**2024 JC1 Promotional Exam**

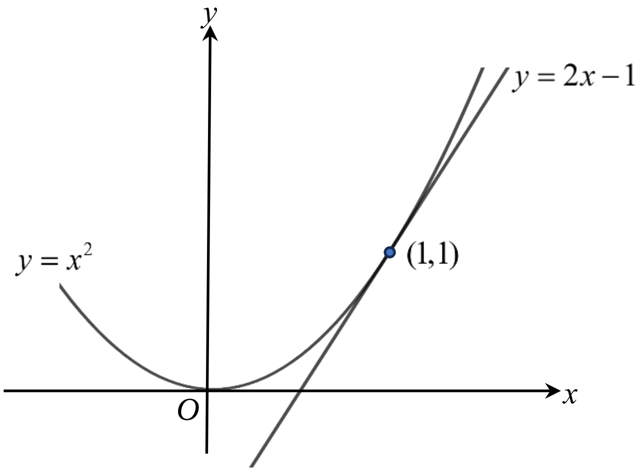
**1** A hyperbola has the equation $a(x-2)^2-b(y+1)^2=c$ where $a$, $b$, and $c$ are positive integers. It passes through the point $(7,10)$. A line $y=x-1$ intersects the hyperbola at $x=1$. Find one possible value for each of $a$, $b$ and $c$. [4]

**2** It is given that $y=\tan 3x$.

**(a)** Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=Ay\dfrac{\mathrm{d}y}{\mathrm{d}x}$, where $A$ is a constant to be determined. [3]

**(b)** Hence show that $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}=B\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+Cy\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, where $B$ and $C$ are constants to be determined. [3]

**3** The graph shows the curve $C$ with equation $y=x^2$ and the line $L$ with equation $y=2x-1$. The line $L$ is tangent to $C$ at (1,1). The region $R$ located in the first quadrant is bounded by the curve $C$, the line $L$ and the $x$-axis.




**(a)** Find the area of region $R$, giving your answer in exact form. [2]

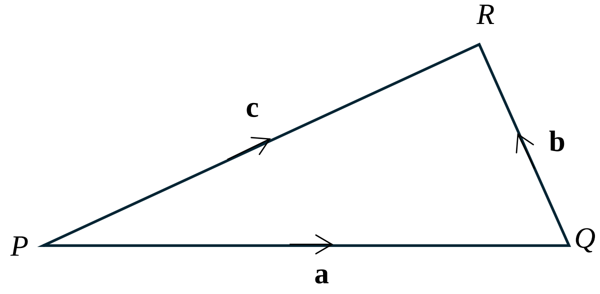
**(b)** The region R is rotated one complete revolution around the $x$-axis. Find the volume of this solid of revolution, giving your answer in exact form. [3]

**4 (a)** Sketch the graph of $y=\dfrac{4-x}{2+3x-2x^2}$. Give the equations of the asymptotes and the coordinates of the point where the curve intersects the $x$-axis. [4]

**(b)** Solve the inequality $\dfrac{4-x}{2+3x-2x^2}>0.$ [1]

**(c)** Hence solve the inequality $\dfrac{4-|x|}{2+3|x|-2x^2}>0.$ [2]

**5** In triangle $PQR$, the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ represent the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ respectively.

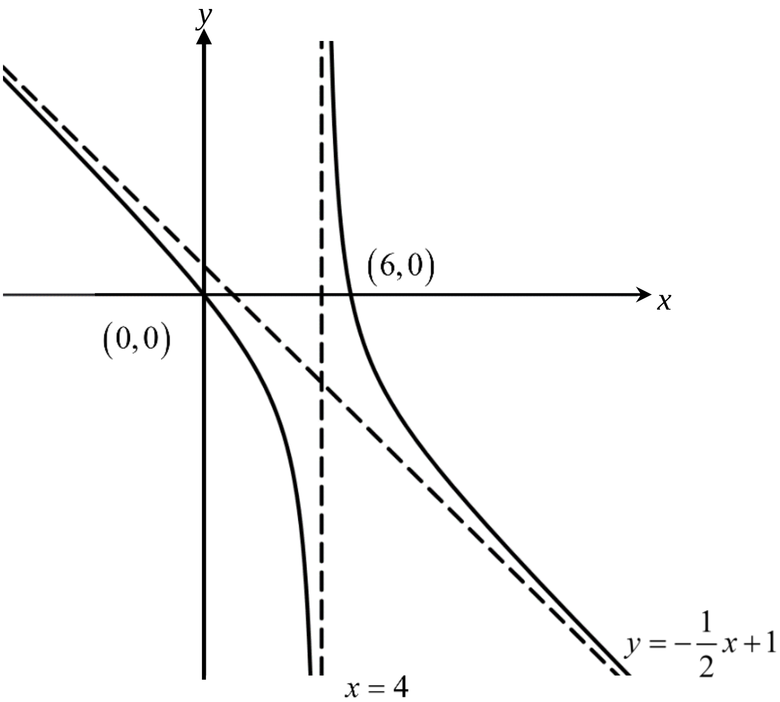



**(a)** State a geometrical interpretation of $|\mathbf{c}\times\hat{\mathbf{a}}|$ and hence show that the area of the triangle $PQR$ can be written as $\frac{1}{2}|\mathbf{c}\times\mathbf{a}|$ units$^2$. [3]

**(b)** With clear explanations, show that $|\mathbf{c}\times\mathbf{a}|=|\mathbf{b}\times\mathbf{a}|$ and deduce that

$$\frac{|\mathbf{c}|}{\sin\angle PQR}=\frac{|\mathbf{b}|}{\sin\angle QPR} \text{ (Sine Rule).}$$ [2]

**6** The diagram below shows the graph of curve $C_1$, with equation $y=\mathrm{f}(x)$, defined for $x\in\mathbb{R}\setminus\{4\}$, with asymptotes $x=4$, and $y=-\frac{1}{2}x+1$, and cuts the $x$-axis at the origin and $(6,0)$.




**(a)** Sketch the graph of $y=\mathrm{f}'(x)$, labelling the equations of asymptotes if any. [2]

**(b)** Given another curve $C_2$ has equation $\dfrac{(y+1)^2}{k^2}-(x-4)^2=1$, where $k$ is a positive constant. Describe a sequence of transformations that transform the graph of $y^2-x^2=1$ onto the graph of $C_2$. [3]

**(c)** Determine the range of values of $k$ for which $C_1$ and $C_2$ intersect exactly 2 times. [1]

**7** The function f is defined by $\mathrm{f}: x \mapsto \sqrt{5+2x}$, for $x \in \mathbb{R}$, $x > -\dfrac{5}{2}$.

**(a)** Find the value of $a$ such that $\mathrm{f}^2(a)=5$. [2]

Another function g is defined by $\mathrm{g}: x \mapsto \left|\dfrac{x}{1-2x}\right|$, for $x \in \mathbb{R}$, $x \le k$.

**(b)** State the largest value of $k$ for $\mathrm{g}^{-1}$ to exist. [1]

It is now given that $k=-1$.

**(c)** Find $\mathrm{g}^{-1}(x)$ and state its domain. [3]

**(d)** Explain why the composite function fg exists. State the exact range of fg. [2]

8 A curve $C$ has parametric equations

$$x=1-2\cos 2\theta, \qquad y=\sin 2\theta-1, \qquad \text{for } 0 \le \theta \le \frac{3\pi}{8}.$$

**(a)** Find the coordinates of the point where $C$ cuts the $y$-axis, giving your answer in exact form. [2]

**(b)** Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}\cot 2\theta$. Find the coordinates of the points and the gradient of $C$ where $\theta=\dfrac{\pi}{4}$. What can be said about the tangent to $C$ as $\theta \to 0$? [5]

**(c)** Hence sketch $C$, labelling the exact coordinates of axial intercepts and endpoints, and showing clearly the features of the curve at the points where $\theta=0$ and $\theta=\dfrac{\pi}{4}$. [3]

**(d)** Find a cartesian equation of $C$. [2]

**9(a)** Find $\displaystyle\int \frac{x+2}{x^2+2x-3}\,\mathrm{d}x$. [2]

**(b)** Find $\displaystyle\int 2x\sin x\,\mathrm{d}x$. [2]

**(c)** Using the substitution $x = 3\sin\theta$, find the exact value of $\displaystyle\int_0^{\frac{3}{2}} \sqrt{9-x^2}\,\mathrm{d}x$. [5]

**10** For positive real numbers $a$, and $b$, the lines $l_1$ and $l_2$ have equations given by

$$l_1\colon\ \mathbf{r} = \begin{pmatrix}1\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\3\end{pmatrix} \text{ for } \lambda \in \mathbb{R} \quad \text{and} \quad l_2\colon\ \mathbf{r} = \begin{pmatrix}a\\9\\9\end{pmatrix} + \mu\begin{pmatrix}1\\7\\b\end{pmatrix} \text{ for } \mu \in \mathbb{R}.$$

It is given that the two lines intersect each other.

**(i)** Show that $3a = b + 2$. [2]

**(ii)** Find the value of $b$ if the angle between the two lines is $\cos^{-1}\left(\frac{11}{\sqrt{660}}\right)$ radians. [3]

The equation of the plane $\Pi_1$ that contains both lines is now given by $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix} = 6$.

**(iii)** A second plane $\Pi_2$ meets $\Pi_1$ in the line $l_1$ and contains the point (2, 1, 3). Find an equation for $\Pi_2$ in scalar product form. [2]

**(iv)** Hence find the angle between the planes $\Pi_1$ and $\Pi_2$. [2]

**(v)** The plane $\Pi_3$ is parallel to the plane $\Pi_1$. Given that the distance between both planes is $\frac{7}{\sqrt{11}}$ units and that $\Pi_3$ is closer to the origin than $\Pi_1$, find a cartesian equation for $\Pi_3$. [3]

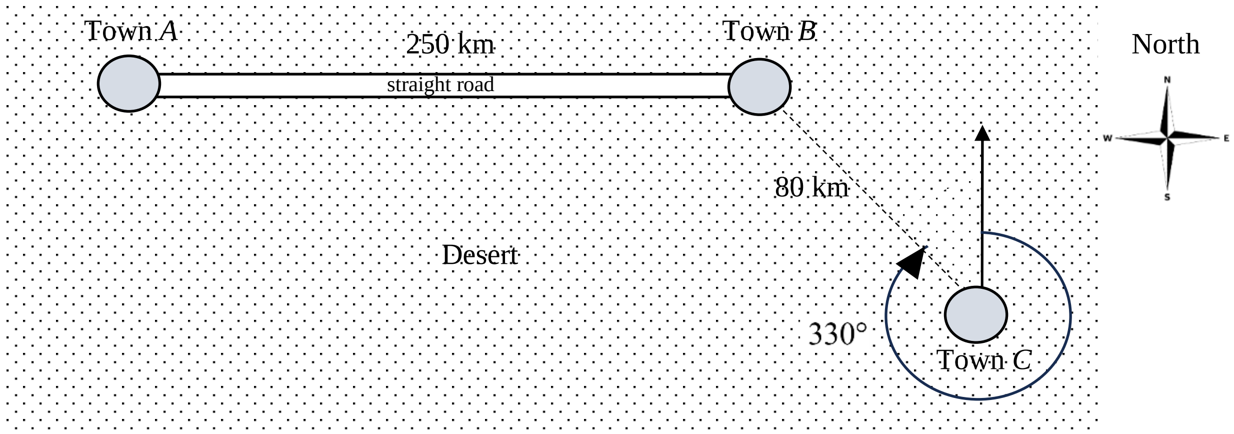
**11 (a)** A sphere has radius $r$ cm, surface area $S$ cm$^2$ and volume $V$ cm$^3$.

**(i)** Show that $\dfrac{\mathrm{d}V}{\mathrm{d}S} = \dfrac{r}{2}$. [2]

**(ii)** Given that the surface area of the sphere is increasing at a constant rate of 3 cm$^2$/s, find the radius when the rate of increase of the volume is 9 cm$^3$/s . [2]

[Volume of sphere, $V = \dfrac{4}{3}\pi r^3$; Surface area of sphere, $S = 4\pi r^2$ ]

**(b)** CarsExtreme is hosting a competition where competitors must drive from Town $A$ to Town $C$ in the shortest time possible.




Town $A$ is located 250 km to the west of Town $B$, while Town $B$ is positioned 80 km away from Town $C$ at a bearing of $330°$. A straight road connects Town $A$ and Town $B$, as shown in the picture above. The three towns are in a desert and competitors must traverse this desert to reach their destinations. To ensure safety, a speed limit of 110 km/h is enforced within the desert, except on the straight road where the speed limit is 130 km/h. It is assumed that competitors will always drive at the maximum permitted speed and will take the shortest possible route when crossing the desert.

Competitor $P$ and $Q$ have different strategies for the competition. Competitor $P$ plans to drive along the straight road from Town $A$ to a point $x$ km before reaching Town $B$, and then cut diagonally across the desert to Town $C$. On the other hand, Competitor $Q$ intends to drive directly from Town $A$ to Town $C$, traveling entirely through the desert.

**(i)** Show that the time taken, $T$ hours, for Competitor $P$ is

$$T = \frac{1}{1430}\left(2750 - 11x + 13\sqrt{x^2 + 80x + 6400}\right)$$ [3]

**(ii)** Use differentiation to find the minimum time that Competitor $P$ will take, giving your answer correct to the nearest minute. (You need not show that your answer gives a minimum.) [4]

**(iii)** However, prior to the start of the competition, weather changes in the desert prompts the organisers to adjust the speed limit to $M$ km/h for desert driving. Competitor $P$ then decides to drive to Town $B$ before heading to Town $C$, while Competitor $Q$ sticks to his original plan. Find the range of values of $M$ if Competitor $P$ arrives at Town $C$ before $Q$. [3]

**12** At a particular bank, a savings account pays an interest of $0.9\%$ per month on the last day of each month. On 1 January 2024 Mr Monie put an initial deposit of \$10000 into the savings account and continues to make a deposit of \$810 at the start of each subsequent month.

**(a)** Find the total amount in Mr Monie's savings account at the end of the 2 months. [1]

**(b)** Show that the total amount in the savings account at the end of the $n^{\text{th}}$ month is given by $\$100900(1.009)^{n-1} - \$90810$. [3]

**(c)** On what date did the value of Mr Monie's account first exceed \$30000? [4]

Mr Pey has \$30000 and is considering whether to keep this amount (with no further deposit) in the savings account to earn interest or invest this amount in another financial product. This financial product gives a disbursement of \$100 at the end of the first month. At the end of every subsequent month, the financial product gives \$10 more than the previous month. For example, at the end of first month, Mr Pey will have \$30100 and at the end of two months, he will have \$30210, and so on.

**(d)** Find, in terms of $N$, the total amount Mr Pey will have at the end of $N$ months if he invests in the financial product. [2]

**(e)** Given that $0 < N < 400$, find the range of values of $N$ such that investing in the financial product is more profitable than keeping the money in the savings account. [2]