RAFFLES INSTITUTION

2022 Year 6

H3 9820 Lecture Test 1

Time allocated: 1 hour **Total Marks: 25**

Instructions: Write your name and CT group on all the work you hand in.
Answer **all** questions.

1 The positive integers a, b, c and d satisfy the equation

$$(ad-bc)^2=(a+b)(c+d).$$

(a) Show that there are positive integers x, y and z, with x and y coprime, such that $a+b=x^2z$ and $c+d=y^2z$. [4]

(b) Find a quadratic equation satisfied by $\frac{y}{x}$ and hence, or otherwise, prove that $4ac+1$ is a perfect square. [6]

(a) **[4]**	Let $z=\gcd(a+b,c+d)$. Then clearly z is a positive integer, and we have also $a+b=pz,\ c+d=qz$, where p and q are coprime. Substituting these into the equation we have $(ad-bc)^2=pqz^2 \Rightarrow \left(\frac{ad-bc}{z}\right)^2=pq$. Since p and q are coprime and their product is a perfect square, they must be perfect squares themselves. Hence there are positive coprime integers x and y such that $p=x^2, q=y^2$ and thus positive integers x, y and z such that $a+b=x^2z$ and $c+d=y^2z$.
(b) **[6]**	We note that $z=\frac{ad-bc}{xy}=\frac{a+b}{x^2}=\frac{c+d}{y^2}$. So we have $\frac{y^2}{x^2}=\frac{c+d}{a+b}$, $\frac{y}{x}=\frac{ad-bc}{a+b}$. Note also that $a\frac{y^2}{x^2}-\frac{y}{x}=\frac{a(c+d)}{a+b}-\frac{ad-bc}{a+b}=\frac{ac+bc}{a+b}=c$ which is the quadratic equation we are looking for. Solving the quadratic equation we have $\frac{y}{x}=\frac{1\pm\sqrt{1+4ac}}{2a}$. Since x and y are coprime positive integers, $\sqrt{1+4ac}$ has to be an integer (otherwise it is irrational since a and c are positive integers), and thus $4ac+1$ is a perfect square.

2 Let u and v be quadratic functions of x and let

$$y = \frac{u}{v}.$$

(a) Use mathematical induction to prove that

$$v\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} + (n+2)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + \binom{n+2}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0,$$

for $n \geq 1$. [8]

(b) Now assume that $v = (\alpha - x)^2$ for some real number α and, for all positive integers n, define

$$z_n = \frac{(\alpha - x)^{n+2}}{n!}\frac{\mathrm{d}^n y}{\mathrm{d}x^n}.$$

Use the result of **(a)** to prove that $z_1, z_2, z_3, ...$ is an arithmetic progression. By writing y as partial fractions, or otherwise, show that the common difference is $u(\alpha)$. [7]

(a) [8]	Let P_n be the statement $v\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} + (n+2)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + \binom{n+2}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0$ for positive integers n. From $y = \frac{u}{v} \Rightarrow yv = u$ and thus differentiating with respect to x we have $\frac{\mathrm{d}y}{\mathrm{d}x}v + y\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}.$ Differentiating once more with respect to x we have $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}v + 2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} + y\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = \frac{\mathrm{d}^2u}{\mathrm{d}x^2}.$ Differentiating once more with respect to x and using the fact that u is a quadratic expression implies that $\frac{\mathrm{d}^3u}{\mathrm{d}x^3} = 0,$ (since $\frac{\mathrm{d}u}{\mathrm{d}x}$ linear, $\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$ constant) $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}v + \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\frac{\mathrm{d}v}{\mathrm{d}x} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\frac{\mathrm{d}v}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + y\frac{\mathrm{d}^3v}{\mathrm{d}x^3} = \frac{\mathrm{d}^3u}{\mathrm{d}x^3} = 0.$ This simplifies to $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}v + 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\frac{\mathrm{d}v}{\mathrm{d}x} + 3\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + y\frac{\mathrm{d}^3v}{\mathrm{d}x^3} = 0.$ Since $(n+2) = \binom{n+2}{2} = 3$ when n = 1, the statement P_1 is true. Now suppose P_n is true for some positive integer n, i.e. for some positive integer n, $v\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} + (n+2)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + \binom{n+2}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0.$

Differentiating once more with respect to x we have

$$v\frac{\mathrm{d}^{n+3}y}{\mathrm{d}x^{n+3}}+\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+(n+2)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+(n+2)\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$$

$$+\binom{n+2}{2}\frac{\mathrm{d}^3v}{\mathrm{d}x^3}\frac{\mathrm{d}^{n}y}{\mathrm{d}x^{n}}+\binom{n+2}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}=0$$

$$\Rightarrow v\frac{\mathrm{d}^{n+3}y}{\mathrm{d}x^{n+3}}+(n+3)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+\left(n+2+\binom{n+2}{2}\right)\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}=0$$

since $\frac{\mathrm{d}^3v}{\mathrm{d}x^3}=0$ as v is a quadratic.

Since

$$n+2+\binom{n+2}{2}=n+2+\frac{(n+2)(n+1)}{2}$$

$$=\frac{2(n+2)+(n+2)(n+1)}{2}$$

$$=\frac{(n+2)(2+n+1)}{2}=\frac{(n+3)(n+2)}{2}$$

$$=\binom{n+3}{2}$$

We thus have $v\frac{\mathrm{d}^{n+3}y}{\mathrm{d}x^{n+3}}+(n+3)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+\binom{n+3}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}=0$. Since P_1 is true and $P_n\Rightarrow P_{n+1}$, by Mathematical Induction, P_n is true for all positive integers n.

(b)
[7]

To show $z_1, z_2, z_3, \ldots$ is an arithmetic progression, we will show that $z_{n+2}-z_{n+1}=z_{n+1}-z_n$ or equivalently $z_{n+2}-2z_{n+1}+z_n=0$. From the definition of $z_n=\frac{(\alpha-x)^{n+2}}{n!}\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$, we have

$$z_{n+2}-2z_{n+1}+z_n=\frac{(\alpha-x)^{n+4}}{(n+2)!}\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}-2\frac{(\alpha-x)^{n+3}}{(n+1)!}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}+\frac{(\alpha-x)^{n+2}}{n!}\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$$

$$=\frac{(\alpha-x)^{n+2}}{(n+2)!}\left[(\alpha-x)^2\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}-2(\alpha-x)(n+1)\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}+(n+2)(n+1)\frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right]$$

Here, we note that with the choice of $v=(\alpha-x)^2$, we have $\frac{\mathrm{d}v}{\mathrm{d}x}=-2(\alpha-x)$ and $\frac{\mathrm{d}^2v}{\mathrm{d}x^2}=2$. Therefore

$$z_{n+2}-2z_{n+1}+z_n$$

$$=\frac{(\alpha-x)^{n+2}}{(n+2)!}\left[v\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+\frac{\mathrm{d}v}{\mathrm{d}x}(n+1)\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}+\frac{(n+2)(n+1)}{2}(2)\frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right]$$

$$=\frac{(\alpha-x)^{n+2}}{(n+2)!}\left[v\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+(n+1)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}+\binom{n+2}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}^ny}{\mathrm{d}x^n}\right]=0. \quad \text{from (a)}$$

Hence $z_1, z_2, z_3, \ldots$ is an arithmetic progression.

	$y=\frac{u}{v}=\frac{u}{(\alpha-x)^2}$. Since u is a quadratic, we can write $y=A+\frac{B}{(\alpha-x)}+\frac{C}{(\alpha-x)^2}$. Thus $y=\frac{A(\alpha-x)^2+B(\alpha-x)+C}{(\alpha-x)^2}=\frac{u}{(\alpha-x)^2}$. Substituting $x=\alpha$ we have $C=u(\alpha)$. We also have $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{B}{(\alpha-x)^2}+\frac{2C}{(\alpha-x)^3}\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{2B}{(\alpha-x)^3}+\frac{6C}{(\alpha-x)^4}$. The common difference of the arithmetic progression is $z_2-z_1=\frac{(\alpha-x)^4}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{(\alpha-x)^3}{1!}\frac{\mathrm{d}y}{\mathrm{d}x}$ $=\frac{(\alpha-x)^4}{2}\left(\frac{2B}{(\alpha-x)^3}+\frac{6C}{(\alpha-x)^4}\right)-(\alpha-x)^3\left(\frac{B}{(\alpha-x)^2}+\frac{2C}{(\alpha-x)^3}\right)$ $=(\alpha-x)B+3C-(\alpha-x)B-2C$ $=C=u(\alpha)$ as required.

[END OF PAPER]