2. Equations and Inequalities (solutions)

<u>(I) Equations</u>

1	$d+f+g=140$ …(1) $g=d+f+20 \Rightarrow d+f-g=-20$ …(2) $21d+42f+10g=2900$ …(3) Solving the 3 equations, $d=20, f=40, g=80$
2	Let *x*, *y*, and *z* be the number of trays of blueberry, strawberry and chocolate cupcakes respectively. Time: $8x+7y+6z=17\times 60=1020$ Amt: $0.6x+0.6y+0.8z=96$ Price: $12x(1)+12y(0.9)+12z(0.8)=1572$ Using GC, $x=50,\ y=50,\ z=45$
3	2) Let the price (per kg) for crab, lobster and bamboo clam for his first visit be c, ℓ, b. $3.20c+1.50\ell+7b=277.50$ $5.60c+1.20(1.1\ell)+6.50b=347$ $4.50c+2(1.1^2\ell)+6.50b=395.18$ From GC, $c=36.20$, $\ell=79.9983$, $b=5.95$ Required price: <u>\$36.20</u>, 79.9983×1.1^2 = <u>\$96.80</u> and <u>\$5.95</u> respectively.
4	Let *x* be no. of chickens. Let *y* be no. of horses. Let *z* be no. of sheep. $z=2x \qquad \Rightarrow 2x+0y-z=0$ --------------(1) $2x+4y+4z=1250 \qquad \Rightarrow 2x+4y+4z=1250$ ---------(2) **<u>Case 1:</u>** If $x+y+z=250$ -------------(3) By GC, $x=-125,\ y=625,\ z=-250$ (rejected) (Alternative): Reject $x+y+z=250$, because any combination of 250 animals will never have 1250 legs. (Maximum no of legs = 250 x 4 = 1000)

Case 2:

If $x+y+z=350$ -------------(3)

By GC, $x = 75$, $y = 125$, $z = 150$

$\therefore$ Correct number of chickens =75, horses =125, sheep = 150

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Finding table values in SGD,

	Cheese/kg	Chocolate/kg	Candy/kg
Price/SGD	4	6	6
Price/SGD	8	10	4
Price/SGD	8	5	7

Let *x, y, z* be the number of three kg packs bought from Denmark, England and Russia respectively.

$4x+8y+8z=84$

$6x+10y+5z=85$

$6x+4y+7z=77$

$x=5,\quad y=3,\quad z=5$

She should buy 13 packs in total.

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Sub $(1,1)$ and $(2,2)$ into $y=\mathrm{h}(x)$.

$a+b+c+d=1$ ----- (1)

$8a+4b+2c+d=2$ -----(2)

Since (2,2) is also the stationary point, $\mathrm{h}'(2)=0$. i.e.

$12a+4b+c=0$ ----- (3)

Using the GC,

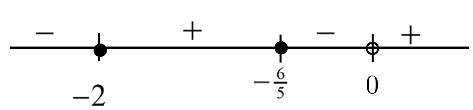
$a=-\frac{1}{2}-\frac{1}{4}d$

$b=\frac{3}{2}+\frac{5}{4}d$

$c=-2d$

$$\frac{ab}{c}\le 0$$

$$\frac{\left(-\frac{1}{2}-\frac{1}{4}d\right)\left(\frac{3}{2}+\frac{5}{4}d\right)}{-2d}\le 0$$



$\{d\in\mathbb{R}: d\le -2 \text{ or } -\frac{6}{5}\le d<0\}$

7	$M\colon\ x^2+y^2+Ax+By+C=0$ $\Rightarrow \left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2-\frac{A^2}{4}-\frac{B^2}{4}+C=0$ Centre of M : $\left(-\frac{A}{2},-\frac{B}{2}\right)$ $y=-2(x+1)$ passes through the centre: $-\frac{B}{2}=-2\left(-\frac{A}{2}+1\right)$ $2A+B=4$ ----- (1) At intersection between $y=\ \lvert x\rvert$ and M, we have $x^2+\lvert x\rvert^2+Ax+B\lvert x\rvert+C=0$ At $x=-2$, $(-2)^2+(\lvert -2\rvert)^2+A(-2)+B(\lvert -2\rvert)+C=0$ $-2A+2B+C=-8$ ----- (2) At $x=-8$, $(-8)^2+(\lvert -8\rvert)^2+A(-8)+B(\lvert -8\rvert)+C=0$ $-8A+8B+C=-128$ ----- (3) Solving (2), (3) and (4) using GC: $A=8,\quad B=-12,\quad C=32$ $M\colon\ x^2+y^2+8x-12y+32=0$
8	Let $x : y : z$ be the ratio for the servings of fish fillet, salad and fries. $150x + 15y + 5z = 4k$ $60x + 30y + 250z = 8k$ $25x + 5y + 110z = 3k$ where k is a constant. $150x + 15y + 5z = 4$ $60x + 30y + 250z = 8$ $25x + 5y + 110z = 3$ Solving matrix or simultaneous equations $x = 0.02$, $y = 0.06$, $z = 0.02$ Ratio is 1:3:1 (ans)
9	(i) Let $u_n=an^3+bn^2+cn+d$ $u_1=a+b+c+d=32.1$

$u_2 = 8a + 4b + 2c + d = 17$

$u_3 = 27a + 9b + 3c + d = 0.7$

$u_4 = 64a + 16b + 4c + d = -7.8$

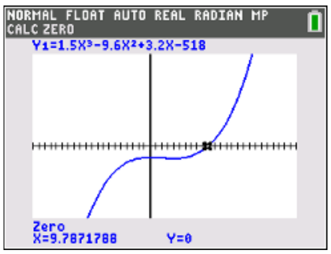
Using GC, $a = 1.5$, $b = -9.6$, $c = 3.2$, $d = 37$

$\therefore u_n = 1.5n^3 - 9.6n^2 + 3.2n + 37$

(ii)

$u_n > 555 \Rightarrow 1.5n^3 - 9.6n^2 + 3.2n - 518 > 0$

Method 1

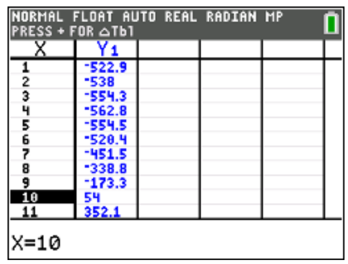


From GC graphing, $n > 9.7871$

$\therefore$ least value of $n = 10$

Method 2

From GC Table,



NORMAL FLOAT AUTO REAL RADIAN MP
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X	Y1
1	-522.9
2	-538
3	-554.3
4	-562.8
5	-554.5
6	-520.4
7	-451.5
8	-338.8
9	-173.3
10	54
11	352.1

X=10

$\therefore$ least value of $n = 10$

(II) Inequalities

10

$\frac{x+3}{x-1} < x < \frac{1}{2}$

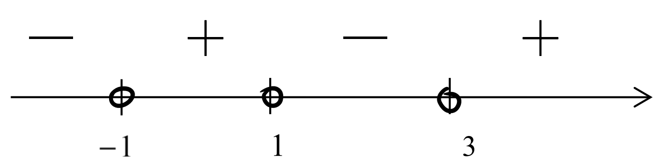
$\frac{x+3}{x-1} < x$ and $x < \frac{1}{2}$

$\frac{x+3-x(x-1)}{x-1} < 0$

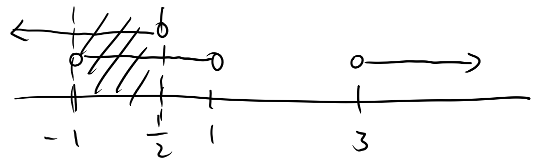
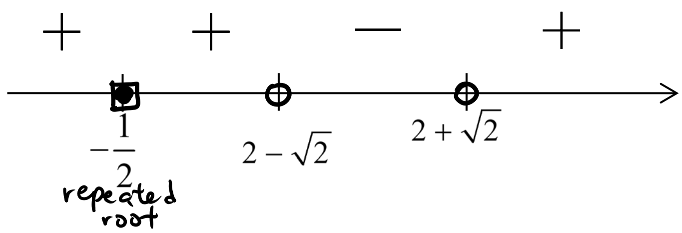
$\frac{3+2x-x^2}{x-1} < 0$

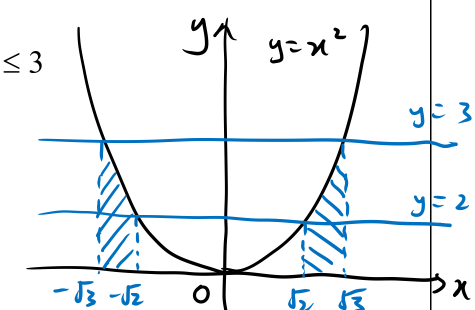
$\frac{x^2-2x-3}{x-1} > 0$

$\frac{(x+1)(x-3)}{x-1} > 0$



$\therefore -1 < x < 1$ or $x > 3$

	Since $x<\frac{1}{2}$, $\quad -1<x<\frac{1}{2}$ 
11	(i) $\frac{(2x+1)^2}{4x-x^2-2}\geq 0$ $\frac{(2x+1)^2}{x^2-4x+2}\leq 0$ $\frac{(2x+1)^2}{(x-2)^2-2}\leq 0$ $\frac{(2x+1)^2}{\left(x-2+\sqrt{2}\right)\left(x-2-\sqrt{2}\right)}\leq 0$  $2-\sqrt{2}<x<2+\sqrt{2}$ or $x=-\frac{1}{2}$ (ii) Replace x with $\sqrt{x}$ $2-\sqrt{2}<\sqrt{x}<2+\sqrt{2}$ or $\sqrt{x}=-\frac{1}{2}$ (rej) $0.343<x<11.7$

12	$\dfrac{6}{x^2} \le \dfrac{x+1}{x}$ $---(*)$ $\dfrac{6-x(x+1)}{x^2} \le 0$ $\dfrac{-x^2-x+6}{x^2} \le 0$ $\dfrac{x^2+x-6}{x^2} \ge 0$ $\dfrac{(x+3)(x-2)}{x^2} \ge 0$ $x \le -3$ or $x \ge 2$ Replace x with $x-2$ in (*), obtain $\dfrac{6}{(x-2)^2} \le \dfrac{(x-2)+1}{x-2}$ $\dfrac{6}{(2-x)^2} \le \dfrac{x-1}{x-2}$ $x-2 \le -3$ or $x-2 \ge 2$ $x \le -1$ or $x \ge 4$
13	$\dfrac{5}{x-2} \le x+2$ $\dfrac{5}{x-2}-(x+2) \le 0$ $\dfrac{5-(x-2)(x+2)}{x-2} \le 0$ $\dfrac{(3-x)(3+x)}{x-2} \le 0$ Solution set = $\{x \in \mathbb{R} : x \ge 3 \text{ or } -3 \le x < 2\}$ **<u>"Hence method"</u>** Replace x by x^2 The solutions for $\dfrac{5}{x^2-2} \ge x^2+2$ are $x^2 \le -3$ or $2 < x^2 \le 3$ $\therefore \sqrt{2} < x \le \sqrt{3}$ or $-\sqrt{3} \le x < -\sqrt{2}$  **<u>"Otherwise method"</u>** Solve algebraically but method is longer.

14

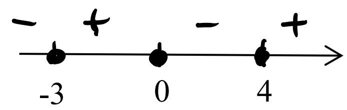
$x(x+4)(x-1) \geq 4x(x+2)$

$x(x^2+3x-4-4x-8) \geq 0$

$x(x^2-x-12) \geq 0$

$x(x+3)(x-4) \geq 0$

$-3 \leq x \leq 0 \;\text{ or }\; x \geq 4$



(ii) Replace x by $|x|$,

$\therefore -3 \leq |x| \leq 0 \;\text{ or }\; |x| \geq 4$

Since $|x| \geq 0$,

$\Rightarrow x = 0 \;\text{ or }\; x \geq 4 \;\text{ or }\; x \leq -4.$

(ii) Replace x by $\frac{x}{2}$,

$\therefore -3 \leq \frac{x}{2} \leq 0 \;\text{ or }\; \frac{x}{2} \geq 4$

$\Rightarrow \therefore -6 \leq x \leq 0 \;\text{ or }\; x \geq 8$

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$$\frac{x^2-2x+15}{x^2-6x+6} \geq 0$$

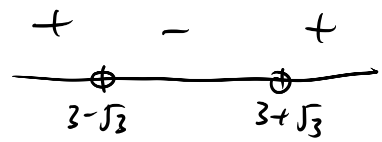
Since the discriminant of $x^2-2x+15 = 4-4(1)(15) = -56 < 0$ and the coefficient of x^2 is positive, we know $x^2-2x+15 > 0$ for all real values of x.

Since $x^2-2x+15 > 0$ for all real values of x,

$$\frac{x^2-2x+15}{x^2-6x+6} \geq 0 \;\Rightarrow\; x^2-6x+6 > 0$$

$$\Rightarrow \left(x-3-\sqrt{3}\right)\left(x-3+\sqrt{3}\right) > 0$$

$$x < 3-\sqrt{3} \quad\text{or}\quad x > 3+\sqrt{3}$$



Replace x by $|x|$,

$|x| < 3-\sqrt{3} \quad\text{or}\quad |x| > 3+\sqrt{3}$

$-3+\sqrt{3} < x < 3-\sqrt{3} \quad\text{or}\quad x < -3-\sqrt{3} \quad\text{or}\quad x > 3+\sqrt{3}$

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From GC, $0 < x < 4.42806 \;\text{ or }\; x > 13.706$

$0 < x < 4.43 \;\text{ or }\; x > 13.7$

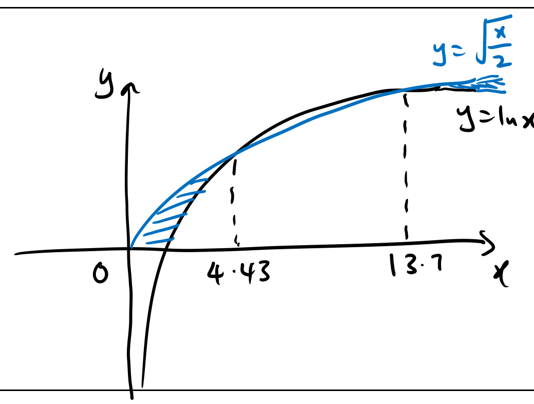
Replacing x by x^2: $\sqrt{\frac{x^2}{2}} > \ln x^2$

$x > 2\sqrt{2}\ln x$

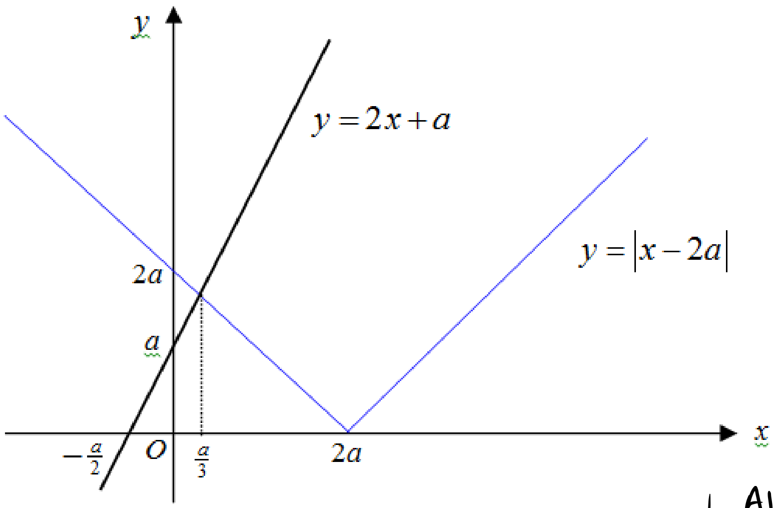
From above, $0 < x^2 < 4.42806 \;\text{ or }\; x^2 > 13.706$

$0 < x < 2.1043 \;\text{ or }\; x > 3.702$

$0 < x < 2.10 \;\text{ or }\; x > 3.70$



17



To find intersection point, $2x + a = -(x - 2a)$

$$x = \frac{a}{3}$$

From the graph, for $|x - 2a| < 2x + a$,

$$x > \frac{a}{3}$$

Replace x by $-x$ and let $a = 2$ in the above inequality,

$|(-x) - 2(2)| < 2(-x) + 2$ becomes $|x + 4| < 2 - 2x$

Thus $-x > \frac{2}{3} \Rightarrow x < -\frac{2}{3}$

Alternative

$|x - 2a| < 2x + a$

$\Rightarrow -2x - a < x - 2a < 2x + a$

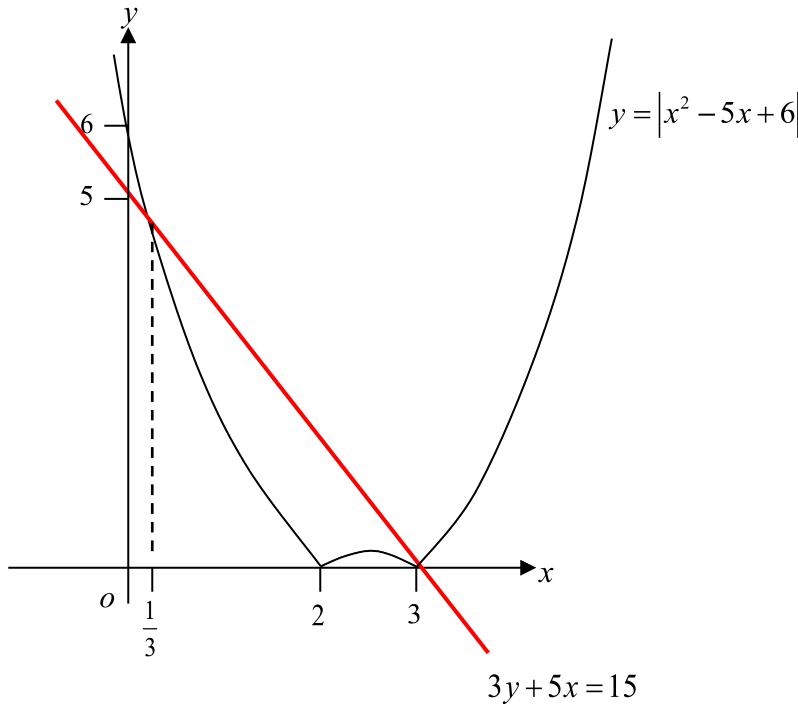
$\Rightarrow -2x - a < x - 2a$ and $x - 2a < 2x + a$

$\Rightarrow x > \frac{a}{3}$ and $x > -3a$

$\Rightarrow x > \frac{a}{3}$ //

18

$3y+5x=15 \Rightarrow y=5-\frac{5}{3}x$



$$3|x-3| \le \frac{15-5x}{|x-2|}, \quad (x \ne 2)$$

$$\Rightarrow \quad |x-3||x-2| \le \frac{15-5x}{3}$$

$$\Rightarrow \quad |(x-3)(x-2)| \le 5-\frac{5}{3}x$$

$$\Rightarrow \quad |x^2-5x+6| \le 5-\frac{5}{3}x$$

From GC, the x-coordinates of the points of the intersections of the 2 graphs are $x=\frac{1}{3}$ or $x=3$.

From the graph, $|x^2-5x+6| \le 5-\frac{5}{3}x$ for $\frac{1}{3} \le x \le 3$

Since $x \ne 2$, therefore the solution for $3|x-3| \le \frac{15-5x}{|x-2|}$

is $\frac{1}{3} \le x < 2$ or $2 < x \le 3$ (<u>OR</u> $\frac{1}{3} \le x \le 3, \quad x \ne 2$)

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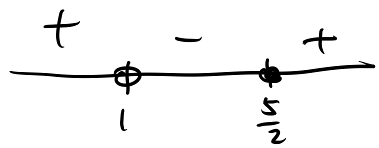
(i) $x^2+4x+5=(x+2)^2+5-2^2=(x+2)^2+1>0$, $\forall x\in\mathbb{R}$. (shown)

(ii)

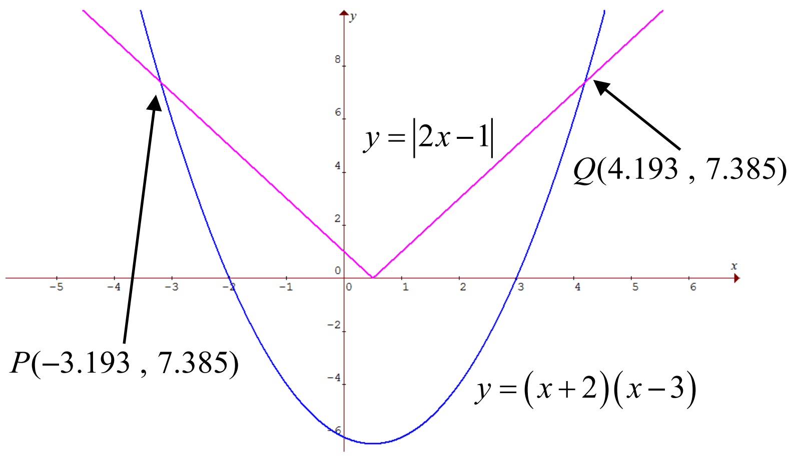
$$\frac{(2x-5)}{(x^2+4x+5)(x-1)}\le 0,\quad x\ne 1$$

$$\because x^2+4x+5>0\Rightarrow\frac{(2x-5)}{(x-1)}\le 0$$

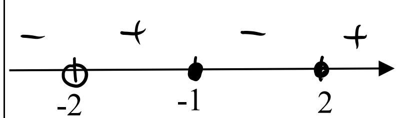


$$\Rightarrow 1<x\le\frac{5}{2}.$$

(b) Sketch the graphs of $y=(x+2)(x-3)$ and $y=|2x-1|$.



From the graph, solution to the inequality $(x+2)(x-3)>|2x-1|$ is $x<-3.19$ or $x>4.19$.

20	$\dfrac{(x+1)(4-x)}{(3x+1)^2} \geq 0, \quad x \neq -\dfrac{1}{3}$ Since $(3x+1)^2 > 0$ for all $x \in \mathbb{R} \setminus \{-\frac{1}{3}\}$, $\Rightarrow (x+1)(4-x) \geq 0$ $\therefore -1 \leq x \leq 4,\ x \neq -\dfrac{1}{3}$ Replace x with $\sqrt{x}$: $\therefore -1 \leq \sqrt{x} \leq 4,\ \sqrt{x} \neq -\dfrac{1}{3}$ Since $\sqrt{x} \geq 0$, $\Rightarrow 0 \leq \sqrt{x} \leq 4$ $\therefore 0 \leq x \leq 16$
21	$\dfrac{x^2}{x+2} \leq 1\ ,\ x \neq -2$ $\dfrac{x^2}{x+2} - 1 \leq 0$ $\dfrac{x^2 - x - 2}{x+2} \leq 0$ $\dfrac{(x-2)(x+1)}{x+2} \leq 0$  $\therefore x < -2$ or $-1 \leq x \leq 2$ (i) $\dfrac{x^2}{\lvert x\rvert + 2} \leq 1 \Rightarrow \dfrac{\lvert x\rvert^2}{\lvert x\rvert + 2} \leq 1$ Replace x with $\lvert x\rvert$: From above result, $\lvert x\rvert < -2$ (no solutions since $\lvert x\rvert \geq 0$ for all $x \in \mathbb{R}$) or $-1 \leq \lvert x\rvert \leq 2 \Rightarrow \lvert x\rvert \geq -1$ and $\lvert x\rvert \leq 2$ $\Rightarrow x \in \mathbb{R}$ and $-2 \leq x \leq 2$ Hence, the solution set = $\{\, x \in \mathbb{R} : -2 \leq x \leq 2 \,\}$.

(ii)

$$\frac{(-\mathrm{e}^x)^2}{-\mathrm{e}^x+2}\le 1$$

$$\frac{\mathrm{e}^{2x}}{2-\mathrm{e}^x}\le 1$$

Replace x with $-\mathrm{e}^x$:

From above result, $-e^x < -2 \Rightarrow e^x > 2 \Rightarrow x > \ln 2$

$or\ \ -1 \le -\mathrm{e}^x \le 2$

$\Rightarrow -\mathrm{e}^x \ge -1 \quad and \quad -\mathrm{e}^x \le 2$

$\Rightarrow \mathrm{e}^x \le 1 \quad and \quad \mathrm{e}^x \ge -2$

$\Rightarrow x \le 0 \quad and \quad x \in \mathbb{R}$

$\Rightarrow x \le 0$

Hence, the solution set = $\{x \in \mathbb{R} : x \le 0 \text{ or } x > \ln 2\}$

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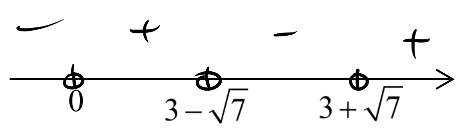
$$\frac{2}{x} < 6 - x.$$

$$\frac{2 - x(6-x)}{x} < 0$$

$$\frac{x^2 - 6x + 2}{x} < 0$$

$$\frac{\left(x-(3-\sqrt{7})\right)\left(x-(3+\sqrt{7})\right)}{x} < 0$$

$\therefore x < 0 \ \text{ or } \ 3-\sqrt{7} < x < 3+\sqrt{7}$

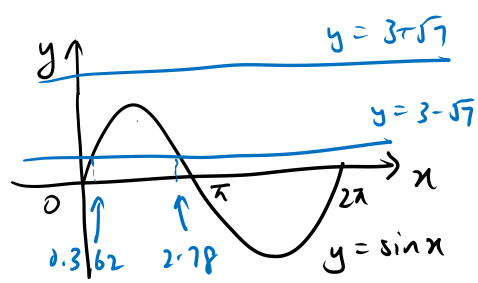


$$\frac{2}{\sin x} < 6 - \sin x$$

Replace x by $\sin x$

From above result, $\sin x < 0, \quad 3-\sqrt{7} < \sin x < 3+\sqrt{7}$

For $0 \le x \le 2\pi$, $0.362 < x < 2.78, \quad \pi < x < 2\pi.$



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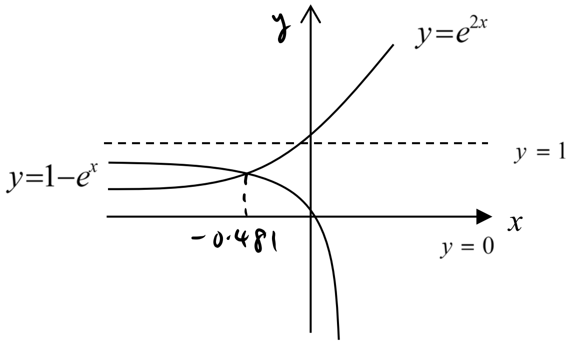
(a) Using G.C.,



x-coordinate of intersection: $x = -0.481$

For $e^{2x} < 1 - e^x$,

$\therefore\ x < -0.481$

Alternative Mtd:

Let $e^x = y$

$y^2 + y - 1 < 0$

$(y - 0.61803)(y + 1.6180) < 0$

$-1.6180 < y < 0.61803$

$-1.6180 < e^x < 0.61803$

$\Rightarrow 0 < e^x < 0.61803$

$\therefore x < -0.481$

(b) $2x^2 - 4x + 3 = 2(x-1)^2 + 1$, $a = -1$, $b = 1$

$$\frac{x^2}{x-3} < 1 - x$$

$$\frac{x^2}{x-3} + x - 1 < 0$$

$$\frac{x^2 + (x-1)(x-3)}{x-3} < 0$$

$$\frac{2x^2 - 4x + 3}{x-3} < 0$$

Since $2x^2 - 4x + 3 > 0$

$\Rightarrow x - 3 < 0$

$\therefore x < 3$

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(a)

For all real x,

$$4x^2-4x+3=4\left(x^2-x\right)+3$$
$$=4\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]+3$$
$$=4\left(x-\frac{1}{2}\right)^2+2>0 \quad \text{since} \quad \left(x-\frac{1}{2}\right)^2\geq 0$$

OR

$$4x^2-4x+3$$
$$=(2x-1)^2+2>0$$

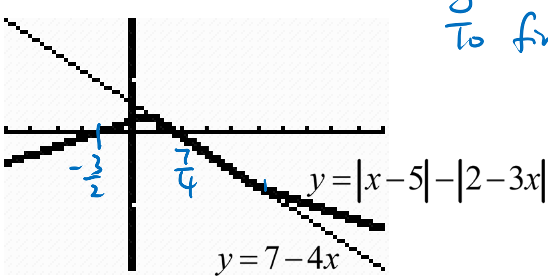
since $(2x-1)^2\geq 0$

$$\frac{32x-243}{x^2+7x-60}>4$$
$$\frac{4x^2+28x-240-32x+243}{x^2+7x-60}<0$$
$$\frac{4x^2-4x+3}{x^2+7x-60}<0$$

Since $4x^2-4x+3>0$ for all real x, then

$$x^2+7x-60<0$$
$$(x+12)(x-5)<0$$
$$-12<x<5$$

(b)



$y=|x-5|-|2-3x|$

To find x-intercepts ⇒ $y=0$

$|x-5|=|2-3x|$

$x-5=2-3x$ or $x-5=-(2-3x)$

$4x=7$ or $-2x=3$

$x=\frac{7}{4}$ or $x=-\frac{3}{2}$

$$\ln\left(|x-5|-|2-3x|\right)\leq \ln(7-4x)$$
$$\Rightarrow 0<|x-5|-|2-3x|\leq 7-4x$$

Using the sketch and the calculator, $-\frac{3}{2}<x<\frac{7}{4}$