2022 H1 MATH (8865/01) JC 2 PRELIMINARY EXAMINATION SOLUTIONS

Qn	Solution
1	**System of Linear Equations**
	Let x, y and z be the original selling price for a waffle, a scoop of ice cream and a cookie in dollars. $2x(0.6)+(3y-3)+z=20.2 \Rightarrow 1.2x+3y+z=23.2$ --- (1) $0.6x+(2y-3)+2z=12.6 \Rightarrow 0.6x+2y+2z=15.6$ --- (2) $3x(0.6)+(6y-3(3))+5z=36.8 \Rightarrow 1.8x+6y+5z=45.8$ --- (3) Using GC, $x=6, y=5, z=1$ $\therefore$ the original selling price for a waffle is \$6.

Qn	Solution
2	**Techniques of Differentiation & Techniques of Integration**
(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2(1+3x)^2}\right)$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(1+3x)^{-2}\right)$ $=\frac{1}{2}(-2)(1+3x)^{-3}(3)$ $=-\frac{3}{(1+3x)^3}$
(b)	$\int\left(\frac{1}{4-5x}+\mathrm{e}^{\frac{1}{2}x+1}\right)\,\mathrm{d}x.$ $=-\frac{1}{5}\ln\lvert 4-5x\rvert+\frac{\mathrm{e}^{\frac{1}{2}x+1}}{\frac{1}{2}}+C$, where C is an arbitrary constant $=-\frac{1}{5}\ln\lvert 4-5x\rvert+2\mathrm{e}^{\frac{1}{2}x+1}\ C$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>3</td><td>Equations and Inequalities</td></tr>
<tr><td>(a)</td><td>

$2x^2+(2-k)x-k=-3x-5$

$2x^2+(2-k)x+3x-k+5=0$

$2x^2+(5-k)x+5-k=0$

Since $y=2x^2+(2-k)x-k$ and $y=-3x-5$ intersect at two distinct points,

Discriminant >0

$(5-k)^2-4(2)(5-k)>0$

$25-10k+k^2-40+8k>0$

$k^2-2k-15>0$

$(k-5)(k+3)>0$

$k<-3 \quad \text{or} \quad k>5$

Set of values = $\{k\in\mathbb{R}: k<-3 \quad \text{or} \quad k>5\}$

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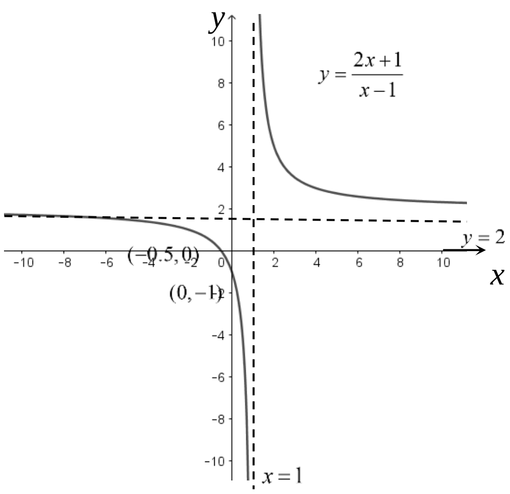
<tr><td>(b)(i)</td><td>



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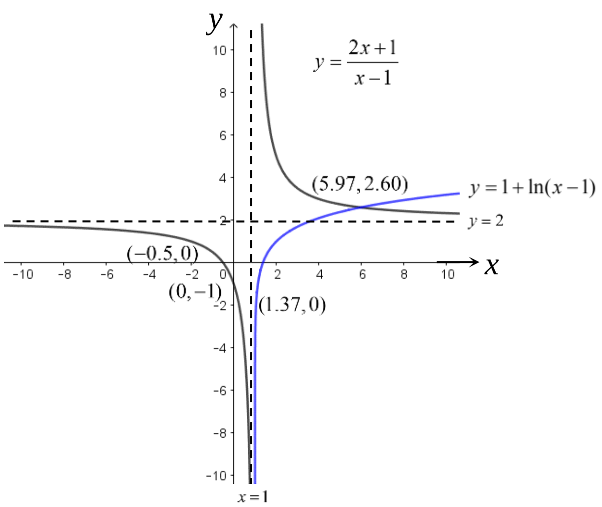
<tr><td>(b)(ii)</td><td>

$\frac{2x+1}{x-1}-\ln(x-1)\geq 1$

$\frac{2x+1}{x-1}\geq 1+\ln(x-1)$



Range of x values: $1<x\leq 5.97$ (3 s.f)

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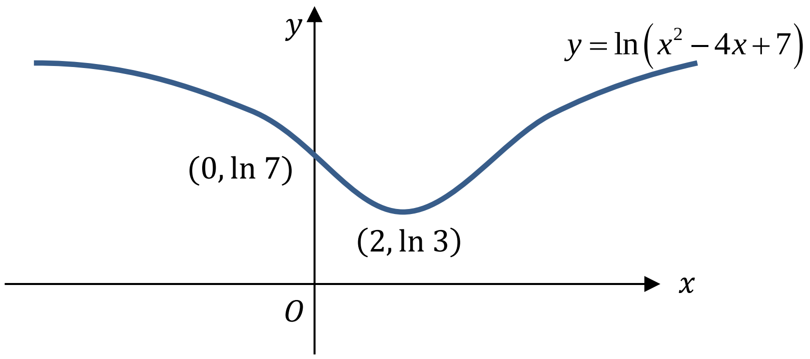
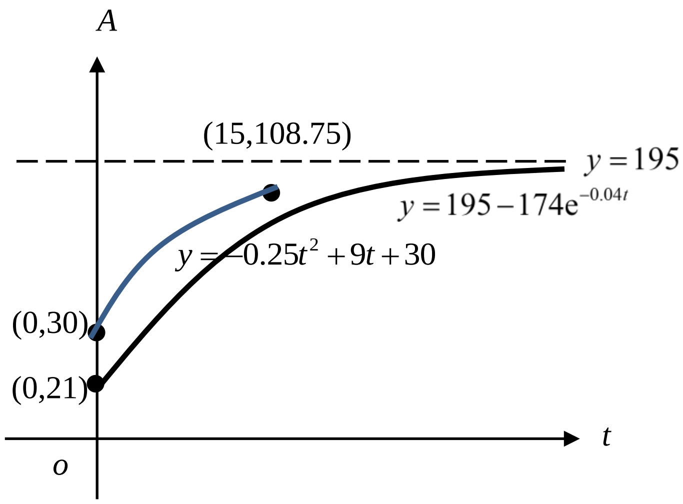
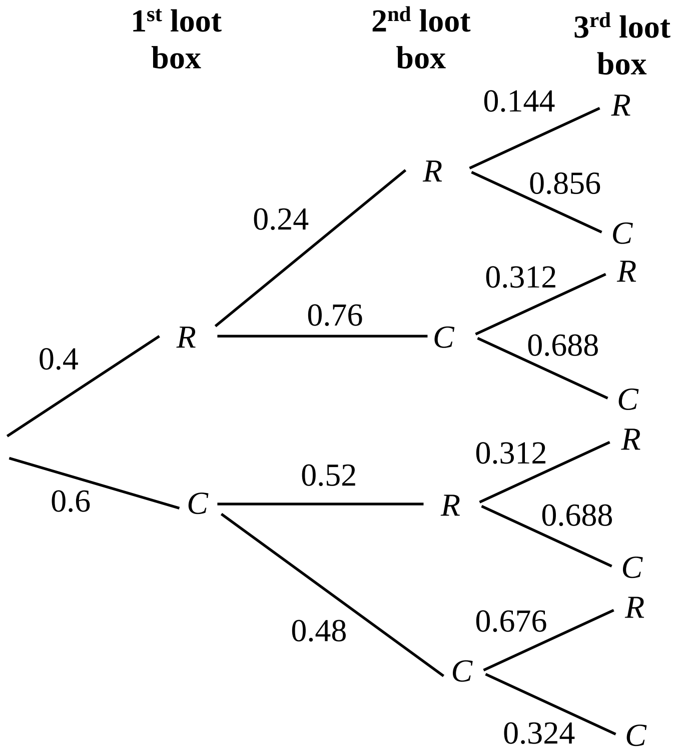
Qn	Solution
4	**Applications of Differentiation**
(i)	$y=\ln\left(x^2-4x+7\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-4}{\left(x^2-4x+7\right)}$ $\frac{2x-4}{\left(x^2-4x+7\right)}=0\Rightarrow x=2, y=\ln 3$ Coordinates of stationary point: $\left(2,\ln 3\right)$
(ii)	
(iii)	When $x=0, y=\ln 7,\ \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4}{7}$ Equation of tangent: $y-\ln 7=-\frac{4}{7}\left(x-0\right)$ $y=-\frac{4}{7}x+\ln 7$
(iv)	$y=-\frac{4}{7}x+\ln 7$ When $x=0, y=\ln 7$ and $y=0, x=\frac{7}{4}\ln 7$ $\therefore P$ has coordinates $\left(\frac{7}{4}\ln 7,0\right)$ and Q has coordinates $\left(0,\ln 7\right)$ Area of triangle $OPQ=\frac{1}{2}\left(\frac{7}{4}\ln 7\right)\left(\ln 7\right)=\frac{7}{8}\left(\ln 7\right)^2$ units^2

Qn	Solution
5	**Cross topical – curve sketching, differentiation, integration**
(i)	$A = 195 - 174e^{-0.04(4)}$ $= 46.727$ sales = \$46727 (nearest dollar)
(ii)	Using GC, $\frac{dA}{dt} = 5.0539973$ Rate of change of weekly sales is \$5054 per week (nearest dollar)
(iii), (iv)	
(iv)	$\int_0^k \left(-0.25t^2 + 9t + 30\right) - \left(195 - 174e^{-0.04t}\right)\ dt$ $= \int_0^k -0.25t^2 + 9t - 165 + 174e^{-0.04t}\ dt$ $= \left[\frac{-0.25}{3}t^3 + \frac{9}{2}t^2 - 165t - \frac{174}{0.04}e^{-0.04t}\right]_0^k$ $= \frac{-0.25}{3}k^3 + \frac{9}{2}k^2 - 165k - \frac{174}{0.04}e^{-0.04k} - \left(-\frac{174}{0.04}\right)$ $= -\frac{1}{12}k^3 + \frac{9}{2}k^2 - 165k - 4350e^{-0.04k} + 4350$ The integral represents the **increase in total sales in the first *k* weeks** due to the advertising campaign.
(v)	For an eight-week advertising campaign, $k = 8$, increase in total sales $= -\frac{1}{12}(8)^3 + \frac{9}{2}(8)^2 - 165(8) - 4350e^{-0.04(8)} + 4350$ $= 116.585$ thousands > \$100 000 which is more than the total cost incurred for the campaign. Hence, the company should release the cereal on the market with the advertising campaign.

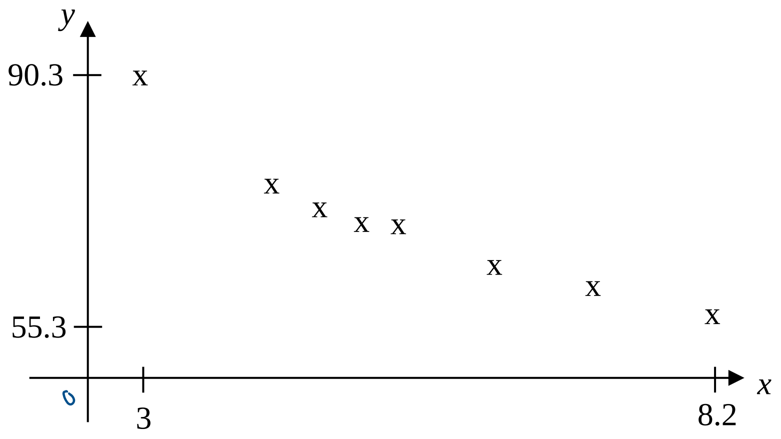
Qn	Solution
6	**Normal Distribution**
	$\mathrm{P}\left(Z > \frac{10-\mu}{\sigma}\right) = 0.748, Z \sim N(0,\ 1)$ $\frac{10-\mu}{\sigma} = -0.66821$ $\mu - 0.66821\sigma = 10 --- (1)$ $\mathrm{P}(X > 18) = 0.748 - 0.725$ $\mathrm{P}(X > 18) = 0.023$ $\mathrm{P}\left(Z > \frac{18-\mu}{\sigma}\right) = 0.023, Z \sim N(0,\ 1)$ $\frac{18-\mu}{\sigma} = 1.9954$ $\mu + 1.9954\sigma = 18 --- (2)$ Solving equations (1) and (2), μ = 12.0 , σ = 3.00

Qn	Solution
7	**Probability**
(i)	Since $\mathrm{P}(A\mid B)=\frac{1}{2}p$ and $\mathrm{P}(A)=2p$ and $p>0$, $\mathrm{P}(A\mid B)\neq \mathrm{P}(A)$ Hence *A* and *B* are not independent events.
(ii)	$\mathrm{P}(A\mid B)=\frac{1}{2}p$ $\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{1}{2}p$ $\mathrm{P}(A\cap B)=\frac{1}{2}p^2$ $\mathrm{P}(A\cup B)=0.6$ $\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)=0.6$ $2p+p-\frac{1}{2}p^2=0.6$ $6p-p^2=1.2$ $p^2-6p+1.2=0$ Using GC, $p=0.207151$ or $p=5.79284$ (reject since $0<p<0.5$) $\therefore p=0.207$
(iii)	$\mathrm{P}(A'\cup B')=1-\mathrm{P}(A\cap B)$ $=1-\frac{1}{2}(0.207151)^2$ $= 0.979$

Qn	Solution
8	**Binomial Distribution**
(i)	The probability that any randomly chosen strawberry is bruised is constant at $\frac{2}{100}=0.02$. Whether a randomly chosen strawberry is bruised is independent of any other strawberries.
(ii)	The probability that a strawberry is bruised may not be the same across all strawberries due to the nature of the harvesting process.
(iii)	$X \sim \mathrm{B}(20, 0.02)$ $\mathrm{P}(X=1)=0.272(3 \text{ s.f.})$
(iv)	$\mathrm{P}(X \geq 2)=1-\mathrm{P}(X \leq 1)=0.059899=0.0599 \quad (3 \text{ s.f.})$
(v)	Let Y be the number of punnets in a crate of n punnets that have at least 2 bruised strawberries. $Y \sim \mathrm{B}(n, 0.059899)$ $\mathrm{P}(Y \leq 1) \leq 0.2$ From GC, when $n=48$, $\mathrm{P}(Y \leq 1)=0.20928>0.2$, when $n=49$, $\mathrm{P}(Y \leq 1)=0.19983<0.2$. Hence, least $n=49$.

Qn	Solution
9	**Probability (Tree Diagram**
(i)	Let R be the event Philip got a rare item from the loot box Let C be the event Philip got a common item from the loot box 
(ii)	P(at least 1 common item from the loot boxes) $=1-\mathrm{P}(RRR)$ $=1-0.4(0.24)(0.144)$ $=0.986 \quad (3\text{ s.f.})$
(iii)	P(at least 1 rare item | common item in the 2nd loot box) $=\dfrac{\mathrm{P}(RC)+\mathrm{P}(CCR)}{\mathrm{P}(\text{common item on }2^{\text{nd}}\text{ loot box})}$ $=\dfrac{0.4(0.76)+0.6(0.48)(0.676)}{0.4(0.76)+0.6(0.48)}$ $=0.842 \quad (3\text{ s.f.})$ Or $=\dfrac{\mathrm{P}(RCR)+\mathrm{P}(RCC)+\mathrm{P}(CCR)}{\mathrm{P}(\text{common item on }2^{\text{nd}}\text{ loot box})}$ $=\dfrac{0.4(0.76)(0.312)+0.4(0.76)(0.688)+0.6(0.48)(0.676)}{0.4(0.76)+0.6(0.48)}$ $=0.842 \quad (3\text{ s.f.})$

Qn	Solution

10	**Correlation and Regression**
(i)	Using GC, $\bar{x} = 5.2875$ Let k be Candidate H's weight (in kg). $\bar{y} = \dfrac{70.3+58.7+90.3+66.7+68.5+55.3+62.3+k}{8}$ $= \dfrac{472.1+k}{8}$ Substitute $(\bar{x}, \bar{y})$ into the regression line y on x, $\bar{y} = 101.68 - 6.1332\bar{x}$ $\dfrac{472.1+k}{8} = 101.68 - 6.1332(5.2875)$ $k = 81.9$ kg (3 s.f.)
(ii)	
(iii)	Using GC, $r = -0.908$ (3 s.f.) Since $r = -0.908$ is close to -1, there is a strong negative linear correlation between x and y. As the hours of sleep per day increases, the weight of the person decreases.
(iv)	Using GC, $y = 98.870 - 5.7766x$ (5 s.f.) $y = 98.9 - 5.78x$ (3 s.f.)
(v)	For $x = 7$, $y = 98.870 - 5.7766(7) = 58.4$ kg (3 s.f.) Since $x = 7$ is within data range and r is close to -1, it indicates a strong negative linear correlation between x and y. Hence, the estimate is reliable.
(vi)	There are other factors like exercising and dieting which contribute to weight loss. Hence, it is incorrect to claim that sleeping more causes weight loss.

Qn	Solution
11	**Hypothesis Testing**
(i)	An unbiased estimate for population mean is $\bar{x} = \frac{17550}{30} = 585$. An unbiased estimate for population variance is $s^2 = \frac{1}{29}\left[10350000 - \frac{17550^2}{30}\right] = \frac{83250}{29}$ (or 2870 to 3 s.f.)
(ii)	Let μ denotes the population mean mass of cakes (in grams). $H_0 : \mu = 600$ $H_1 : \mu < 600$ Under H_0, since $n = 30$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(600, \frac{\frac{83250}{29}}{30}\right)$ approximately. Using GC, $p-\text{value} = 0.062587$ Since H_0 is not rejected, $p-\text{value} > \frac{\alpha}{100}$ $0.062587 > \frac{\alpha}{100}$ $6.2587 > \alpha$ $\therefore \{\alpha \in \mathbb{R} : 0 < \alpha < 6.25\}$
(iii)	$H_0 : \mu = 600$ $H_1 : \mu \neq 600$ At 5% significance level, reject H_0 if $z-\text{value} < -1.95996$ or $z-\text{value} > 1.95996$ $\frac{\bar{x} - 600}{\sqrt{\frac{2000}{40}}} < -1.95996$ or $\frac{\bar{x} - 600}{\sqrt{\frac{2000}{40}}} > 1.95996$ $\bar{x} < 586.14$ or $\bar{x} > 613.859$ $\therefore \bar{x} < 586$ or $\bar{x} > 614$

Qn	Solution
12	**Normal and Sampling Distribution**
(i)	Let X be the weight of a randomly chosen packet of chocolate chip cookies. $X \sim N\left(400,10^2\right)$ Let Y be the weight of a randomly chosen packet of red velvet cookies. $Y \sim N\left(280,3^2\right)$ $P\left(X<380\right)P\left(Y<275\right)=0.00109$
(ii)	$X+Y \sim N\left(400+280,10^2+3^2\right)$ $\therefore X+Y \sim N\left(680,109\right)$ $\mathrm{P}\left(X+Y<655\right)=0.0083200224=0.00832$
(iii)	The event in **(i)** is a proper subset of the event in **(ii)**.
(iv)	Let B be the cost of a randomly chosen packet of chocolate chip cookies. $B=0.012X \sim N\left(4.8,0.0144\right)$ Let C be the cost of a randomly chosen packet of red velvet cookies. $C=0.04Y \sim N\left(11.2,0.0144\right)$ $B_1+B_2+B_3-C \sim N\left(3(4.8)-11.2,3(0.0144)+0.0144\right)$ $\therefore B_1+B_2+B_3-C \sim N\left(3.2,0.0576\right)$ $\mathrm{P}\left(B_1+B_2+B_3-C>3\right)=0.798$
(v)	. Let $\overline{B}$ be the mean mass of ten randomly chosen packets of chocolate chip cookies. $\overline{B} \sim N\left(4.8,\dfrac{0.0144}{10}\right)$ $\mathrm{P}\left(4.70<\overline{B}<4.90\right)=0.992$