**JURONG PIONEER JUNIOR COLLEGE**
**9749 H2 PHYSICS/ 8867 H1 PHYSICS**

# KINEMATICS

## Content

- **Rectilinear motion**
- **Non-linear motion**

## Learning Outcomes

Candidates should be able to:

(a) show an understanding of and use the terms distance, displacement, speed, velocity and acceleration.
(b) use graphical methods to represent distance, displacement, speed, velocity and acceleration.
(c) identify and use the physical quantities from the gradients of displacement-time graphs and areas under and gradients of velocity-time graphs, including cases of non-uniform acceleration.
(d) derive, from the definitions of velocity and acceleration, equations which represent uniformly accelerated motion in a straight line.
(e) solve problems using equations which represent uniformly accelerated motion in a straight line, including the motion of bodies falling in a uniform gravitational field without air resistance.
(f) describe qualitatively the motion of bodies falling in a uniform gravitational field with air resistance.
(g) describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

## Introduction

The study of physics often begins with *Newtonian Mechanics* which investigates relationships between force and motion. The study of motion can be divided into two aspects: how objects move **(kinematics)** and why objects move in different ways **(dynamics)**. Models and representations such as motion diagrams, graphs and equations are used to quantify, describe and predict motion. Motion in the real world is rather complex. The study of motion at this level is made simple with the use of assumptions. One valid assumption is to model the moving body as a point with no size where effects such as rotation or change of shape are not considered. It is also appropriate to ignore air resistance in solving most problems.

**Key Questions:**
1. How do we describe the motion of objects?
2. How can the motion of objects be represented, quantified and predicted?
3. How can we tell if an object is moving with a constant acceleration?
4. For an object falling freely in a gravitational field, how would it move?

## 1 Kinematics Quantities

*(a) Show an understanding of and use the terms distance, displacement, speed, velocity and acceleration.*

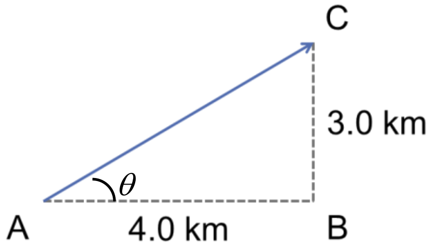
### 1.1 Distance and Displacement

**Distance** travelled (scalar) is the total length moved along the path of motion.
**Displacement** (vector) is the linear distance in a specified direction from a reference point.

**Example 1**

A man at point A travels 4.0 km due East to point B. He then travels a further 3.0 km due North to point C.

Determine the total distance travelled and his displacement from the original position (point A).




Total distance travelled

$= 4.0 + 3.0 = 7.0$ km

AC is the displacement $s$

$s^2 = 4.0^2 + 3.0^2$

$s = 5.0$ km

$tan\,\theta = \frac{3.0}{4.0}$

$\theta = 37°$

The displacement from A is 5.0 km in a direction 37° North of East.

### 1.2 Speed and Velocity

**Speed** (scalar) is the rate of change of distance moved by an object.

**Velocity** (vector) is defined as the rate of change of displacement.

Instantaneous speed is the speed of a moving object at a particular instant.

If an object travels along a path with varying speeds, we can always identify the smallest and largest speeds attained during its motion. The average speed lies between these two extremes. Normally when we mention about velocity, we are referring to instantaneous or constant velocity, unless otherwise stated.

| | **speed** | **velocity** |
|---|---|---|
| **instantaneous** | speed at a particular instant<br>$= \frac{\text{small distance travelled}}{\text{short time taken}}$<br>$= \frac{dx}{dt}$ | velocity at a particular instant<br>$= \frac{\text{small displacement}}{\text{short time taken}}$<br>$= \frac{ds}{dt}$ |
| **average** | average speed between 2 instants<br>$= \frac{\text{total distance travelled}}{\text{total time taken}}$<br>$= \frac{\Delta x}{\Delta t}$ | average velocity between 2 instants<br>$= \frac{\text{displacement}}{\text{time taken}}$<br>$= \frac{\Delta s}{\Delta t}$ |

**Example 2**

The man in Example 1 took 1 hour to move 4.0 km due East to point B, and another 1 hour to move 3.0 km due North to point C.

Determine the man's average speed and average velocity for the entire journey.

$$\text{Avg speed} = \frac{\text{total distance travelled}}{\text{total time taken}} = \frac{7.0}{2} = 3.5 \text{ km h}^{-1}$$

$$\text{Avg velocity} = \frac{\text{displacement}}{\text{time taken}} = \frac{5.0}{2} = 2.5 \text{ km h}^{-1}$$

The average velocity is 2.5 km $h^{-1}$ in a direction 37° North of East.

## 1.3 Acceleration

**Acceleration** (vector) is defined as the rate of change of velocity.

- Instantaneous acceleration, $a = \dfrac{dv}{dt}$
- Average acceleration , $\langle a \rangle = \dfrac{\Delta v}{\Delta t}$ where $\Delta v$ = change in velocity

  = final velocity – initial velocity

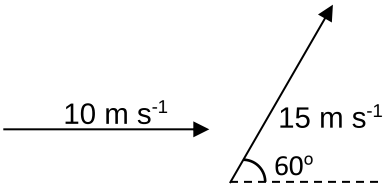
  $= v_f - v_i$

  or $= v - u$

Since velocity is a vector quantity, change in velocity $\Delta v$ is also a vector with a magnitude and a direction. A change in velocity occurs when either the magnitude or the direction changes, or both the magnitude and direction change.

**Example 3**

A particle has an initial horizontal velocity $v_i$ of 10 m $s^{-1}$ towards the right. A short time later, its velocity $v_f$ is 15 m $s^{-1}$ at an angle of 60° to the horizontal, as shown.




Calculate the change in velocity that has taken place.

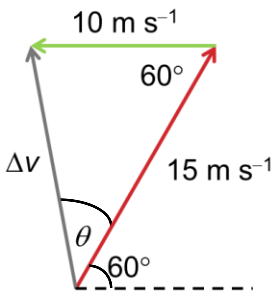
$\Delta\vec{v} = \vec{v}_f - \vec{v}_i = \vec{v}_f + (-\vec{v}_i)$

$(\Delta v)^2 = 10^2 + 15^2 - 2(10)(15)\cos 60°$

$\Delta v = 13.2$ m $s^{-1}$

$\dfrac{10}{\sin\theta} = \dfrac{13.2}{\sin 60°} \rightarrow \theta = 41°$

The change in velocity is 13.2 m $s^{-1}$ at an angle of 41° to the left of the final velocity.



| physical quantity | symbol | definition | nature | SI base unit |
|---|---|---|---|---|
| displacement | *s* | linear distance from a reference point, along a specified direction. | vector | m |
| speed | *v* | rate of change of distance travelled. | scalar | $m\ s^{-1}$ |
| velocity | *v* | rate of change of displacement. | vector | $m\ s^{-1}$ |
| acceleration | *a* | rate of change of velocity. | vector | $m\ s^{-2}$ |

## 1.4 Using + and – to represent directions

For 1-D (straight line) motion, directions of displacement, velocity and acceleration can be represented using ‘+’ and ‘–’ signs. The figure below shows the usual sign convention to denote directions.

| displacement with respect to reference point | | |
|---|---|---|
| + | right | above |
| – | left | below |
| **velocity and acceleration** | | |
| + | towards right | upwards |
| – | towards left | downwards |

Note that the sign convention can be reversed. It is valid as long as it is consistently applied within the same context.

## 1.5 Acceleration vs Deceleration

Deceleration is a term used to describe an object slowing down (speed decreases). It is not the same as negative acceleration as the negative sign denotes direction only. The sign of the acceleration is not sufficient to provide information on whether the object is moving faster or slower. For example, if upwards is taken as positive, a ball then falls with an acceleration of $-9.81\ m\ s^{-2}$. The negative sign denotes the downward direction of acceleration, hence the ball speeds up as it falls.

To determine if an object speeds up or slows down, we need to look at both the velocity and acceleration at the same time.

- When an object’s velocity and acceleration are in the same direction, the speed of the object increases with time.
- When an object’s velocity and acceleration are in the opposite direction, the speed of the object decreases with time. The object is said to be undergoing deceleration.

## 1.6 Free fall

An object is said to be in free fall if the only force acting on it is the gravitational force of the Earth.

The downward acceleration of a body undergoing free fall is known as *acceleration of free fall, g,* and has a constant value of $9.81\ m\ s^{-2}$. All free-falling objects have this acceleration, independent of their masses and direction of motion.

For a falling object, true free fall only exists in a vacuum where there is no air resistance.

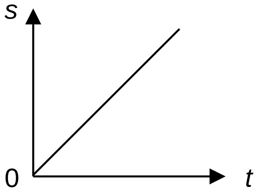
(b) *Use graphical methods to represent distance, displacement, speed, velocity and acceleration.*
(c) *Identify and use the physical quantities from the gradients of displacement-time graphs and areas under and gradients of velocity-time graphs, including cases of non-uniform acceleration.*

## 2.1 Displacement-time graph

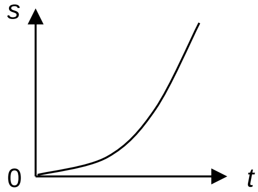
The motion of a moving object can be represented and analysed using a displacement-time (*s*-*t*) graph, which charts the variation of displacement with time.

- Gradient of a *s*-*t* graph = $\frac{ds}{dt}$ = velocity at a particular instant

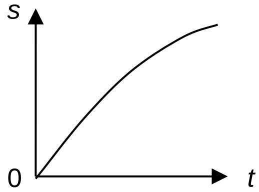
The diagrams below are some common displacement-time graphs:



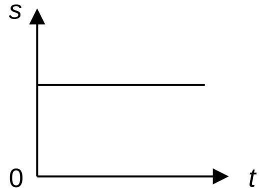
(a) constant velocity



(b) increasing velocity



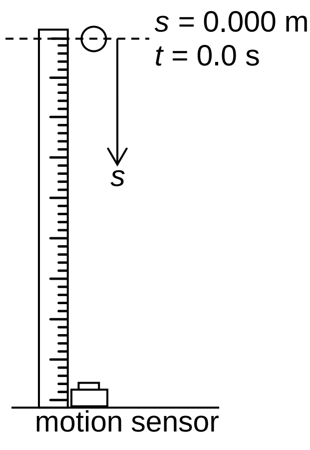
(c) decreasing velocity

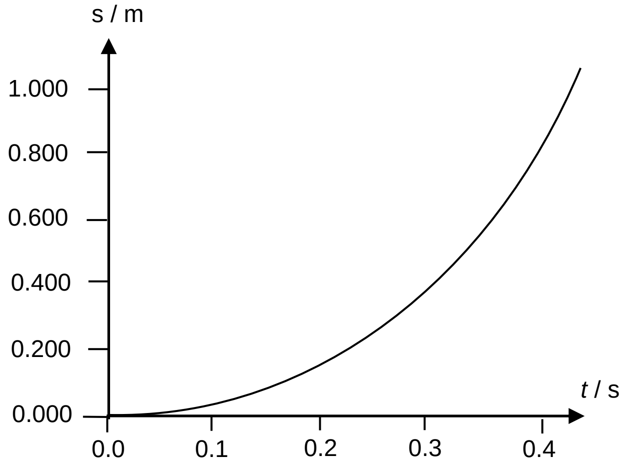


(d) zero velocity

**Example 4**

A ball placed at the zero mark of a vertical rule is released from rest. A motion sensor records the ball's displacement (with reference to the rule's zeroth mark) and corresponding time. The displacement-time graph is shown below.



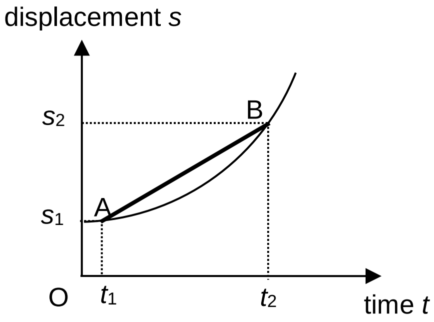

(a) With reference to the graph, briefly describe how the ball's velocity varies with time.
(b) Estimate the velocity of the ball when *t* = 0.3 s.

(a) The gradient of the graph increases with time. Since the gradient of a displacement-time graph represents the velocity, the ball's velocity increases with time.

(b) When $t = 0.3$ s,

$$\text{gradient of tangent} = \text{velocity} = \frac{0.72 - 0.00}{0.40 - 0.15} = 2.9 \text{ m s}^{-1}$$

## 2.2 Graphical interpretation of average velocity and instantaneous velocity



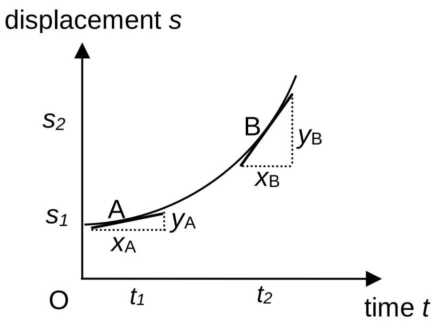
*A displacement-time graph to illustrate average velocity*

- For a journey taken from point A to point B,

$$\textit{Average velocity} = \frac{\text{change in displacement}}{\text{time taken}}$$

- Mathematically, $\langle v \rangle = \dfrac{\Delta s}{\Delta t} = \dfrac{s_2 - s_1}{t_2 - t_1}$

  Graphically, the average velocity between times $t_1$ and $t_2$ is derived from the gradient of the line joining these two points.



*A displacement-time graph to illustrate instantaneous velocity*

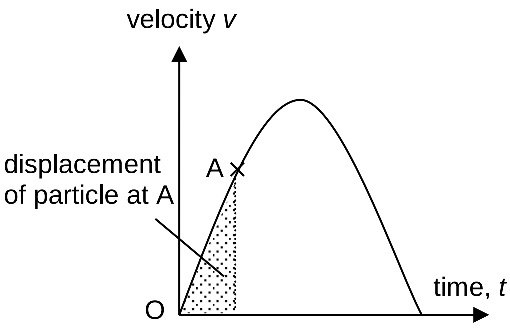
- *Instantaneous velocity* refers to velocity at a particular instant.
- Mathematically, $v = \dfrac{ds}{dt}$
- Graphically, it is determined from the gradient of the tangent at that instant.

  $\Rightarrow$ Instantaneous velocity at A = $\dfrac{y_A}{x_A}$

  $\Rightarrow$ Instantaneous velocity at B = $\dfrac{y_B}{x_B}$

## 2.3 Velocity-time graph

The motion of a moving object can also be represented and analysed using a velocity-time (*v*-*t*) graph, which charts the variation of velocity with time.



*A velocity-time graph*

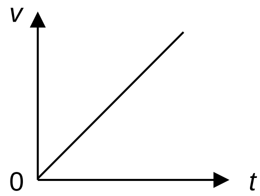
**Instantaneous acceleration**

$a = \dfrac{dv}{dt}$ is given by the gradient of the *v*-*t* graph at that instant.

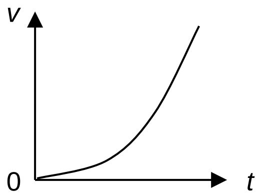
**Displacement**

$s = \int v dt$ is given by the area under the *v*-*t* graph.

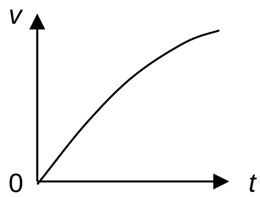
**Examples of velocity-time graphs**



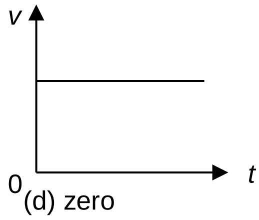

(a) constant acceleration



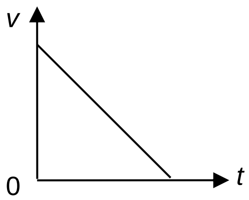

(b) increasing acceleration



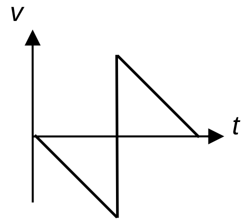

(c) decreasing acceleration



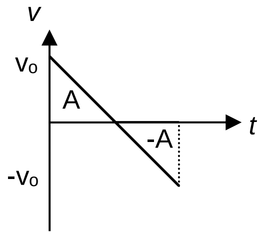

(d) zero acceleration




(e) constant deceleration




(f) Ball 1

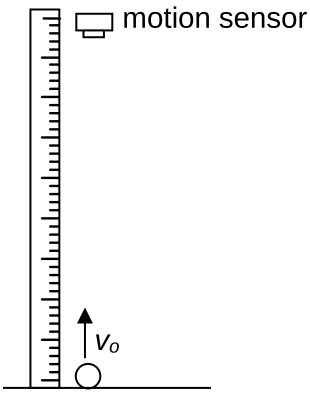



(g) Ball 2

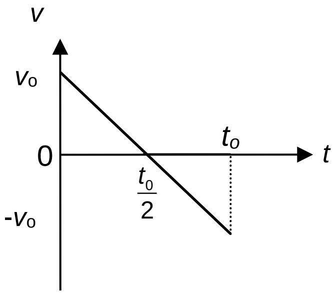
Taking upward to be positive,
Ball 1: released from rest from a height above a horizontal hard surface and rebounds once.
Ball 2: being thrown vertically upwards and falling back to the original position (in free fall).

**Example 5**

A ball is projected vertically upwards with a speed of $v_o$. The ball reaches a maximum height and falls down. A motion sensor records the ball's velocity and time, and a velocity-time graph is shown below.




Using the graph,
(a) describe how the ball's acceleration varies with time,
(b) describe how the ball's displacement varies with time,
(c) determine in terms of $v_o$ and $t_o$,
 (i) the displacement for the whole journey,
 (ii) the distance covered for the whole journey.




(a) The acceleration, which is represented by the gradient of the velocity-time graph, is constant, directed downward (negative gradient).

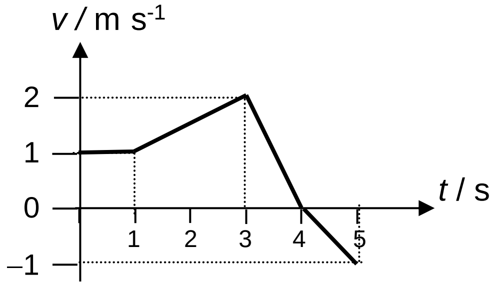
(b) The displacement increases to a maximum value in time $\frac{t_o}{2}$ and then falls back to zero in the same amount of time $\frac{t_o}{2}$.

(c) (i) displacement = 0

 (ii) distance $= 2\left(\frac{1}{2}\right)v_o\left(\frac{t_o}{2}\right) = \frac{v_o t_o}{2}$ (total area of the two triangles)

**Example 6**

A toy car is moving along a straight path, and its motion is represented by a velocity-time graph as shown.




(a) Describe the motion of the toy car.
(b) Sketch the displacement-time and acceleration-time graphs.

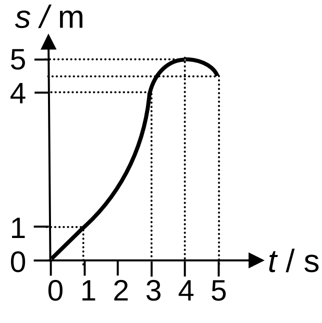
(a) For the first 1 s, the car is travelling at a constant velocity of 1 m $s^{-1}$.

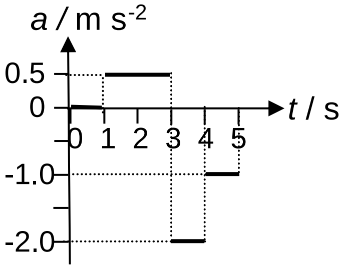
In the next 2 s, it accelerates uniformly with an acceleration of 0.5 m $s^{-2}$ to 2 m $s^{-1}$.

From $t$ = 3 s to $t$ = 4 s, it then decelerates uniformly at 2 m $s^{-2}$ to rest momentarily.

In the last 1 s, it moves in the opposite direction and accelerates at 1 m $s^{-2}$ until it reaches a velocity of –1 m $s^{-1}$.

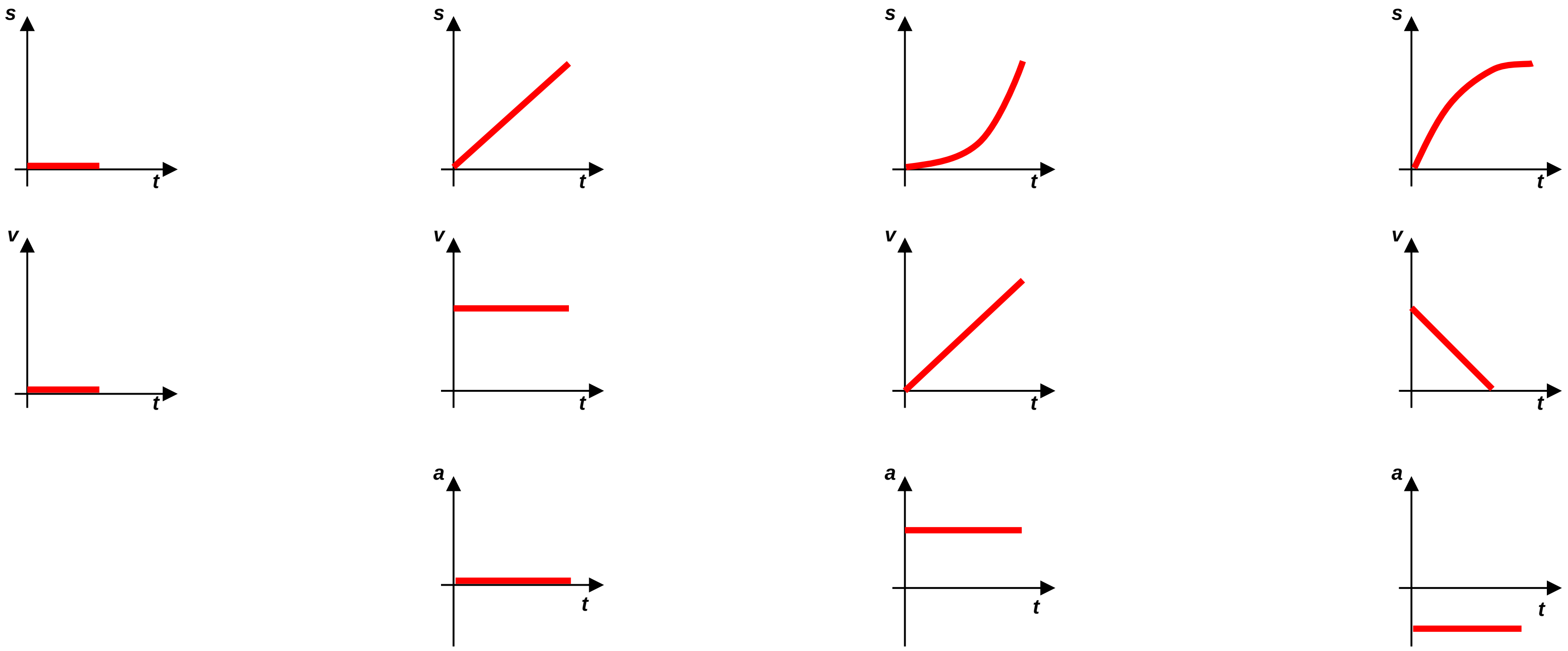
(b)



**Graphical Representations**

**No motion** | **Constant velocity motion** | **Uniformly accelerated motion** | **Uniformly decelerated motion**




*This figure illustrates that acceleration-time graphs and velocity-time graphs can be deduced from the displacement-time graphs. Please note that this is by no means an exhaustive list of graphs.*

# 3 Kinematics Equations of Motion

*(d) Derive, from the definitions of velocity and acceleration, equations which represent uniformly accelerated motion in a straight line.*

## 3.1 Uniformly accelerated motion

When an object's velocity changes at a constant rate, its acceleration is uniform. Two examples of uniformly accelerated motion are:

- a ball thrown up in the air (air resistance is ignored)
- a toy car sliding down a frictionless slope.

Uniformly accelerated motion can be analysed using Equations of Motion.

## 3.2 Derivation of Equations of Motion from definitions

Consider a body moving with a **constant** acceleration *a*. Equations used to describe its motion can be derived from first principles:

*Acceleration* is defined as the rate of change of velocity.

In equation form, $a = \frac{v-u}{t}$

where $a$ = acceleration
$u$ = initial velocity
$v$ = final velocity
$t$ = time taken

Making *v* the subject,

$$\boxed{v = u + at} \qquad \text{- - - - (1)}$$

*Average velocity* can be calculated by the ratio of total displacement *s* to the total time taken.

In equation form, $<v> = \frac{s}{t}$

$\rightarrow \quad s = <v>t$

Since the acceleration is constant, the average velocity is midway between *u* and *v*.

i.e. $<v> = \frac{u+v}{2}$

Equating and making *s* the subject,

$$\boxed{s = \frac{1}{2}(u+v)t} \qquad \text{- - - - (2)}$$

Substituting (1) into (2):

$$s = \frac{1}{2}[u + (u + at)]t$$

Making *s* the subject,

$$\boxed{s = ut + \frac{1}{2}at^2} \qquad \text{- - - - (3)}$$

From (1), $t = \frac{v-u}{a}$

Substituting this expression for *t* into (2):

$$s = \frac{1}{2}(u+v)\left(\frac{v-u}{a}\right)$$

$$= \left(\frac{v+u}{2}\right)\left(\frac{v-u}{a}\right)$$

$$= \frac{v^2 - u^2}{2a}$$

Making *v* the subject,

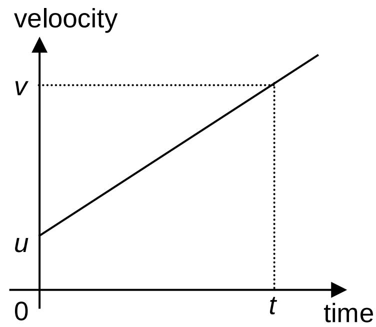
$$\boxed{v^2 = u^2 + 2as} \qquad \text{- - - - (4)}$$

Equations of Motion (1), (2), (3) and (4) only apply under the following two conditions:

- acceleration must be uniform;
- the vectors act along the same line (i.e. motion is rectilinear).

## 3.3 Derivation of Equations of Motion from graphs

For kinematics graphs, (1) gradient of velocity-time graph = acceleration $a$
and (2) area under velocity-time graph = displacement $s$




*A velocity-time graph illustrating uniform acceleration*

For the velocity-time graph shown,

gradient = $a = \dfrac{v-u}{t-0}$

area = $s = \dfrac{1}{2}(v+u)t$

Hence, $a = \dfrac{v-u}{t} \Rightarrow v = u + at$ - - - - (1)

$s = \dfrac{1}{2}(v+u)t$ - - - - (2)

Substituing (1) into (2) by replacing $v$, $s = ut + \dfrac{1}{2}at^2$ - - - - (3)

Substituting (1) into (2) by replacing $t$, $v^2 = u^2 + 2as$ - - - - (4)

## 3.4 Sign Convention

Since $u$, $v$, $a$ and $s$ are vector quantities, their directions must be taken into account when solving problems. Appropriate signs must be assigned to them when using these equations. Once determined, the sign convention must apply for the equation, for all the vectors.

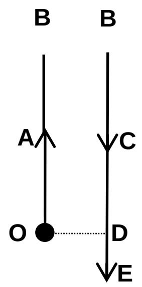
If, say, upward direction is taken to be positive, then

- displacement is negative for points below the initial position
- velocity of body moving downward is negative
- acceleration directed downward is negative

**Example 7**

A ball is projected vertically upwards from point O. Taking the upward direction as positive, identify the sign of the displacement $s$, velocity $v$ and acceleration $a$, during its flight from O, through A, B, C, D to E. (Ignore air resistance)

*The trajectory of a ball being thrown upwards:*




upward motion: O, A, B

downward motion: B, C, D, E

| | *s* | *v* | *a* |
|---|---|---|---|
| **O** | 0 | + | – |
| **A** | + | + | – |
| **B** | + | 0 | – |
| **C** | + | – | – |
| **D** | 0 | – | – |
| **E** | – | – | – |

## 4 Problem Solving

*(e) Solve problems using equations which represent uniformly accelerated motion in a straight line, including the motion of bodies falling in a uniform gravitational field without air resistance.*

### 4.1 Techniques in problem solving

In solving kinematics questions, the step-by-step process is to:

1. Draw a diagram to show the object's path.
2. State the sign convention.
3. List the known quantities and identify the unknown quantities.
4. Select the appropriate equations to apply.
5. Make the unknown quantities the subject of the equations and solve them.

**Example 8**

A car is travelling with uniform acceleration along a straight road. The road has marker posts every 100 m. When the car passes the first post, it has a speed of 10 m $s^{-1}$. When it passes the second post, its speed is 20 m $s^{-1}$.
(a) Calculate the car's acceleration.
(b) Calculate the car's velocity when it passes the third post?

(a) For the motion between the first and the second posts:

$u = 10$
$v = 20$
$a = ?$
$s = 100$
$t = ?$

$$v^2 = u^2 + 2as$$
$$20^2 = 10^2 + 2a(100)$$
$$a = 1.5 \text{ m s}^{-2}$$

(b) For the motion between the second and the third posts:

$u = 20$
$v = ?$
$a = 1.5$
$s = 100$
$t = ?$

$$v^2 = u^2 + 2as$$
$$v^2 = 20^2 + 2(1.5)(100)$$
$$v = 26.5 \text{ m s}^{-1}$$

### 4.2 Necessary information not made explicit in question

There are occasions where deductions must be made so that a problem can be solved.

For example, for vertical motion,

- $a = -g = -9.81$ m $s^{-2}$ when an object is under free fall, taking upwards as positive
- $a = g = 9.81$ m $s^{-2}$ when an object is under free fall, taking downwards as positive
- $v = 0$ m $s^{-1}$ when an object thrown vertically upwards is at maximum height
- $u = 0$ m $s^{-1}$ when an object is released from rest
- $s = 0$ m when an object thrown vertically upwards falls back to its initial position

**Example 9**

A man standing at the edge of a cliff throws a ball vertically upwards with a velocity of 20 m $s^{-1}$.

(a) Determine the maximum height above the cliff reached by the ball.

(b) If the cliff is 30 m high, determine

(i) the time taken for the ball to reach the base of the cliff;

(ii) the ball's speed just before it hits the base of the cliff.

Taking upwards as positive:

(a) $u = 20$    $v^2 = u^2 + 2as$

$v = 0$    $0 = 20^2 + 2(-9.81)h$

$a = -g = -9.81$    $h = 20.4$ m

$s = ?$

(b)(i) $u = 20$    $s = ut + \frac{1}{2}at^2$

$a = -9.81$    $-30 = 20t + \frac{1}{2}(-9.81)t^2$

$s = -30$    $4.905t^2 - 20t - 30 = 0$

$t = ?$    $t = \dfrac{20 \pm \sqrt{(-20)^2 - 4(4.905)(-30)}}{2(4.905)} = 5.24$ s

(ii) $u = 20$    $v^2 = u^2 + 2as$    OR    $v = u + at$

$v = ?$    $v^2 = 20^2 + 2(-9.81)(-30)$    $v = 20 + (-9.81)(5.24)$

$a = -9.81$    $v = \pm 31.4$    $= -31.4$ m $s^{-1}$

$s = -30$    $v = -31.4$ m $s^{-1}$

$t = 5.24$

**Example 10**

A parachuter falls from a hovering helicopter at an altitude of 1.5 km. For the first 10 s, he falls freely, and then he pulls the ripcord. The parachute opens and he falls with an upward acceleration of 20 m $s^{-2}$ until the downward speed is 5.0 m $s^{-1}$. Thereafter, he falls at a constant velocity. How long does the entire trip take?

The trip may be divided into 3 stages. Taking downward as positive:

For stage 1 (first 10 s, free fall),

$u_1 = 0$    $s = ut + \frac{1}{2}at^2$

$v_1 = ?$    $s_1 = 0 + \frac{1}{2}(9.81)(10)^2 = 490.5$ m

$a_1 = 9.81$    $v = u + at$

$s_1 = ?$    $v_1 = 0 + (9.81)(10) = 98.1$ m $s^{-1}$

$t_1 = 10$

For stage 2 (upward $a$),

$u_2 = 98.1$    $v = u + at$

$v_2 = 5.0$    $5 = 98.1 + (-20)t_2$

$a_2 = -20$    $t_2 = 4.655$ s

$s_2 = ?$    $v^2 = u^2 + 2as$

$t_2 = ?$    $5^2 = 98.1^2 + 2(-20)s_2$

$s_2 = 240.0$ m

For stage 3 (constant $v$), $t_3 = \dfrac{1500 - 490.5 - 240.0}{5.0} = 153.9$ s

Total time = 10 + 4.655 + 153.9 = 169 s

## 5 The effect of air resistance

*(f) Describe qualitatively the motion of bodies falling in a uniform gravitational field with air resistance.*

### 5.1 Air resistance

When an object moves through air, it experiences an opposing force called air resistance. The following 2 points must be kept in mind when analysing motion involving air resistance:

- Magnitude of air resistance is proportional to the object's speed (at low speed) or square of its speed (at high speed).
- Direction of air resistance is always opposite to the object's direction of motion.

### 5.2 Object thrown vertically up without air resistance (free fall)

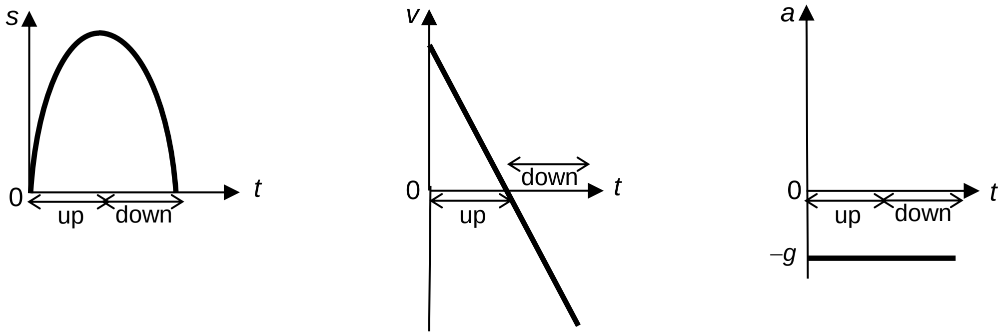
If an object is thrown vertically upwards and air resistance can be ignored, the object is said to be in free fall motion.

As the object rises, its velocity decreases steadily at a rate of 9.81 m $s^{-2}$.

When the object has reached a maximum height, it will stop momentarily before falling down.

Taking upward to be positive,

Combining both upward and downward motion on the same graph, we get:




### 5.3 Object thrown vertically up with air resistance (non-uniform acceleration)

If an object is thrown vertically upwards and air resistance is considered, the object is not in free fall motion. At the instant it is thrown, its velocity is the greatest and the opposing air resistance is the largest. Since both Earth's gravitational pull and air resistance are directed downwards, this causes the object's velocity to decrease at a faster rate (larger deceleration). The maximum height reached by the object is hence lower than the case without air resistance.

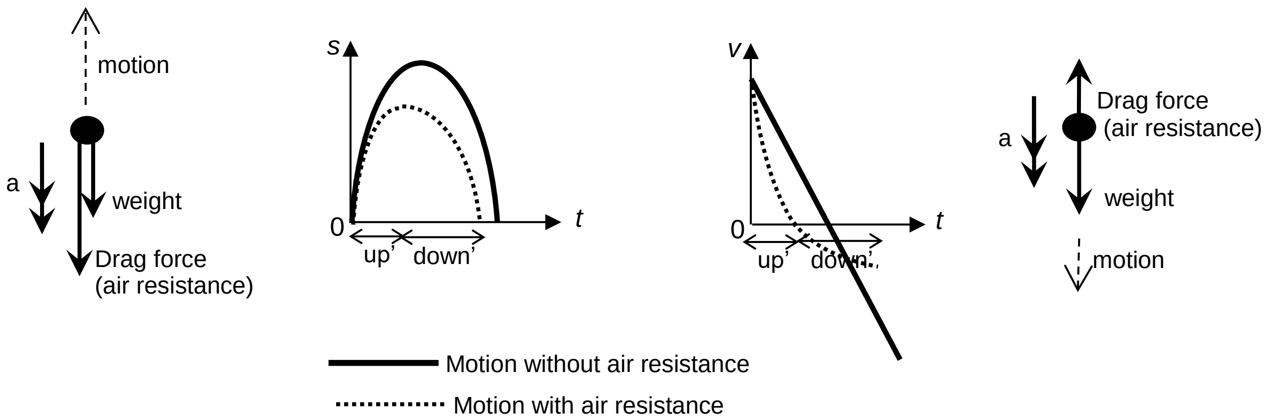
As the object travels up with decreasing velocity, the magnitude of the air resistance decreases, causing the velocity to decrease at a slower rate than its initial motion. Hence, the deceleration also decreases until the object reaches maximum height.

When the object is at maximum height, its velocity and air resistance at this instant are both zero. The only force acting on the object is the Earth's gravitational pull, so the value of deceleration is 9.81 m $s^{-2}$ .

As the object travels down, its velocity and air resistance both increase. However, the air resistance is now acting upwards, opposite to Earth's gravitational pull. This causes the object's velocity to increase at a slower rate than the case without air resistance (smaller acceleration). Hence, as compared to the free fall case, it acquires a lower final velocity.

As compared to its upward motion, the object hence takes a longer time, on its downward motion to return to its starting position, than it took to travel up.

Comparing the *s*-*t* and *v*-*t* graphs with and without air resistance, we get the following:

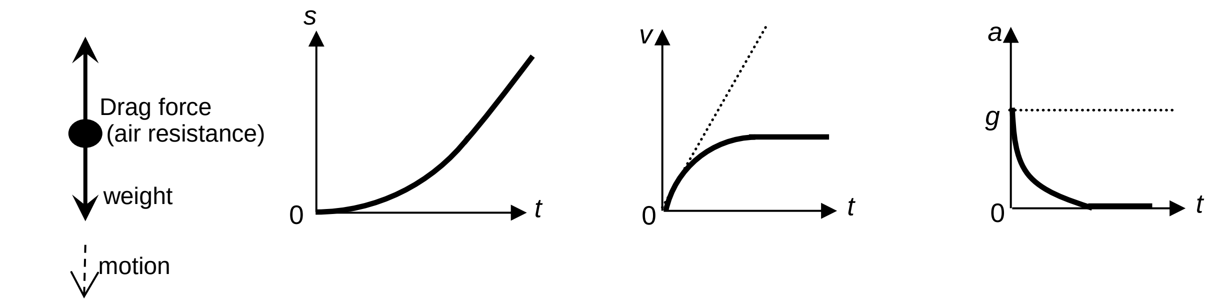



## 5.4 Terminal velocity

If an object falls through a long enough distance, and air resistance is present, the object's velocity will increase until it reaches a maximum velocity known as *terminal velocity*.

As the object falls, its velocity increases. This causes the opposing air resistance to increase. The air resistance increases until it is equal to the object's weight. The resultant force on the object is now zero. Hence the downward acceleration is zero and the object continues to move down with a constant velocity (terminal velocity).

Taking downward to be positive, the following graphs represent motion leading to terminal velocity.



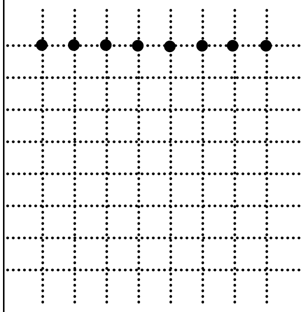
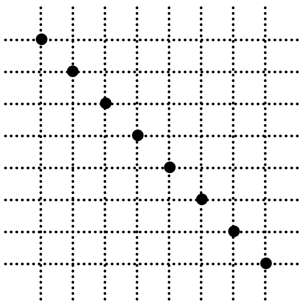
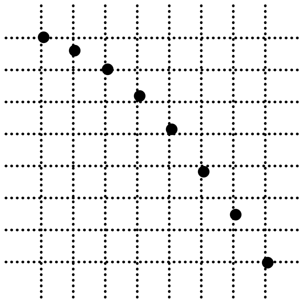

*Graphs for terminal velocity (under air resistance)*

## 6 Two Dimensional Motion

*(g) Describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.*

### 6.1 Comparison between 1-D motion and 2-D motion

Below shows the snapshots of three different types of motion:

| | | | |
|---|---|---|---|
| Path (snapshots of an object against a grid as background) |  |  |  |
| Description | Uniform velocity in horizontal direction<br>No velocity in vertical direction | Uniform velocity in horizontal direction<br>Uniform velocity in vertical direction | Uniform velocity in horizontal direction<br>Uniform *acceleration* in vertical direction |
| Variables | $v_x$ = constant<br>$a_x = 0$ | $v_x$ = constant, $v_y$ = constant<br>$a_x = 0$ $a_y = 0$ | $v_x$ = constant, $v_y \neq$ constant<br>$a_x = 0$ $a_y \neq 0$ |

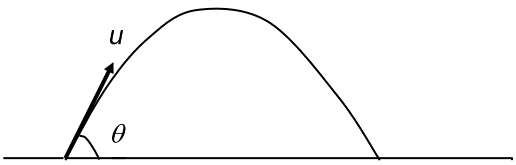
An example of the third type of motion shown above is projectile motion, where an object moving in air (with no air resistance) has acceleration only in the vertical direction.

Galileo, the first physicist to describe projectile motion accurately, stated that the horizontal and vertical components of the motion can be analysed separately.

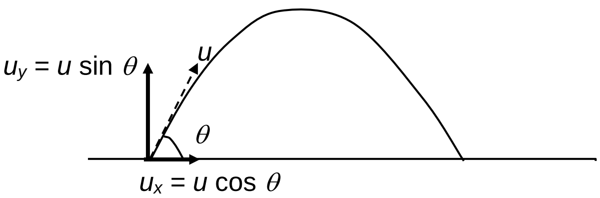
Motion along each direction is independent of each other.

### 6.2 Analysing Projectile Motion

Consider an object projected in air at an angle of $\theta$ above the horizontal. The path traced out by the object is parabolic in shape, as shown below:



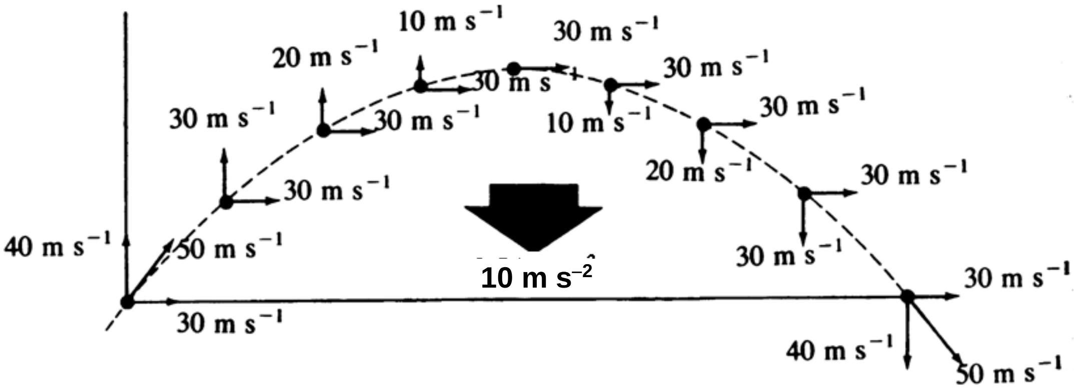

The initial velocity $u$ can be resolved into 2 perpendicular components.
The horizontal component is $u_x = u \cos\theta$ and the vertical component is $u_y = u \sin\theta$.



Assume that air resistance is ignored,

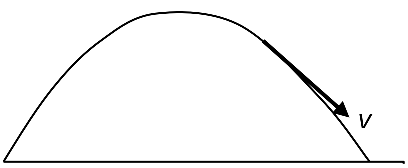
- there is no horizontal acceleration and therefore horizontal velocity $u_x$ is constant;
- vertical acceleration is due to gravity only ($a_y = g$ or $a_y = -g$), therefore vertical motion is uniformly accelerated;
- the equations of motion can be applied to solve for the quantities in the *x* (horizontal) and *y* (vertical) directions;
- time *t* is the same for both *x, y* components.

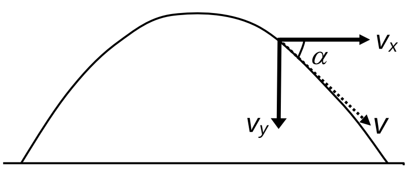
Example of a second-by-second exposure of an object projected at an angle to the horizontal is shown below. Assume $g$ = 10 m s$^{-2}$.




| missing variable | general equations of motion | equations for projectile motion: | |
|---|---|---|---|
| | | **horizontal component** $(a_x = 0)$ | **vertical component** $(a_y \neq 0)$ |
| $s$ | $v = u + at$ | $v_x = u_x$ | $v_y = u_y + a_y t$ |
| $v$ | $s = ut + \frac{1}{2}at^2$ | $s_x = u_x t$ | $s_y = u_y t + \frac{1}{2}a_y t^2$ |
| $a$ | $s = \left(\frac{u+v}{2}\right)t$ | $s_x = u_x t$ | $s_y = \left(\frac{u_y + v_y}{2}\right)t$ |
| $t$ | $v^2 = u^2 + 2as$ | $v_x = u_x$ | $v_y^{\,2} = u_y^{\,2} + 2a_y s_y$ |

The final velocity *v* (at any point along the path) is determined by recombining $v_x$ and $v_y$:




$$v^2 = v_x^{\,2} + v_y^{\,2}$$

$$\tan\alpha = \frac{v_y}{v_x}$$

The object's velocity is *v,* at an angle of $\alpha$ below the horizontal.

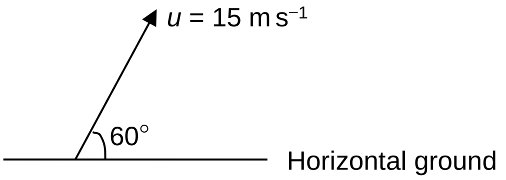
The direction of the velocity vector *v* is the tangent to the path at the point.

## 6.3 Problem solving strategy for Projectile Motion

1. Sketch the path of the projectile, including initial and final positions.
2. Resolve the initial velocity vector into *x*- and *y*-components.
3. Treat the horizontal motion and the vertical motion independently.
4. Time is the only quantity that is common for both *x*- and *y*- directions.
5. Follow the techniques for solving problems with constant velocity to analyse the horizontal motion of the projectile.
6. Follow the techniques for solving problems with constant acceleration to analyse the vertical motion of the projectile. (Note: vertical acceleration is 9.81 m $s^{-2}$ downwards)

**Example 11**

A ball is projected from horizontal ground with an initial velocity of 15 m $s^{-1}$ at an angle of 60° to the horizontal, as shown in the figure on the right.




(a) Calculate, for this ball, the initial values of
  (i) the vertical component of the velocity,
  (ii) the horizontal component of the velocity.

(b) Assuming that air resistance can be neglected, use your answers in (a) to determine
  (i) the maximum height *H* to which the ball rises,
  (ii) the time of flight, i.e. the time interval between the instant when the ball is projected and the instant when it returns to the ground level,
  (iii) the range, i.e. the displacement along the same horizontal level (distance between the point from which the ball is projected and the point where it strikes the ground).

(c) (i) Assuming air resistance is negligible, sketch the path of the ball. Label this path **N**.
  (ii) Assuming air resistance cannot be neglected, sketch the path of the ball. Label it **A**.
  (iii) Suggest an explanation for any differences between the paths **N** and **A**.

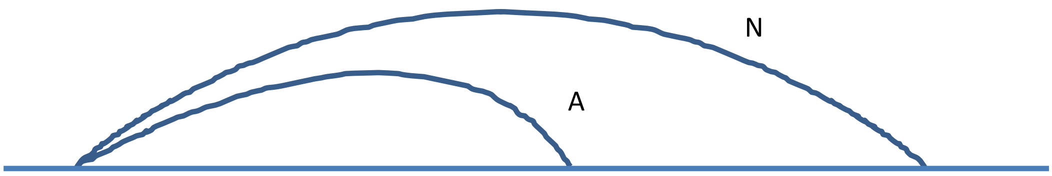
(a) (i) $u_y = 15 \sin 60° = 13$ m $s^{-1}$  (ii) $u_x = 15 \cos 60° = 7.5$ m $s^{-1}$

(b) (i) $v_y^2 = u_y^2 + 2a_y s_y$
$0 = 13^2 + 2(-9.81)H$
$H = 8.6$ m

(ii) $s = ut + \frac{1}{2}at^2$
$0 = 13t + \frac{1}{2}(-9.81)t^2$
$4.905t - 13 = 0$
$t = 2.7$ s

(iii) $R = u_x t$
$= (7.5)(2.7)$
$= 20$ m

(c) (i)
(ii)

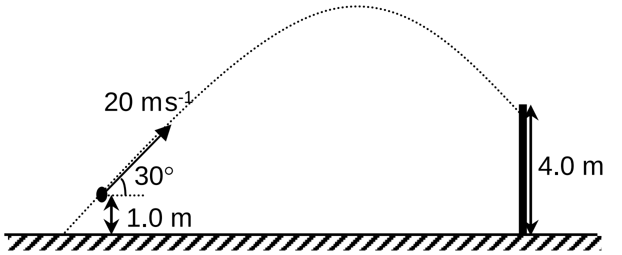



(iii) For path A, air resistance is taken into account and hence the maximum height attained by the ball is lower than that of path N.

Moreover, the range covered by the ball is shorter than that of path N due to a correspondingly shorter time of flight and smaller average horizontal velocity.

**Example 12**

A tennis ball is thrown from a height of 1.0 m above the ground with a speed of 20 m s$^{-1}$ at an angle of 30° above the horizontal. On its downward motion, it strikes the top of a fence which is 4.0 m above the ground, as shown.




Calculate
(a) the greatest height above the ground reached by the ball,
(b) the time taken by the ball to reach this height,
(c) the time taken by the ball to reach the point of impact with the fence,
(d) the ball's velocity just before it hits the fence.

(a) $v_y^2 = u_y^2 + 2as_y$

$0 = (20\sin 30°)^2 + 2(-9.81)h$

$h = 5.1\text{ m}$

Above ground, $H = 5.1 + 1.0 = 6.1\text{ m}$

(b) $v_y = u_y + at$

$0 = 20\sin 30° + (-9.81)t$

$t = 1.02\text{ s}$

(c) $s_y = u_y t + \frac{1}{2}at^2$

$3.0 = (20\sin 30°)t + \frac{1}{2}(-9.81)t^2$

$4.905t^2 - 10t + 3 = 0$

$t = 0.366\text{ s}$ or $1.67\text{ s}$

(N.A.)

(d) $v_y = u_y + at$

$v_y = 20\sin 30° + (-9.81)(1.67)$

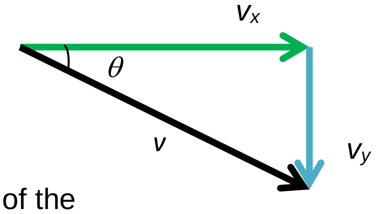
$v_y = -6.41\text{ m s}^{-1}$

$v_x = 20\cos 30° = 17.3\text{ m s}^{-1}$

$v = \sqrt{v_x^2 + v_y^2} = 18.4\text{ m s}^{-1}$

$\tan\theta = \frac{v_y}{v_x}$

$\theta = 20.3°$




i.e. the velocity of the ball is 18.4 m s$^{-1}$ directed at 20.3° below horizontal

**Example 13**

A dive bomber, diving at an angle of 60° with the vertical, releases a bomb at an altitude of 600 m. The bomb hits the ground 5.0 s after being released.
(a) Calculate the speed of the bomber.
(b) Determine the horizontal displacement of the bomb.
(c) Determine the velocity of the bomb just before striking the ground,

Take downward as positive.

(a) Consider vertical component,

$s_y = u_y t + \frac{1}{2}at^2$

$600 = u\cos 60°\,(5.0) + \frac{1}{2}(9.81)(5.0)^2$

$u = 191\text{ m s}^{-1}$

(b) $s_x = u_x t$

$= 191\sin 60°\,(5.0)$

$= 827\text{ m}$

(c) $v_x = 191\sin 60° = 165\text{ m s}^{-1}$

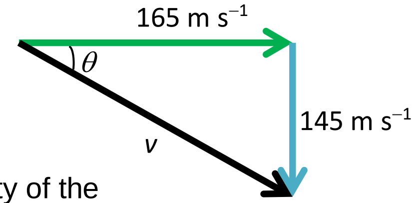
$v_y = 191\cos 60° + (9.81)(5.0) = 145\text{ m s}^{-1}$

$v = \sqrt{145^2 + 165^2}$

$v = 220\text{ m s}^{-1}$

$\tan\theta = \frac{145}{165}$

$\theta = 41.2°$




i.e. the velocity of the bomb is 220 m s$^{-1}$ directed at 41.2° below horizontal