n-2} + b\text{]} \\
&= a(au_{n-2} + b) + b && \text{simplify.} \\
&= a^2u_{n-2} + b(1+a) && \text{[by applying } u_{n-2} = au_{n-3} + b\text{]} \\
&= a^2(au_{n-3} + b) + b(1+a) && \text{simplify.} \\
&= a^3u_{n-3} + b(1+a+a^2) \\
&\ \ \vdots \\
&= a^nu_o + b(1+a+a^2+\ldots+a^{n-1}) \\
&= a^nu_o + b\left(\frac{1-a^n}{1-a}\right)
\end{aligned}
$$

GP: $S_n = \frac{a(1-r^n)}{1-r}$.

Note that $a \neq 1$ for the above result to hold.
If $a = 1$, we have $u_n = u_0 + nb$.

Result 1A:
The solution for a 1st order linear recurrence relation of the form

$$u_n = au_{n-1} + b \text{ where } a, b \in \mathbb{R},\ a \neq 0 \text{ and } n \geq 1, \quad n \in \mathbb{Z}^+$$

is $u_n = a^nu_o + b\left(\frac{1-a^n}{1-a}\right)$, where $a \neq 1$.

If $a = 1$, $u_n = u_0 + nb$.

Exercise: $b=0$
Show that the solution for a 1st order linear homogeneous recurrence relation $u_n = au_{n-1}$ is $u_n = a^nu_0$.

$$
\begin{aligned}
u_n &= au_{n-1} \\
&= a(au_{n-2}) \\
&= a^2(au_{n-3}) \\
&= a^3(u_{n-4}) \\
&= a^n(u_0). \\
&= a^nu_0.
\end{aligned}
$$

vs $u_n = ar^{n-1}$ (for GP).

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Example 3

A sequence is given by the recurrence relation $\boxed{u_{n+1} = 0.5u_n + 25,}$ for $n = 0, 1, 2, \ldots$.

Show that $u_n = 0.5^n(u_0 - 50) + 50$.

(i) Find the limit of u_n.

(ii) State the value of u_0 which would result in a constant sequence.

Solution:

$u_n = 0.5u_{n-1} + 25$

$= 0.5(0.5u_{n-2} + 25) + 25$

$= 0.5^2 u_{n-2} + 25(0.5 + 1)$ simplify. [by applying $u_{n-1} = 0.5u_{n-2} + 25$]

$= 0.5^2(0.5u_{n-3} + 25) + 25(0.5 + 1)$

$= 0.5^3 u_{n-3} + 25(0.5^2 + 0.5 + 1)$ simplify. [by applying $u_{n-2} = 0.5u_{n-3} + 25$]

$\vdots$

$= 0.5^n u_0 + 25\left(\frac{1 - 0.5^n}{1 - 0.5}\right)$ [Result 1A]

$= 0.5^n u_0 + 50(1 - 0.5^n)$

$= 0.5^n(u_0 - 50) + 50$

This can be proven by Mathematical Induction in Chapter 8.

(i) As $n \to \infty$, $0.5^n \to 0$, $u_n \to 50$. Therefore limit of u_n is 50.

(ii) The value of u_0 which would result in a constant sequence is 50.

Recall: $u_n = \underbrace{\underbrace{0.5^n(u_0 - 50)}_{0}}_{\text{doesn't matter anymore.}} + 50$

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Example 4

A poultry farmer raised geese in his farm. Taking into consideration various conditions, such as the birth of new geese, sale of poultry, deaths caused by infection, etc. that affect the population of geese in his farm, the number of geese at the end of the nth month is modeled by u_n, where $u_{n+1}=0.75u_n+80$ and u_0 represents the initial number of geese.

(i) Find u_1, u_2 and u_3 in the cases when
(a) $u_0=576$, **(b)** $u_0=320$ and **(c)** $u_0=318$.

(ii) Using the GC, describe the behaviour of the sequence in each case.

(iii) Show that $u_n=0.75^n(u_0-320)+320$ and deduce that as n becomes very large, $u_n \to 320$.

Solution:

(i)(a) $u_1=512,\ u_2=464,\ u_3=428$

(i)(b) $u_1=320,\ u_2=320,\ u_3=320$

(i)(c) $u_1=318.5,\ u_2=318.875,\ u_3=319.15625$

Note:
The model may produce numbers that have fractional value even though the number of geese must be a whole number.

(2nd window).
change table intervals to 10

(ii)(a) When $u_0=576$, the sequence $u_1,u_2,u_3,\ldots$ strictly decreases and converges to 320.

(ii)(b) When $u_0=320$, the sequence $u_1,u_2,u_3,\ldots$ is a constant sequence.

(ii)(c) When $u_0=318$, the sequence $u_1,u_2,u_3,\ldots$ strictly increases and converges to 320.

(iii)

$$\left.\begin{aligned}
u_n &= 0.75u_{n-1}+80\\
&= 0.75(0.75u_{n-2}+80)+80\\
&= 0.75^2u_{n-2}+80(1+0.75)\\
&= 0.75^2(0.75u_{n-3}+80)+80(1+0.75)\\
&= 0.75^3u_{n-3}+80(1+0.75+0.75^2)
\end{aligned}\right\}\text{repeated substitution.}$$

$$\begin{aligned}
&= \ldots\\
&= 0.75^n u_{n-n}+80(\underline{1+0.75+0.75^2+\ldots+0.75^{n-1}})\\
&= 0.75^n u_0+80\left(\frac{1-0.75^n}{1-0.75}\right) \leftarrow \text{apply GP formula.}\\
&= 0.75^n u_0+\underline{320}(1-0.75^n)\\
&= 0.75^n(u_0-320)+320. \text{ (shown)}
\end{aligned}$$

As $n\to\infty$, $0.75^n \to 0$, so $0.75^n(u_0-320)\to 0$.

Hence $u_n=0.75^n(u_0-320)+320 \to 320$. (deduced)

generalised for all n.

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$u_1 \quad u_{r+1}$

$u_1 = 20000$

Example 5

On 1 January 2001 Mr X puts \$20000 into an educational fund, and on the 1st day of each subsequent year he makes a withdrawal of \$1000. The interest rate was 2% per annum, so that on the last day of each year the amount in the account increases by 2%.

The amount of money in the fund at the beginning of n th year after $(n-1)$th withdrawal is denoted by u_n.

(i) Write down an expression for u_{n+1} in terms of u_n and hence find an expression for u_{n+1} in terms of n.

(ii) Find the amount in the fund to the nearest dollar after the 10th withdrawal.

(iii) Calculate the number of withdrawals that can be made.

Solution:

(i) $u_{n+1} = 1.02u_n - 1000$, where $u_1 = 2000$

Using result A, we have

$u_{n+1} = (1.02)^{n} u_1 + (-1000)\left(\frac{1-1.02^n}{1-1.02}\right)$ — no. of terms -1

$= 20000(1.02)^n + 50000(1-1.02^n)$

$= 50000 - 30000(1.02)^n$

* from 1 to n+1th term, there are n+1 terms ∴ index should be n

Recall: $u_n = a^{n} u_0 + b\left(\frac{1-a^n}{1-a}\right)$ — no. of terms -1 $= n+1-1$

where $a \neq 1$

from 0 to n, there are n+1 terms ∴ index should be n.

(ii) $u_{11} = 50000 - 30000(1.02^{10})$

$= 13430$ (nearest dollar)

Alternatively, from the GC:

nMin = 1

$u(n) = 1.02u(n-1) - 1000$

$u(1) = 20000$

Recall: $u_{n+1} = 1.02u_n - 1000$

(iii) $50000 - 30000(1.02^n) < 0$

$1.02^n > \frac{5}{3}$

$n > \frac{\ln(\frac{5}{3})}{\ln 1.02} = 25.8$ — greater than 0.

∴ 25 withdrawals are possible.

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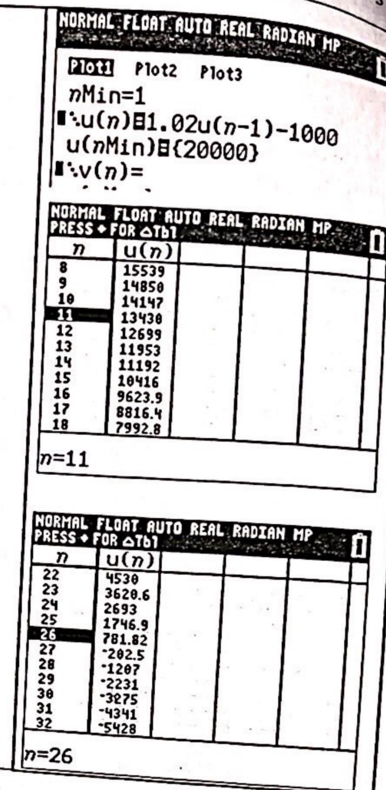

Result 1B:

The general solution for a 1st order linear recurrence relation of the form

$$u_n = au_{n-1} + b \text{ where } a,b \in \mathbb{R},\ a \neq 0,1 \text{ and } n \geq 1,$$

is $u_n = Aa^n + B$ where A and B are real constants.

Proof:

$B = \frac{b}{1-a}$

$A = u_0 - \frac{b}{1-a}$

If $a \neq 1$, the 1st order linear recurrence relation of the form $x_n = ax_{n-1} + b$ can be re-written as $\underbrace{x_n - k}_{u_n} = a\underbrace{\left(x_{n-1} - k\right)}_{u_{n-1}}$ where $k = \frac{b}{1-a}$ homogeneous!

This can be done if we let $u_n = x_n - k$ and $u_n = au_{n-1}$, a 1st order linear homogeneous recurrence relation. Equating the two and solving for k gives the required form above.

We have seen that the solution for $u_n = au_{n-1}$ is $u_n = a^n u_0$ G.P. and hence $x_n - k = a^n u_0$. G.P.

Rearranging the terms, we obtain $\boxed{x_n = u_0 a^n + k}$, a general solution for the 1st order linear recurrence relation.

no. of terms − 1

Remarks:

Result 1A is useful but is hard to remember. Instead of trying to remember the formula, we could start with the general solution of the form $x_n = Aa^n + B$.

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Alternative solution for example 5(i) is as follows:

Solution:

first order, linear, non-homogeneous relation

$$\underbrace{u_{n+1} = 1.02u_n - 1000}_{\text{first order, linear, non-homogeneous relation}} \text{ where } u_1 = 20000$$

We find $u_2 = 1.02u_1 - 1000$
$= 1.02(20000) - 1000 = 19400$

The general solution is given by $u_{n+1} = A(1.02)^n + B$

Now, $u_1 = A + B = 20000$ — (1) ($n=0$ so $1.02^0 = 1$)

and $u_2 = 1.02A + B$ 19400 — (2) ($n=1$ so $1.02^1 = 1.02$)

Solving, $A = -30000$ and $B = 50000$

$\therefore u_{n+1} = -30000(1.02)^n + 50000$

4 2nd ORDER LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

Recall that any recurrence relation of the form

otherwise it becomes 1st order R.R

in order for u_n to be defined.

$$\boxed{u_n = au_{n-1} + bu_{n-2}}, \text{ where } a, b \in \mathbb{R},\ b \neq 0 \text{ and } n \geq 2, \quad \text{———— (1)}$$

is called a **second order homogeneous linear recurrence relation.**

$u_n = au_{n-1} + bu_{n-2}$

$u_n = s_n$ $u_n = t_n$

4.1 Characteristic equation of a 2nd order linear homogeneous recurrence relation

(function in terms of n.)

Let $u_n = s_n$ and $u_n = t_n$ be two solutions for (1),

i.e., $s_n = as_{n-1} + bs_{n-2}$ and $t_n = at_{n-1} + bt_{n-2}$.

Then for constants A and B, we have

linear combination

$$As_n + Bt_n = A\overbrace{(as_{n-1} + bs_{n-2})}^{s_n} + B\overbrace{(at_{n-1} + bt_{n-2})}^{t_n}$$

$$= a\underbrace{(As_{n-1} + Bt_{n-1})}_{u_{n-1}} + b\underbrace{(As_{n-2} + Bt_{n-2})}_{u_{n-2}}$$

group $n-1$ and $n-2$ terms tgt.

which means that $\boxed{u_n = As_n + Bt_n}$ a linear combination of solutions of the homogenous recurrence relation (1) is also a solution of (1).

We thus note that **any linear combination of solutions of a homogeneous recurrence linear relation is also a solution.**

In solving the first order homogeneous recurrence relation $u_n = au_{n-1}$, we have established that the general solution is $u_n = a^n u_0$. This implies that $u_n = a^n$ is also a solution, since any linear combination of solutions is also a solution.

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This suggests that, for a second order homogeneous recurrence linear relation (1), we may have the solutions of the form $u_n = \lambda^n$.

then $u_{n-1} = \lambda^{n-1}$, $u_{n-2} = \lambda^{n-2}$.

Substituting this into equation (1), we will have

to $u_n = a u_{n-1} + b u_{n-2}$.

$$\lambda^n = a\lambda^{n-1} + b\lambda^{n-2}$$

$$\lambda^{n-2}(\lambda^2 - a\lambda - b) = 0$$

Thus, either $\lambda = 0$ or $\lambda^2 - a\lambda - b = 0$ --------- (2)

Equation (2) is called the **characteristic equation** of (1).

Eg. $u_n = 5u_{n-1} - 4u_{n-2}$

the characteristic equation is:

$\lambda^2 - 5\lambda + 4 = 0$.

$(\lambda - 4)(\lambda - 1) = 0$

$\lambda = 4$ or $\lambda = 1$

4^n and 1^n are both solutions to u_n

Expressing 4^n and 1^n as a linear combination

$\Rightarrow u_n = A4^n + B1^n$

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4.2 General solution of a 2nd order linear homogeneous recurrence relation

Result 2:

For a second order homogeneous linear recurrence relation of the form

$$u_n = au_{n-1} + bu_{n-2}, \text{ where } a, b \in \mathbb{R},\ b \neq 0 \text{ and } n \geq 2,$$

its characteristic equation is $\boxed{\lambda^2 - a\lambda - b = 0}$ and its solution is of the form

(a) $u_n = A\lambda_1{}^n + B\lambda_2{}^n$ if the characteristic equation has two distinct roots λ_1 and λ_2, (can be complex or real roots)

(b) $u_n = A\lambda^n + B\boxed{n\lambda^n}$ if the characteristic equation has only one root λ, → 2 repeated roots.

~~(c) complex~~

cannot be parallel ⇒ if λ^n is a root the $n\lambda^n$ is also a root.

where A and B are constants.

~~complex roots~~

For (a), if the roots are complex, i.e. $\lambda_1 = re^{i\theta}$ and $\lambda_2 = re^{-i\theta}$, then the solution can be written as

(a.2) $u_n = A\left(re^{i\theta}\right)^n + B\left(re^{-i\theta}\right)^n = r^n\left[(A+B)\cos(n\theta) + (A-B)i\sin(n\theta)\right]$

$= r^n\left[C\cos(n\theta) + D\sin(n\theta)\right]$, $C, D \in \mathbb{R}$.

why are A and B conjugate pairs? ⇒ eg. 8.

"because Mathematics is a beautiful system".

proof!

Proof: $z + z^* = 2\mathrm{Re}(z)$, $z - z^* = \mathrm{Re}(z)$

(a) When the characteristic equation $\lambda^2 - a\lambda - b = 0$ has two distinct roots λ_1 and λ_2, it is clear that both $u_n = \lambda_1{}^n$ and $u_n = \lambda_2{}^n$ are solutions of (1). Thus, a linear combination of these two solutions, $A\lambda_1{}^n + B\lambda_2{}^n$ is also a solution of (1).

(b) Now, $\lambda^2 - a\lambda - b = 0$

$$\lambda = \frac{a \pm \sqrt{a^2 + 4b}}{2}. \quad = 0$$

discriminant = 0 ⇒ 2 repeated roots.

$u_n = A(re^{i\theta})^n + B(re^{-i\theta})^n$

$= r^n[(A+B)\cos n\theta + (A-B)i\sin n\theta]$

$u_n = A r^n e^{in\theta} + B r^n e^{-in\theta}$

$= r^n(Ae^{in\theta} + Be^{-in\theta})$

$= r^n[(A+B)\cos n\theta + (A-B)i \ldots]$

Assume that (1) has only one root λ, then $a^2 + 4b = 0$ and $\lambda = \dfrac{a}{2}$.

Thus, $\boxed{b = -\dfrac{a^2}{4}}$ and $\boxed{\lambda = \dfrac{a}{2}}$.

Using the above, we next verify that $\boxed{u_n = n\lambda^n}$ is indeed a solution of (1).

$$\text{R.H.S of (1)} = a\boxed{u_{n-1}} + b\boxed{u_{n-2}}$$

$u_{n-1} = (n-1)\lambda^{n-1}$

$u_{n-2} = (n-2)\lambda^{n-2}$.

$$= a(n-1)\left(\frac{a}{2}\right)^{n-1} + \left(-\frac{a^2}{4}\right)(n-2)\left(\frac{a}{2}\right)^{n-2}$$

$$= n\left(\frac{a}{2}\right)^n = n\lambda^n \quad \text{since}$$

$$= u_n = \text{L.H.S of (1)}$$

Hence, if the characteristic equation has only one root, λ, the linear combination of λ^n and $n\lambda^n$, ie $A\lambda^n + Bn\lambda^n$ is also a solution of (1).

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$|x+iy$
$re^{i\theta}$
$r(\cos\theta + i\sin\theta)$

Exercise:
Show that the solution can be written as $u_n = r^n\left[(A+B)\cos(n\theta)+(A-B)i\sin(n\theta)\right]$ if the characteristic equation has two distinct complex roots.

why are A and B are complex no.s.

$$U_n = A\lambda_1^n + B\lambda_2^n$$
$$= A(re^{i\theta})^n + B(re^{-i\theta})^n$$
$$= r^n\left[Ae^{in\theta} + Be^{-in\theta}\right]$$
$$= r^n\left[A(\cos n\theta + i\sin n\theta) + B(\cos(-n\theta) + i\sin(-n\theta))\right]$$
$$= r^n\left[(A+B)\cos(n\theta) + (A-B)i\sin(n\theta)\right] \quad \text{(shown)}$$

combine real and imaginary parts

$$= r^n\left[C\cos(n\theta) + D\sin(n\theta)\right]$$

where $C = A+B$, $D = (A-B)i$ are real constants.

For an alternative proof of the general solution of a 2[nd] order linear recurrence relation, refer to Annex 1.

Example 6

2nd order, linear, homogeneous eqn.

initial conditions.

Find the solution for the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ where ① $f_0 = 0$ ($n=0$) and ② $f_1 = 1$ ($n=1$).

Solution:

The characteristic equation for $f_n = f_{n-1} + f_{n-2}$ is $\lambda^2 - \lambda - 1 = 0$

$$\lambda = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \quad \text{(two distinct and real roots).}$$

*Result 2

Recall: $U_n = A\lambda_1^n + B\lambda_2^n$

Hence the general solution is $f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Finding A and B: (2 eqns for 2 unknowns)

Given $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = 0 = A + B \Rightarrow B = -A.$$

sub in to this.

$$\text{and } f_1 = 1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1$$

Solving, $A = -B = \frac{1}{\sqrt{5}}$

$$\text{Thus, } f_n = \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^n}_{A\cdot\lambda_1} - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{B\cdot\lambda_2}, \quad n \geq 0$$

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solve for λ

solve for A and B.

esent.

Example 7 ✓ 2nd order u_{n-1}, u_{n-2}. ✓ homogeneous no constant at the back

Find the solution for the recurrence relation $u_n = 6u_{n-1} - 9u_{n-2}$ where $u_0 = 2$ and $u_1 = 3$.

Solution:

The characteristic equation for $u_n = 6u_{n-1} - 9u_{n-2}$ is $\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$.

$\lambda = 3$ (equal real roots)

Recall, $n\lambda^n$ is also a solution if λ^n is a solution.

Hence the general solution is $u_n = A(3^n) + Bn(3^n)$

Given $u_0 = 2$ and $u_1 = 3$, we have $u_n = A\lambda^n + Bn\lambda^n$

$u_0 = 2 = A$

and $u_1 = 3 = 3A + 3B \Rightarrow B = -1$

Thus, $u_n = 2(3)^n - n(3)^n = (2-n)3^n$, $\boxed{n \geq 0}$

Example 8

Find the solution for the recurrence relation $u_n = 2u_{n-1} - 5u_{n-2}$ where $u_0 = 1$ ① and $u_1 = 5$ ②.

$n=0$, $n=1$

Solution:

The characteristic equation for $u_n = 2u_{n-1} - 5u_{n-2}$ is $\lambda^2 - 2\lambda + 5 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = \frac{2 \pm 4i}{2}$

$\lambda = 1+2i$ or $\lambda = 1-2i$ (2 distinct complex roots)

$u_n = A\lambda_1^n + B\lambda_2^n$.

Hence the general solution is $u_n = A(1+2i)^n + B(1-2i)^n$

Alternatively, you may write $u_n = (\underset{r}{\sqrt{5}})^n [C\cos(n\underbrace{\tan^{-1}2}_{\theta}) + D\sin(n\overbrace{\tan^{-1}2}^{\theta})]$

$\theta = \tan^{-1}(\frac{2}{1})$.

Given $u_0 = 1$ and $u_1 = 5$, we have

$u_0 = 1 = A + B$

and $u_1 = 5 = A(1+2i) + B(1-2i)$ conjugate pairs!

Solving, we have $A = \frac{1-2i}{2}$ and $B = \frac{1+2i}{2}$

$u_0 = 1 = C$ and because $\sin 0 = 0$

$u_1 = 5 = \sqrt{5}[C\cos(\tan^{-1}2) + D\sin(\tan^{-1}2)]$

$= \sqrt{5}[\frac{1}{\sqrt{5}} + D\frac{2}{\sqrt{5}}]$ solve for D

so $5 = 1 + 2D \Rightarrow D = 2$

$u_n = (\sqrt{5})^n [\cos(n\tan^{-1}2) + 2\sin(n\tan^{-1}2)]$

$A\lambda_1^n + B\lambda_2^n$.

$re^{i\theta} = r(\cos\theta + i\sin\theta)$

split into real and imaginary parts.

Thus, $u_n = \left(\frac{1-2i}{2}\right)(1+2i)^n + \left(\frac{1+2i}{2}\right)(1-2i)^n$

$1 = A + B$

(polar form) $(\sqrt{5})^n [\cos(n\tan^{-1}2) + (-2i)i\sin(n\tan^{-1}2)]$

$= (\sqrt{5})^n [\cos(n\tan^{-1}2) + 2\sin(n\tan^{-1}2)]$, $n \geq 0$, $n \in \mathbb{Z}^+$.

Interesting observation:

The sequence is obviously a real sequence. However its general formula involves complex numbers.

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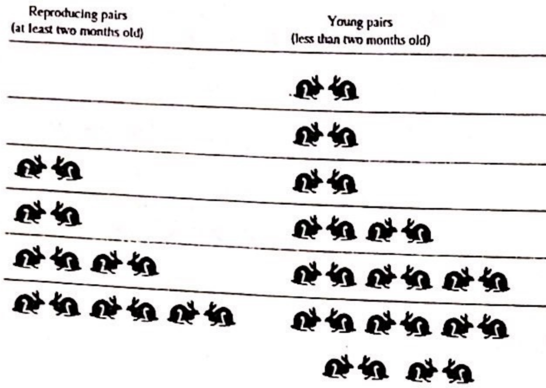
Example 9 (Fibonacci Sequence)
A pair of rabbits does not breed until they are two months old. After they are two months old, each pair of rabbits produces another pair each month, as shown in the diagram below. Find the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die. (Rosen, 2007)



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Solution:

Let u_n be the number of pairs of rabbits on the island at the end of the n^{th} month.
u_n = number of pairs of rabbits at the end of the $(n-1)^{th}$ month + number of pairs of rabbits at the end of the $(n-2)^{th}$ month (fertile rabbits that will reproduce), $n \geq 3$.

Second order recurrence relation: $u_n = u_{n-1} + u_{n-2}$, $n \geq 3$ → depends on how you define your RR. eg. if you say $u_{n+2} = u_{n+1} + u_n$, $n \geq 1$

Initial conditions: $u_1 = 1$, $u_2 = 1$

eg. 6.

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ diff from eg. 6.

$\lambda = \frac{1+\sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$ or $\lambda_2 = \frac{1-\sqrt{5}}{2}$ $(\lambda_2 \neq \lambda_1)$

General solution is $u_n = A\lambda_1^n + B\lambda_2^n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Since $u_1 = 1$, $u_2 = 1$,

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 \quad \text{and} \quad 1 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$2 = A\left(1+\sqrt{5}\right) + B\left(1-\sqrt{5}\right) \qquad 4 = A\left(1+\sqrt{5}\right)^2 + B\left(1-\sqrt{5}\right)^2 \quad \cdots(2)$$

$$B = \frac{2 - A\left(1+\sqrt{5}\right)}{1-\sqrt{5}} \quad \cdots(1)$$

Substitute (1) into (2),

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$$4 = A\left(1+\sqrt{5}\right)^2 + \left[\frac{2-A\left(1+\sqrt{5}\right)}{1-\sqrt{5}}\right]\left(1-\sqrt{5}\right)^2$$

$$4 = A\left(1+\sqrt{5}\right)^2 + \left[2-A\left(1+\sqrt{5}\right)\right]\left(1-\sqrt{5}\right)$$

$$4 = A\left(6+2\sqrt{5}\right)+2-2\sqrt{5}-A\left(1-5\right)$$

$$2+2\sqrt{5} = A\left(6+2\sqrt{5}+4\right)$$

$$2\left(1+\sqrt{5}\right) = 2A\left(5+\sqrt{5}\right)$$

$$A = \frac{1+\sqrt{5}}{5+\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$B = \frac{2-\frac{1}{\sqrt{5}}\left(1+\sqrt{5}\right)}{1-\sqrt{5}} = \frac{2-\frac{1}{\sqrt{5}}-1}{1-\sqrt{5}} = -\frac{1}{\sqrt{5}}.$$

$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\therefore u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 1$$

$\cancel{u_n = At}$

$\cancel{u_n = A\lambda^n + B\lambda^n}$

$u_n =$

u_n not to the power of 1. – eg. square roots.

5 NON-LINEAR 1st ORDER RECURRENCE RELATIONS

In the next few examples, we are not able to get an explicit expression for the nth term of the sequence. However we are still able to determine its "long run behavior".

Example 10

A sequence of real numbers u_1, u_2, u_3, ... satisfies the recurrence relation $u_{n+1} = \sqrt{u_n + 3}$ for $n = 1, 2, 3, ...$

$= (u_n + 3)^{\frac{1}{2}}$

(i) Use a graphic calculator to determine the behaviour of the sequence in each of the cases **(a)** $u_1 = 1$ and **(b)** $u_1 = 6$

(ii) Given that as $n \to \infty$, $u_n \to l$, find the exact value of l using an algebraic method.

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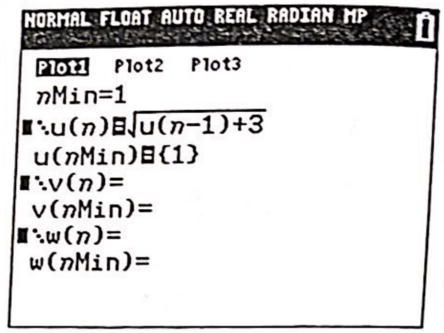
Solution:

Using GC

1. Change to **SEQ** mode.

2. Go to [Y=] menu.



3a. Key in the starting value of n : $n\text{Min}=1$

3b. $u_{n+1}=\sqrt{u_n+3}$ given in question

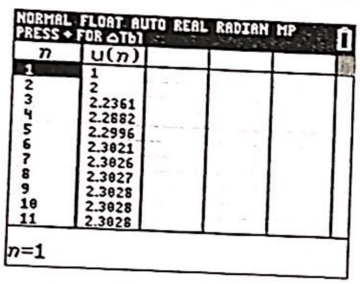
Replacing n by $n-1$, $u_n=\sqrt{u_{n-1}+3}$. What you key into GC

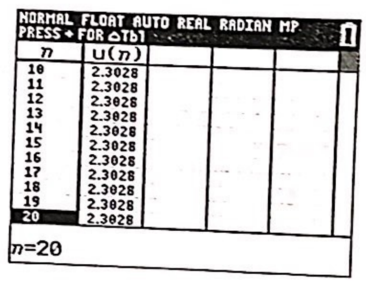
Key in the recurrence relation $u(n)=\sqrt{u(n-1)+3}$.

4. Key in the initial condition: $u(n\text{Min})=1$

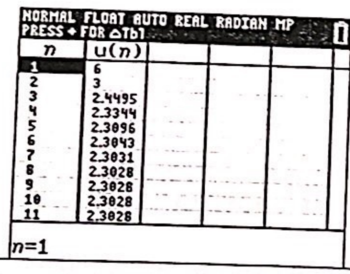
5. Scroll down the **TABLE** to check the behaviour of the sequence.

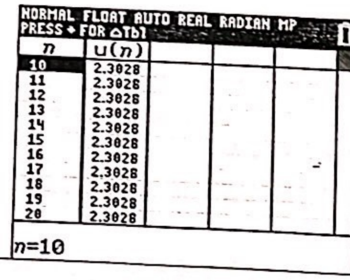
increasing and converging.





6. Repeat Steps 4 and 5 with $u(n\text{Min})=6$





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Solution:

(i) (a) From the GC, the sequence is strictly increasing and converges to 2.3028 (to 5sf) that is, 2.30 (to 3sf) when $u_1 = 1$

(b) From the GC, the sequence is strictly decreasing and converges to 2.3028 (to 5sf) that is 2.30 (to 3sf) when $u_1 = 6$

(ii) Given that $n \to \infty$, $u_n \to l$, we can conclude that

From $u_{n+1} = \sqrt{u_n + 3}$, as $n \to \infty$, we have $u_{n+1} \to l$ as well

$$l = \sqrt{l+3}$$

$$l^2 = l+3$$

$$l^2 - l - 3 = 0$$

$$l = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(3)}}{2} = \frac{1 \pm \sqrt{13}}{2}$$

As the sequence consists of positive numbers, ie. $u_n > 0$,

$$l = \frac{1+\sqrt{13}}{2} = 2.3028$$

Note:

A sequence has to be known to converge before we can apply this method to find the exact value of the limit. For example, if we apply this method to $u_{n+1} = 2u_n + 25$, we will erroneously obtain the limit to be -25 even though it is easy to see that the sequence diverges given any initial value. except at -25

In the previous example, we used a table of values to ascertain the behavior of the sequence generated from a ***specified*** initial value and observed that it was convergent in both cases. We next use a graphical approach to show how the sequence of numbers will evolve given ***any*** initial value.

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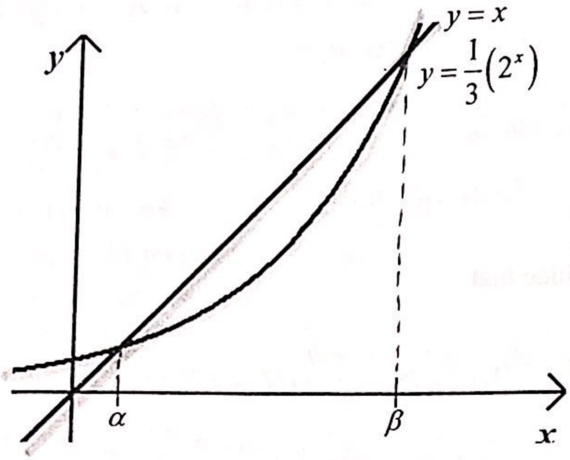
Example 11 [RJC JC2 Term 3 CT Q7 (modified)]

The diagram shows the graphs of $y=\frac{1}{3}\left(2^{x}\right)$ and $y=x$. The two graphs intersect at $x=\alpha$ and $x=\beta$ where $\alpha<\beta$. Find the values of α and β.



A sequence of real numbers x_1, x_2, x_3,... satisfies the recurrence relation

$$x_{n+1}=\frac{1}{3}(2^{x_n}) \quad \text{for } n\geq 1.$$

(i) Determine the behaviour of the sequence using a calculator for the following cases :
(a) $x_1=4$, **(b)** $x_1=-2$, **(c)** $x_1=2.8$.

(ii) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(iii) By using the graphs of $y=\frac{1}{3}\left(2^{x}\right)$ and $y=x$, prove that

if $\alpha<x_n<\beta$, then $\alpha<x_{n+1}<x_n$
if $x_n<\alpha$, then $x_n<x_{n+1}<\alpha$
if $x_n>\beta$, then $x_n<x_{n+1}$

(iv) State briefly how the results in part **(iii)** relate to the behaviours determined in part **(i)**.

* change back to functions mode

Solution:

From the GC, $\alpha = 0.458$, $\beta = 3.31$ (3 sf)

(i) From the GC,
(a) the sequence strictly increases and diverges when $x_1 = 4$

(b) the sequence strictly increases and converges when $x_1 = -2$

(c) the sequence strictly decreases and converges to 0.458 when $x_1 = 2.8$.

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basically the eqn $y = \frac{1}{3}(2^x)$

(ii) If the sequence converges to some constant l,

$x_n \to l$ and $x_{n+1} \to l$ as $n \to \infty$

From $x_{n+1} = \frac{1}{3}(2^{x_n})$, as $n \to \infty$, we have $l = \frac{1}{3}(2^l)$ blue graph. pink graph

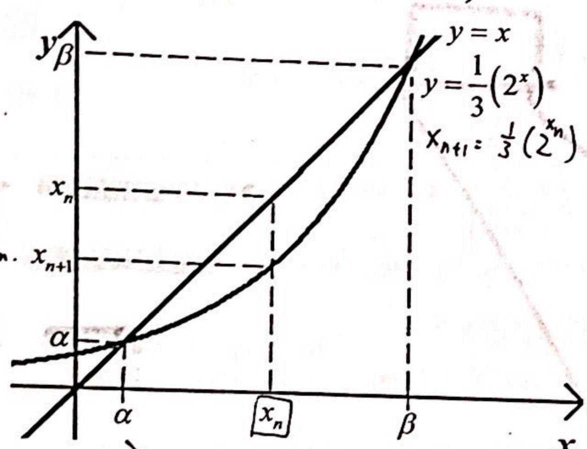
$\Rightarrow l$ satisfies the equation $x = \frac{1}{3}(2^x)$.

$\therefore$ The sequence converges to either α or β.

(iii) If $\alpha < x < \beta$, we can see that the graph of $y = \frac{1}{3}(2^x)$ lies below the graph of $y = x$

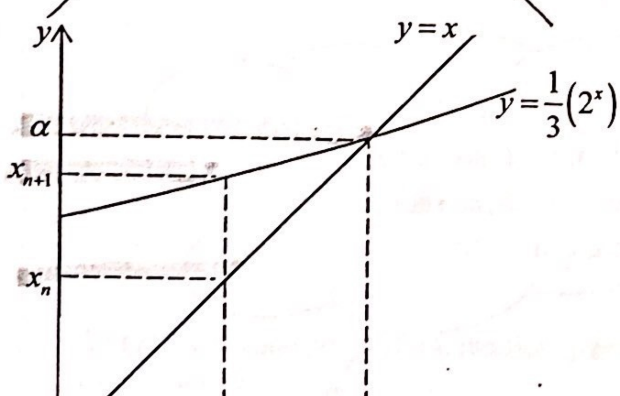
That is, if $\alpha < x_n < \beta$, then $x_{n+1} = \frac{1}{3}(2^{x_n}) < x_n$

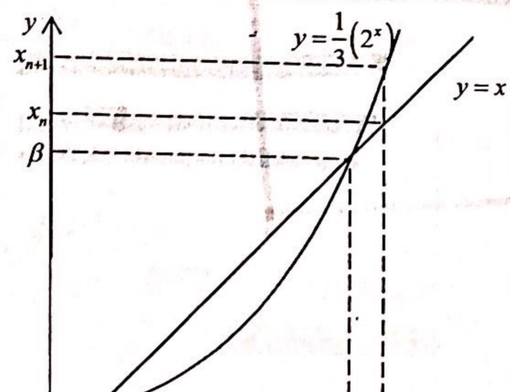
It is clear from the graph that $\alpha < x_{n+1} < x_n$ $(< \beta)$



$\therefore \alpha < x_{n+1} < x_n.$

If $x < \alpha$ or $x > \beta$, we see that the graph of $y = \frac{1}{3}(2^x)$ lies above the graph of $y = x$





For $x_n < \alpha$, it is clear from the graph that $x_n < x_{n+1} < \alpha$.

For $x_n > \beta$, it is clear from the graph that $\beta < x_n < x_{n+1}$.

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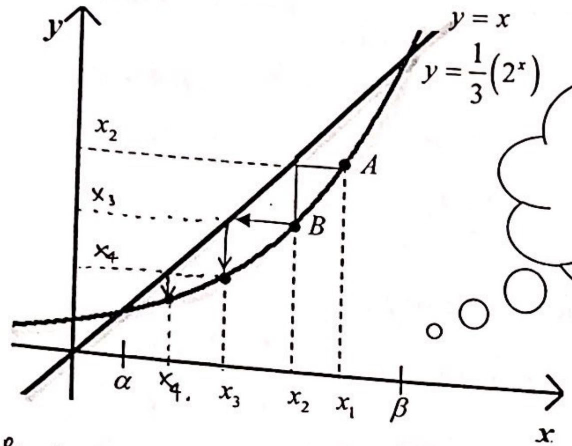

When $\alpha < x < \beta$, we see that $x_1 > x_2 > x_3 > \ldots$ and x_n converges

Recall $\alpha = 0.458$, $\beta = 3.31$ (3sf)

(iv) When $\boxed{x_1 = 4}$, we see that $x_1 > \beta$ and we will expect $x_{n+1} > x_n$

This results in a strictly increasing and divergent series

Recall: if $x_n > \beta$, then $x_n < x_{n+1}$

When $\boxed{x_1 = -2}$, we see that $x_1 < \alpha$ and we will expect $x_n < x_{n+1} < \alpha$

This results in a strictly increasing sequence, which converges to α

Recall: if $x_n < \alpha$, then $x_n < x_{n+1} < \alpha$

When $x_1 = 2.8$, we see that $\alpha < x_1 < \beta$ and we will expect $\alpha < x_{n+1} < x_n$

This results in a strictly decreasing sequence, which converges to α.

Recall: if $\alpha < x_n < \beta$, then $\alpha < x_{n+1} < x_n$

When $x_1 = 4$, note that $x_1 > \beta$
When $x_1 = -2$, note that $x_1 < \alpha$
When $x_1 = 2.8$, note that $\alpha < x_1 < \beta$

- When $x_1 = 0$, $x_1 < \alpha$ and we will expect $x_{n+1} > x_n$
  $\Rightarrow$ the sequence is increasing
  but Why does it converge to α?
- When $x_1 = 6$, $x_1 > \beta$ and we will expect $x_{n+1} > x_n$
  $\Rightarrow$ the sequence is increasing and diverges.
- When $x_1 = 2$, $\alpha < x_1 < \beta$ and we will expect $x_{n+1} < x_n$
  $\Rightarrow$ the sequence decreasing but why does it converge to α.

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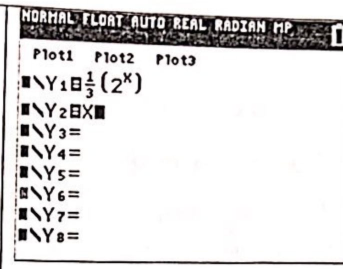
Using GC to find intersection points

1. Press MODE and then scroll down to change to **FUNC** mode.

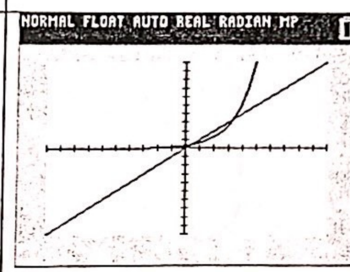
2. Press Y=.

3. Key in the equations:

   $Y_1 = \frac{1}{3}(2^X)$ and $Y_2 = X$.

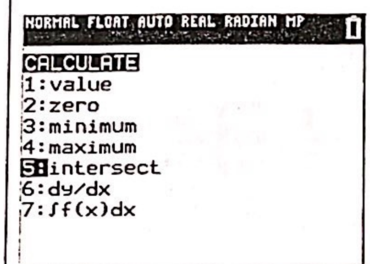


4. Press GRAPH to see the graphs.

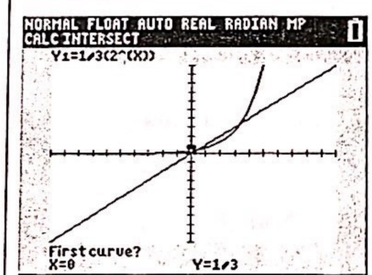


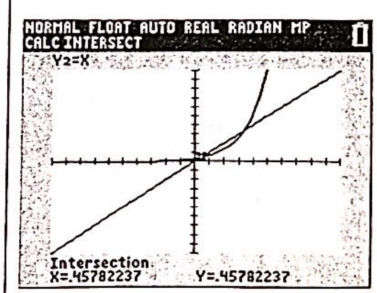
5. Press 2nd TRACE.

6. Select **5: intersect**.



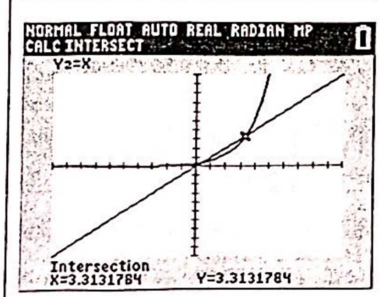
7. Shift the cursor close to the first point of intersection, press ENTER 3 times to get its coordinates.





8. To get the second point of intersection, repeat Steps 5 and 6.

9. Shift the cursor close to the second point of intersection, press ENTER 3 times to get its coordinates.



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Example 12 [Note that the following involves a 1st order linear recurrence relation]

The numbers x_n satisfy the recurrence relation $x_{n+1} = -\frac{1}{2}x_n + 2$ for $n = 1,2,3,...$

Given that $x_n \to l$ as $n \to \infty$, find the exact value of l, and show that $x_{n+1} - l = -\frac{1}{2}(x_n - l)$.

↓ start from one side LHS RHS.

Show that if $0 < x_n < l$, then

(i) $l < x_{n+1}$ $\Rightarrow x_{n+1} - l > 0$

(ii) $x_n < x_{n+2} < l$

Solution:	Remarks:
As $n \to \infty$, $x_n \to l$ and $x_{n+1} \to l$. Hence, as $n \to \infty$, $\boxed{l = -\frac{1}{2}l + 2} \Leftrightarrow \frac{3}{2}l = 2$ $x_{n+1} = -\frac{1}{2}x_n + 2$ $l = \frac{4}{3}$ //	Recall that if a sequence converges to some constant l, $x_n \to l$ and $x_{n+1} \to l$ as $n \to \infty$.
LHS = $x_{n+1} - l = x_{n+1} - \frac{4}{3}$ $= \left(-\frac{1}{2}x_n + 2\right) - \frac{4}{3}$) sub (so it can be expressed in terms of x_n) $= -\frac{1}{2}x_n + \frac{2}{3}$) expand $= -\frac{1}{2}\left(x_n - \boxed{\frac{4}{3}}\right)$) factorise $\to l$. $= -\frac{1}{2}(x_n - l)$ (shown) = RHS. shown in previous part.	For such "show questions", start from one side, usually the L.H.S, and make use of the information given to reach the other side.
If $0 < x_n < l$, show $l < x_{n+1} \Rightarrow x_{n+1} - l > 0$. **(i)** $x_{n+1} - l = \underbrace{-\frac{1}{2}}_{<0}\underbrace{(x_n - l)}_{<0 \text{ as } x_n < l} > 0$ $\because x_n < l$ $\Rightarrow x_{n+1} > l$ show $x_n < x_{n+2} < l$ $\Rightarrow$ show $x_n < x_{n+2}$ and $x_{n+2} < l$ **(ii)** $x_{n+2} - l = \underbrace{-\frac{1}{2}}_{<0}\underbrace{(x_{n+1} - l)}_{>0} < 0$ $\because x_{n+1} > l$ $\Rightarrow x_{n+2} < l$ $x_{n+2} - l = -\frac{1}{2}\left(-\frac{1}{2}(x_n - l)\right)$ do one more substitution. $\frac{1}{4}$, <0 $= \left(\frac{1}{2}\right)^2 (x_n - l) > (x_n - l)$ $\because x_n - l < 0$ $\Rightarrow x_n < x_{n+2}$	Strategy here again is to start from what we already know, to prove what is needed. $x_{n+2} - l < 0$

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Example 13 **[9740/2012TJC/Promo/Q10]**

A sequence of numbers is defined by $x_{n+1} = \frac{1}{2}x_n + \sqrt{a + x_n}$ for $n = 1, 2, 3, \ldots$

Suppose the sequence converges to 3, find the value of a.

(i) By considering $x_{n+1} - x_n$ and using a graphical method, show that if $1 < x_n < 3$, then $x_{n+1} > x_n$. $\Rightarrow x_{n+1} - x_n > 0$.

(ii) Show that if $1 < x_n < 3$, then $1 < x_{n+1} < 3$.

(iii) Use the results in **(i)** and **(ii)** to deduce the behaviour of the sequence when $x_1 = 2$. Show your explanation clearly.

Solution:

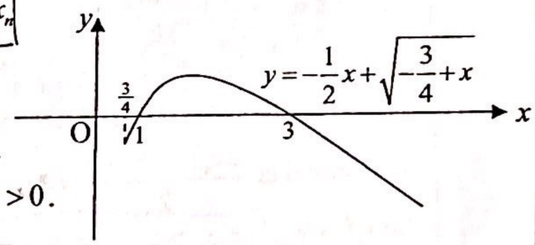
As $n \to \infty$, $x_n \to 3$, $x_{n+1} \to 3$, $x_{n+1} = \frac{1}{2}x_n + \sqrt{a + x_n}$

we have $3 = \frac{3}{2} + \sqrt{a+3}$ 3 3 3

One way to show what is needed is to sketch the graph and look at the appropriate regions

$$a + 3 = \left(\frac{3}{2}\right)^2 \Rightarrow a = -\frac{3}{4}$$

(i) $x_{n+1} - x_n = \left(\frac{1}{2}x_n + \sqrt{-\frac{3}{4} + x_n}\right) - x_n = \boxed{-\frac{1}{2}x_n + \sqrt{-\frac{3}{4} + x_n}}$



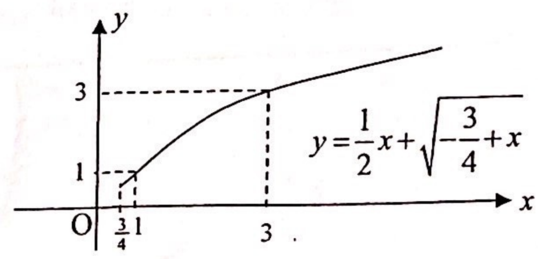
Consider the graph of $y = -\frac{1}{2}x + \sqrt{-\frac{3}{4} + x}$

From the graph, if $1 < x < 3$, then $y = -\frac{1}{2}x + \sqrt{-\frac{3}{4} + x} > 0$.

Let $x = x_n$. Hence if $1 < x_n < 3$, then $-\frac{1}{2}x_n + \sqrt{-\frac{3}{4} + x_n} > 0$

$$\Rightarrow x_{n+1} - x_n > 0 \Rightarrow x_{n+1} > x_n \text{ (shown)}$$

(ii) Consider graph of $y = \frac{1}{2}x + \sqrt{-\frac{3}{4} + x}$



From the graph,

$1 < x < 3 \Rightarrow 1 < y = \frac{1}{2}x + \sqrt{-\frac{3}{4} + x} < 3$.

Hence if $1 < x_n < 3$, then $1 < x_{n+1} < 3$.

(iii) When $x_1 = 2$, $1 < x_1 < 3 \Rightarrow x_2 > x_1$ from **(i)** and $1 < x_2 < 3$ from **(ii)**

$1 < x_2 < 3 \Rightarrow x_3 > x_2$ and $1 < x_3 < 3$

Continuing in this manner, $x_1 < x_2 < x_3 < \ldots\ldots < 3$, hence the sequence increases and converges to 3.

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Example 14

The numbers x_n satisfy the recurrence relation $x_{n+1}=\sqrt{\dfrac{3x_n+5}{2}}$ for $n=1,2,3,\ldots$.

(a) Given that $x_n \to l$ as $n \to \infty$, find the exact value of l.

(b) By considering $x_n^{\,2}-x_{n+1}^{\,2}$ and the graph of $y=2x^2-3x-5$, show that if $x_n > l$, then $x_{n+1} < x_n$ for all $n=1,2,3,\ldots$.

Solution:

(a) As $n \to \infty$, $x_n \to l$ and $x_{n+1} \to l$.

Hence, as $n \to \infty$, $x_{n+1}=\sqrt{\dfrac{3x_n+5}{2}}$ becomes $l=\sqrt{\dfrac{3l+5}{2}}$.

So $l^2=\dfrac{3l+5}{2}$

$\Rightarrow 2l^2-3l-5=0$

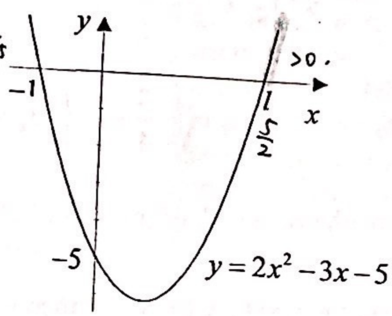
$\Rightarrow (2l-5)(l+1)=0$

$\Rightarrow l=\dfrac{5}{2}$ or $l=-1$ (N.A. as $l \geq 0$)

(b) $x_n^{\,2}-x_{n+1}^{\,2}=x_n^{\,2}-\dfrac{3x_n+5}{2}$

$=\dfrac{2x_n^{\,2}-3x_n-5}{2}$

Recall : $x_{n+1}=\sqrt{\dfrac{3x_n+5}{2}}$



$x_n > \frac{5}{2}$

From the graph of $y=2x^2-3x-5$, when $x_n > l$,

one sees that $\dfrac{2x_n^{\,2}-3x_n-5}{2} > 0$ must consider both cases.

So $x_n^{\,2} > x_{n+1}^{\,2} \Rightarrow \boxed{x_n > x_{n+1} \text{ or } x_n < -x_{n+1}}$

But $x_n > \dfrac{5}{2} > 0$ and $x_{n+1}=\sqrt{\dfrac{3x_n+5}{2}} > 0$.

So it is impossible to have $x_n < -x_{n+1}$.

Thus $x_n > x_{n+1}$.

Sketch the graph and look at the appropriate regions

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