Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 11 Definite Integrals
Learning Package

Resources

- [ ] Core Concept Notes
- [ ] Discussion Questions
- [ ] Extra Practice Questions

SLS Resources

- [ ] Recordings on Core Concepts
- [ ] Exploration Activities
- [ ] Quick Concept Checks

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 11 Definite Integrals
Core Concept Notes

Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Evaluate definite integral using anti-derivatives i.e. $\int_a^b f(x)\ dx = F(a) - F(b)$, where $\frac{d}{dx}F(x) = f(x)$ (anti-derivative). ☐ Evaluate the approximate values of definite integrals using a graphing calculator. ☐ Interpret definite integrals as the area under a curve. ☐ Understand that definite integral is negative when the curve is below the *x*-axis.	☐ Evaluate an integrand of the form $\lvert f(x)\rvert$ without the use of GC. ☐ Find the area of a region bounded by a curve and lines parallel to the coordinate axes. ☐ Find the area of a region between a curve and a line. ☐ Find the area of a region between two curves. ☐ Find the volume of revolution about the *x*- or *y*-axis when it is bounded by one curve.	☐ Explain that the area under a curve is the limit of a sum of the areas of rectangles and relate definite integral as a limit of sums i.e. $\int_a^b f(x)\ dx = \lim_{\delta x \to 0} \sum y\ \delta x$, where $y = f(x)$. ☐ Use a graphing calculator to illustrate the limiting process for simple cases. ☐ Find the volume of revolution about the *x*- or *y*-axis when it is bounded by two curves.

Definition of Learning (John Hattie):
The process of developing sufficient surface knowledge to then move to deeper understanding such that one can appropriately transfer this learning to new tasks and situations.

§1 Definite Integral is a Process of Summation

1.1 Definite Integral as the Limit of a Sum

Please watch the video in SLS before this segment:

You may refer to Annex A for a more comprehensive explanation.

Suppose we wish to find the area bounded by the curve $y = x^2$, the x-axis, the lines $x = 1$ and $x = 2$ as shown in Figure 1.

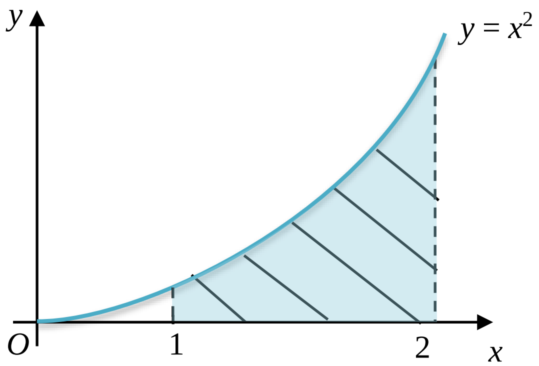
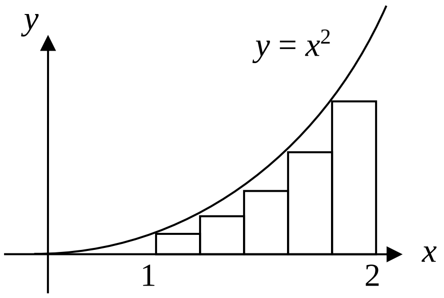


Figure 1

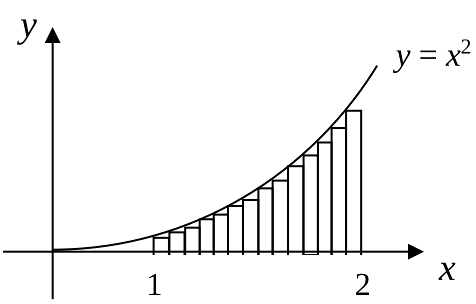
Let us estimate this area by dividing it into 5, 20 and 50 equal rectangular strips as shown in the figures below and finding the total area of the rectangular strips.

When $n = 5$



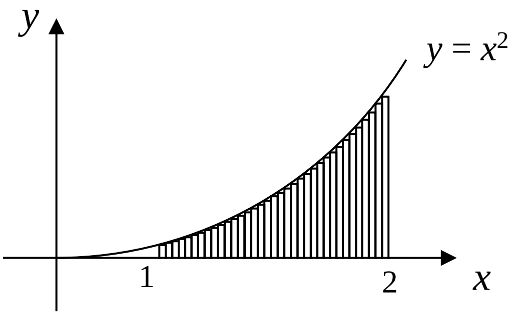
Total area of the rectangles = 2.04 units^2

When $n = 20$



Total area of the rectangles = 2.25875 units^2

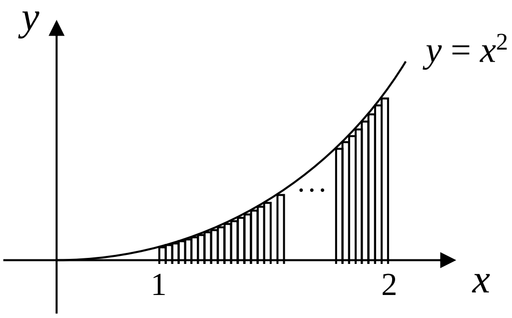
When n = 50



Total area of the rectangles = 2.3034 units^2

From the diagrams above, the total area of the rectangular strips gives an approximate value for the required area. Observe that *increasing the number of rectangular strips gives better approximation* to the required area.

When n tends to infinity



When n tends to infinity, the total area of the rectangles will approach the actual area bounded by the graph $y=x^2$ and $x=1$, $x=2$ and the x-axis, which is $\int_1^2 x^2\,\mathrm{d}x = 2.33\ (3\text{ s.f})$

Explanation: Let $\mathrm{f}(x)=x^2$.

Using integration, the actual area bounded by $y=x^2$, $x=1$, $x=2$ and the x-axis is given by

$$\int_1^2 x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_1^2 = \frac{7}{3}\ (\text{or } 2.33\ (3\text{s.f})).$$

Also, the same area can be estimated by first, dividing x values from 1 to 2 into n equal parts, forming n rectangles of equal width of $\frac{1}{n}$. Hence, the same area is estimated by finding the total areas of the rectangles

$$=\frac{1}{n}\times\mathrm{f}(1)+\frac{1}{n}\times\mathrm{f}\left(1+\left(\frac{1}{n}\right)\right)+\frac{1}{n}\times\mathrm{f}\left(1+2\left(\frac{1}{n}\right)\right)+\ldots+\frac{1}{n}\times\mathrm{f}\left(1+(n-1)\left(\frac{1}{n}\right)\right)$$

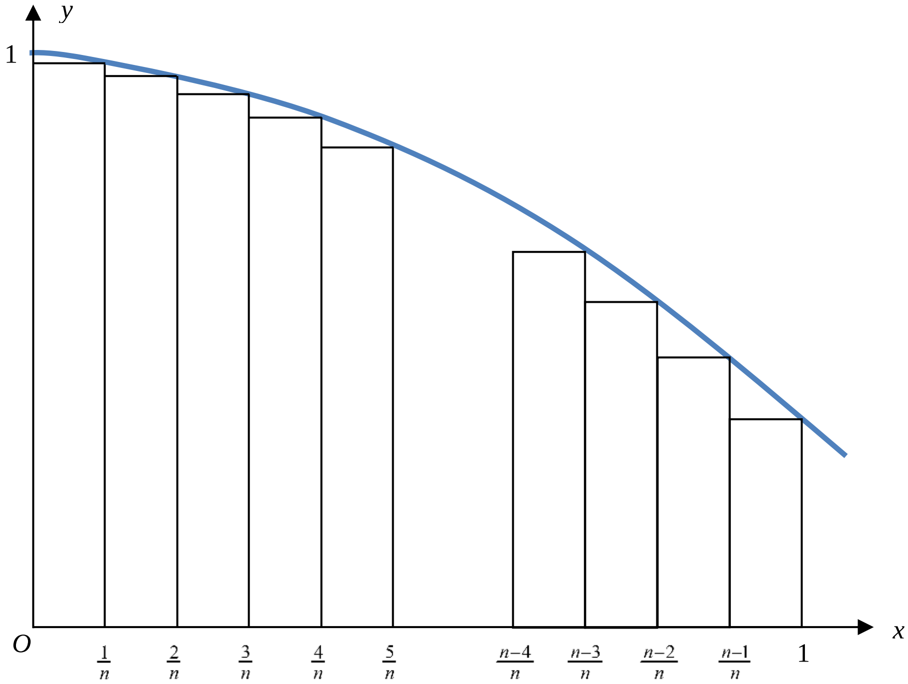
As the number of rectangles, n, increases to infinity,

$$\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}(1)+\mathrm{f}\left(1+\left(\frac{1}{n}\right)\right)+\mathrm{f}\left(1+2\left(\frac{1}{n}\right)\right)+\ldots+\mathrm{f}\left(1+(n-1)\left(\frac{1}{n}\right)\right)\right\}=\int_1^2\mathrm{f}(x)\,\mathrm{d}x=2.33\ (3\text{ s.f})$$

Example 1 [N2001/1/18]

(i) Find the exact value of $\int_0^1 \frac{1}{1+x^2}\,dx$.

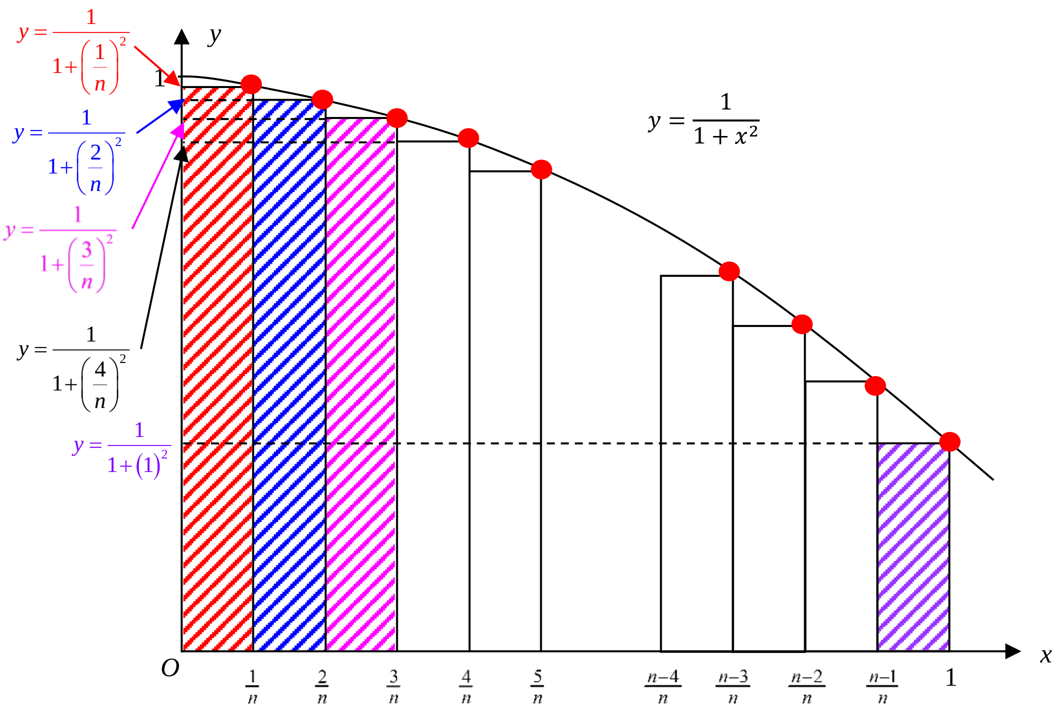
(ii)



The graph of $y=\frac{1}{1+x^2}$, for $0 \le x \le 1$, is shown in the diagram. Rectangles, each of width $\frac{1}{n}$, are drawn under the curve. Show that the total area A of all n rectangles is given by $A=\frac{1}{n}\left\{\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\frac{1}{1+\left(\frac{3}{n}\right)^2}+\ldots+\frac{1}{2}\right\}$. State the limit of A as $n \to \infty$.

(i)	$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$ $=\left[\tan^{-1} x\right]_0^1$ $=\frac{\pi}{4}$

(ii)



To find the height of each rectangle, let $x=\frac{1}{n},\frac{2}{n},\frac{3}{n},\ldots,\frac{n-1}{n},1$.

$$A=\frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^2}\right]+\frac{1}{n}\left[\frac{1}{1+\left(\frac{2}{n}\right)^2}\right]+\frac{1}{n}\left[\frac{1}{1+\left(\frac{3}{n}\right)^2}\right]+\ldots\frac{1}{n}\left(\frac{1}{2}\right)$$

$$A=\frac{1}{n}\left\{\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\frac{1}{1+\left(\frac{3}{n}\right)^2}+\ldots+\frac{1}{2}\right\} \quad \text{(shown)}$$

As $n\to\infty$, sum of area of all n rectangles will tend to area under graph.

$$\therefore \lim_{n\to\infty} A=\frac{\pi}{4}$$

§2 Area under a Curve

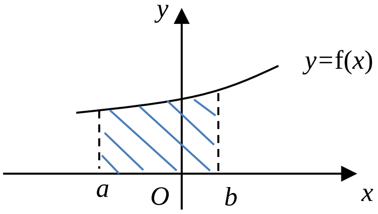
2.1 Area Bounded by a Curve and the *x*-axis

(I) Area Above the *x*-axis

The curve $y = \mathrm{f}(x)$ is continuous on the interval $a \le x \le b$.

The area bounded by the curve $y = \mathrm{f}(x)$ where $\mathrm{f}(x) \ge 0$, the *x*-axis, the lines $x = a$ and $x = b$, *A*, is given by

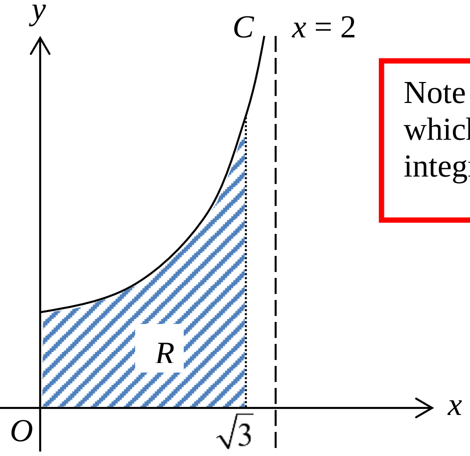
$$A = \int_a^b y \, \mathrm{d}x$$



Example 2 [2014/JJC Prelim/I/4 modified]

The diagram shows a region *R* in the first quadrant bounded by the curve *C* with equation $y = \frac{2}{\sqrt{4-x^2}} + 3$, the *x*-axis, the *y*-axis and the line $x = \sqrt{3}$. Calculate the **exact area** of region *R*.

Note that **exact answer** is required, which means detailed working of integration must be shown.



Region *R* is bounded by *C* and *x*-axis from $x = 0$ to $x = \sqrt{3}$

$$\text{Area} = \int_0^{\sqrt{3}} y \, \mathrm{d}x = \int_0^{\sqrt{3}} \frac{2}{\sqrt{4-x^2}} + 3 \, \mathrm{d}x$$

Observation: $\frac{2}{\sqrt{4-x^2}}$ is $\frac{\text{constant}}{\sqrt{\text{quadratic}}} \Rightarrow$ MF27

$$= \left[2\sin^{-1}\left(\frac{x}{2}\right) + 3x \right]_0^{\sqrt{3}}$$

MF27 Pg 4

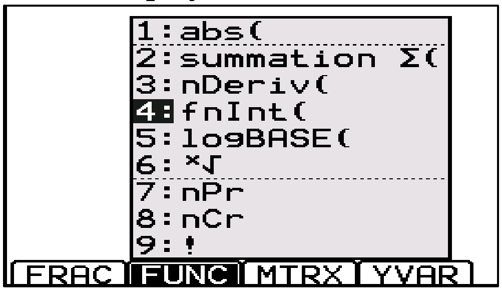
$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right)$$

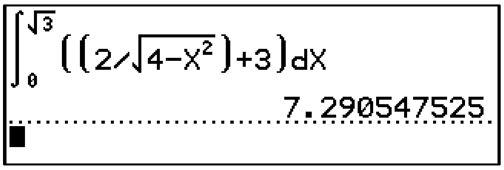
$$= \left[2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 3\sqrt{3} \right] - \left[2\sin^{-1}\left(\frac{0}{2}\right) + 3(0) \right]$$

$$= \frac{2\pi}{3} + 3\sqrt{3}$$

Good Strategy: Check your final answer using GC (Pg 7)

Method 1: Using **GC Home Screen** (Recommended when the quadrant the area lies in is known)

<table>
<tr><th>Steps/Keystrokes/Explanations</th><th>Screen Display</th></tr>
<tr><td>1. Press [ALPHA][WINDOW] and select 4: fnInt(.</td><td>


</td></tr>
<tr><td>2. Key in the lower and upper limits, integrand and variable of integration and press [ENTER]</td><td>


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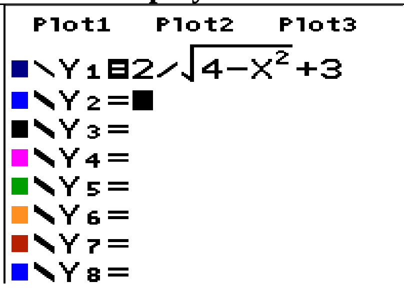
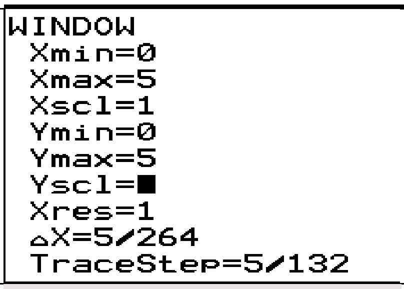
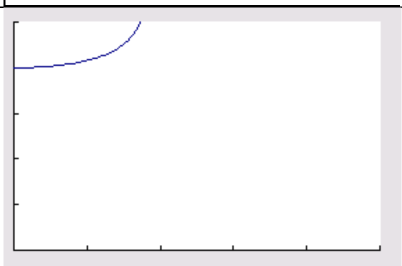
</table>

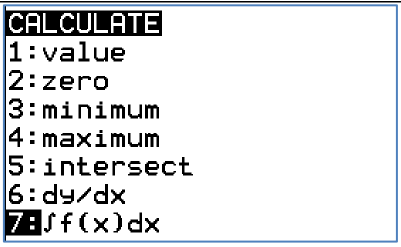
Method 2: Using **GC Graph Screen**

<table>
<tr><th>Steps/ Keystrokes/ Explanations</th><th>Screen Display</th></tr>
<tr><td>1. Press [Y=] to graph the function $Y_1 = \dfrac{2}{\sqrt{4-x^2}}+3$.</td><td>


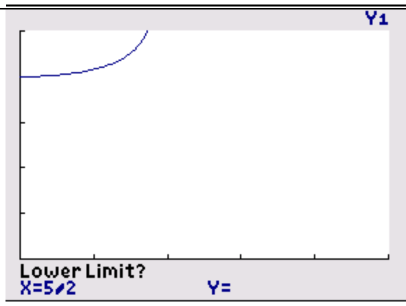
</td></tr>
<tr><td>2. Press [WINDOW] to set the settings as shown on the screen. This is to have a better view of the graph. Basic knowledge of $y = \dfrac{2}{\sqrt{4-x^2}}+3$ is required.</td><td>


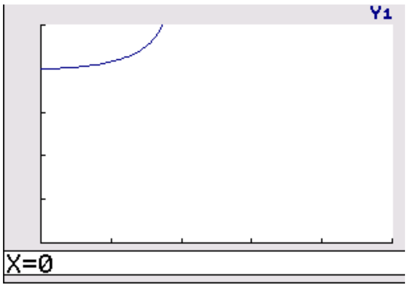
</td></tr>
<tr><td>3. Press [GRAPH] to view the graph.</td><td>

</td></tr>
</table>

4. Press [2nd][TRACE]. Select **7:** $\int f(x)\,dx$, the numeric integral option. This returns the user to the graph screen.



5. Key in the lower limit value and press [ENTER].

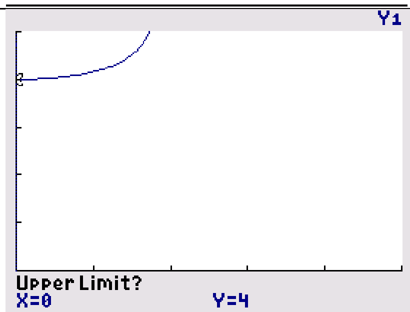


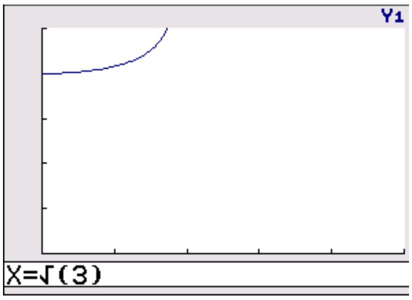


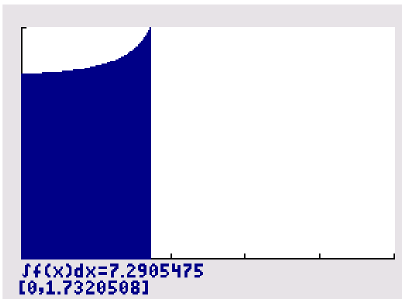
6. Key in the upper limit value and press [ENTER]. This then calculates the value of the integral between the limits and shades the relevant region.

$\therefore$ Using GC, Area = $\displaystyle\int_0^{\sqrt{3}} \frac{2}{\sqrt{4-x^2}}+3\ dx$

= 7.29 units ^{2} (3. s. f)







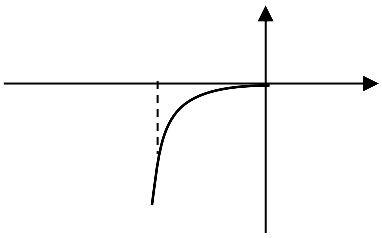
(II) Area Below the *x*-axis

Example 3

With the help of a Graphing Calculator, find the area bounded by the curve $y = x^3$, the x-axis, the line $x = -1$ and $x = 0$.

Using GC,

$$\int_{-1}^{0} x^3 \, dx = -0.25$$



Since **area** is a scalar quantity and must be a **positive** value, the required area is

$$\left|\int_{-1}^{0} x^3 \, dx\right| = |-0.25| = 0.25 \text{ units}^2$$

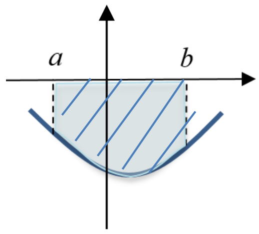
[Alternatively, the required area is $-\int_{-1}^{0} x^3 \, dx = -(-0.25) = 0.25 \text{ units}^2$ **]**

* Good Practice: always draw a **graph** when finding **area**.

Important Conclusion:

If the curve $y = f(x)$ is continuous on the interval $a \le x \le b$, then the area bounded by the curve $y = f(x)$, where $f(x) \le 0$ and the x-axis from $x = a$ to $x = b$ is given by

$$A = \left|\int_a^b y \, dx\right| = -\int_a^b y \, dx$$



*You can choose to put either **modulus** or **negative sign** to ensure the area obtained is a **positive** scalar.

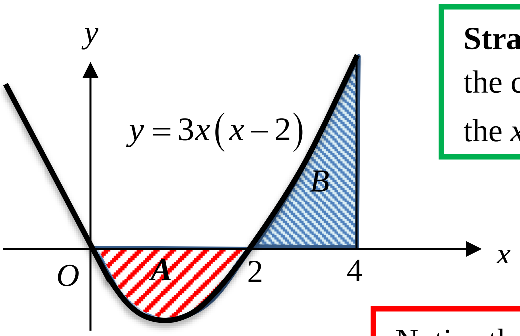
Realise the difference between

Evaluate $\int_{-1}^{0} x^3 \, dx$.	Find the area bounded by the curve $y = x^3$, the x-axis, the line $x = -1$ and $x = 0$.
$\int_{-1}^{0} x^3 \, dx = -0.25$	Required area $= \left|\int_{-1}^{0} x^3 \, dx\right| = |-0.25| = 0.25 \text{ units}^2$

Example 4 (Area Above and Below the *x*-axis)

Find the **exact area** of the region bounded by the curve $y=3x(x-2)$, the x-axis and the line $x=4$.

Note that **exact answer** is required, which means detailed working of integration must be shown.



Strategy: Always **sketch the graph** to identify the correct region bounded by $y=3x(x-2)$, the x-axis and the line $x=4$.

Notice that region A is **below the *x*-axis**, there is a need to put a **negative sign (or modulus)** to ensure the Area of A is **positive.**

Required area

$= \text{Area of } A + \text{Area of } B$

$= -\int_0^2 y \, dx + \int_2^4 y \, dx$

$= -\int_0^2 3x(x-2) \, dx + \int_2^4 3x(x-2) \, dx$

$= -\int_0^2 (3x^2-6x) \, dx + \int_2^4 (3x^2-6x) \, dx$

$= -\left[\frac{3x^3}{3}-\frac{6x^2}{2}\right]_0^2 + \left[\frac{3x^3}{3}-\frac{6x^2}{2}\right]_2^4$

$= -\left[x^3-3x^2\right]_0^2 + \left[x^3-3x^2\right]_2^4$

$= -\left[(8-12)-0\right] + \left[(64-48)-(8-12)\right]$

$= -(-4)+20$

$= 4+20 = 24 \text{ units}^2$

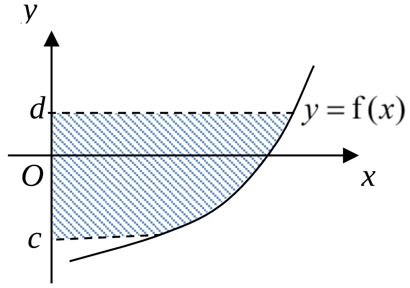
2.2 Area Bounded by a Curve and the *y*-axis

(I) Area on the Right of *y*-axis

The curve $y = \mathrm{f}(x)$ is continuous on the interval $c \le y \le d$.

The area bounded by the curve $y = \mathrm{f}(x)$, where $x \ge 0$, the *y*-axis, the lines $y = c$ and $y = d$, *A*, is given by

$$A = \int_c^d x \, \mathrm{d}y$$



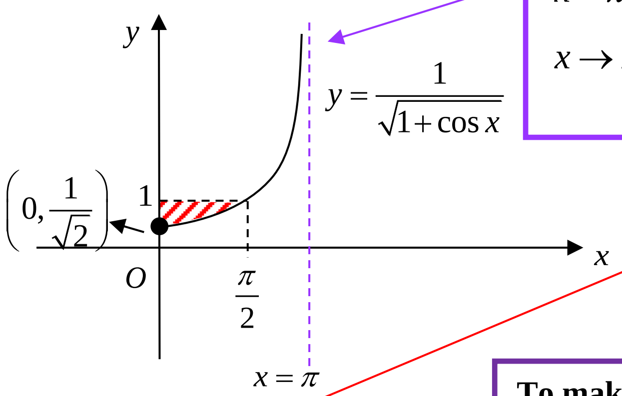
Example 5 [2008/YJC Prelim/I/10 part]

Sketch the curve $y = \dfrac{1}{\sqrt{1+\cos x}}$ for $0 \le x < \pi$. [1]

R is the region enclosed by the curve $y = \dfrac{1}{\sqrt{1+\cos x}}$, the *y*-axis and the line $y = 1$. Calculate the area *R*. [2]

Note that **exact answer is not required**, which means final answer can be obtained directly from GC.

(i)



$x = \pi$ is an asymptote since as $x \to \pi, y = \dfrac{1}{\sqrt{1+\cos \pi}} = \dfrac{1}{\sqrt{1+(-1)}} \to \infty$

Identify the region on the graph and write down the integral corresponding to the area of the required region.

(ii)

Area $R = \int_{\frac{1}{\sqrt{2}}}^{1} x \, \mathrm{d}y$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1}\left(\frac{1}{y^2} - 1\right) \mathrm{d}y$$

$= 0.324$ (using GC)

To make *x* the subject:

$$y = \frac{1}{\sqrt{1+\cos x}}$$

$$\sqrt{1+\cos x} = \frac{1}{y}$$

$$1 + \cos x = \left(\frac{1}{y}\right)^2$$

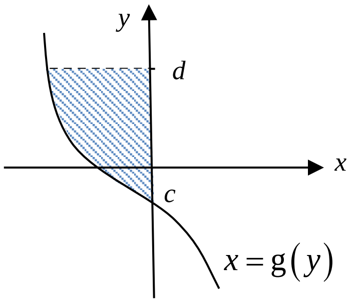
$$x = \cos^{-1}\left(\frac{1}{y^2} - 1\right)$$

(II) **Area on the Left of y-axis**

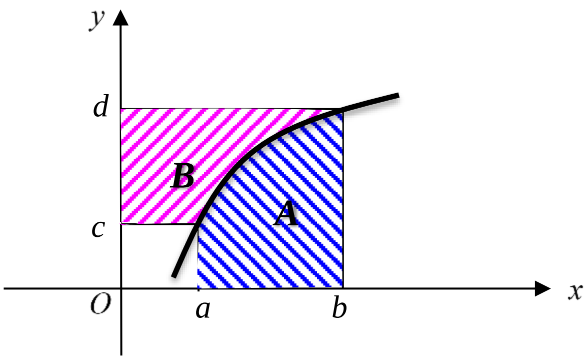
The curve $y = \mathrm{f}(x)$ is continuous on the interval $c \le y \le d$.

The area bounded by the curve $x = \mathrm{g}(y)$, where $x \le 0$, the y-axis and $y = d$, A, is given by

$$A = \left| \int_c^d x \, \mathrm{d}y \right| = -\int_c^d x \, \mathrm{d}y$$



Quick Recap:



Area of a region bounded by a curve, x-axis, lines $x = a$, $x = b$

Area of region $\boldsymbol{A}$ $= \int_a^b y \, \mathrm{d}x$

Area of a region bounded by a curve, y-axis, lines $y = c$, $y = d$

Area of region $\boldsymbol{B}$ $= \int_c^d x \, \mathrm{d}y$

2.3 <u>Area Bounded by Two Curves</u>

(I) **Area bounded by 2 graphs such that one graph y_1=f(x) is entirely above another graph y_2=g(x) over the interval $a \le x \le b$**

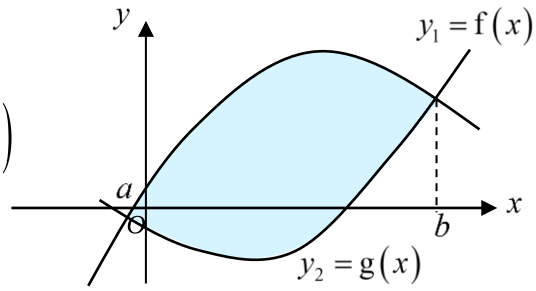
Area between the curves $y_1 = \mathrm{f}(x)$ and $y_2 = \mathrm{g}(x)$

$= \left[\begin{array}{l}\text{Area under the curve } y_1 = \mathrm{f}(x) \\ \text{between } x = a \text{ and } x = b\end{array}\right] - \left[\begin{array}{l}\text{Area under the curve } y_2 = \mathrm{g}(x) \\ \text{between } x = a \text{ and } x = b\end{array}\right]$

$= \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^b \mathrm{g}(x)\,\mathrm{d}x$

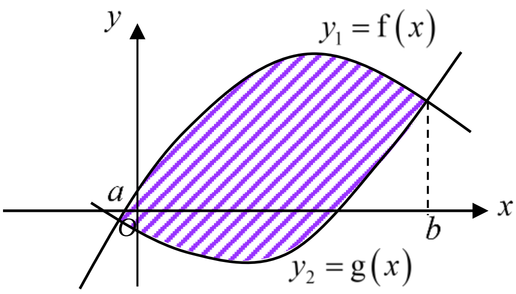
$= \int_a^b \left[\mathrm{f}(x) - \mathrm{g}(x)\right]\mathrm{d}x \quad \left(\text{or} \quad \int_a^b (y_1 - y_2)\,\mathrm{d}x\right)$

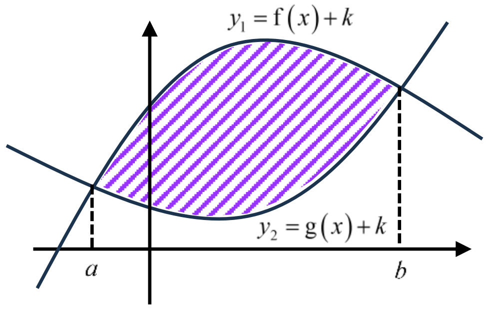
Bigger Area **Smaller Area**



<u>For your Understanding:</u>

To find the area bounded by 2 graphs where one graph $y_1 = \mathrm{f}(x)$ is entirely above another graph $y_2 = \mathrm{g}(x)$ over the interval $a \le x \le b$, we can understand it by the idea of translating both graphs by k units (where k is a large real number) in the positive y-direction.





Area between the curves $y_1 = \mathrm{f}(x)$ and $y_2 = \mathrm{g}(x)$

= Area between the curves $y_1 = \mathrm{f}(x) + k$ and $y_2 = \mathrm{g}(x) + k$

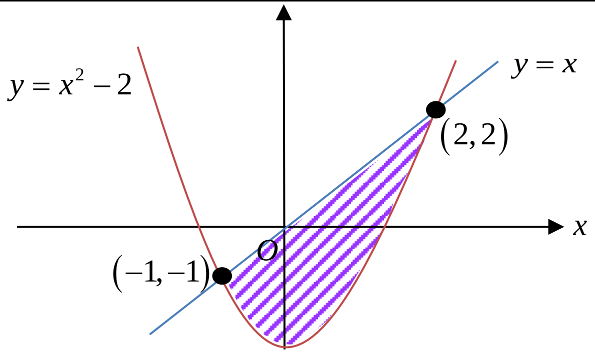
$= \int_a^b \left[\mathrm{f}(x) + k\right]\mathrm{d}x - \int_a^b \left[\mathrm{g}(x) + k\right]\mathrm{d}x$

$= \int_a^b \left[\mathrm{f}(x) + k - \mathrm{g}(x) - k\right]\mathrm{d}x$

$= \int_a^b \mathrm{f}(x) - \mathrm{g}(x)\,\mathrm{d}x$

Example 6 (Area bounded by 2 curves)

Find the area bounded by the line $y = x$ and the curve $y = x^2 - 2$.



Using GC, the intersection points are $(-1, -1)$ and $(-1, -1)$.

Using GC, area bounded by line and curve

$= \int_{-1}^{2} x - \left(x^2 - 2\right) \, \mathrm{d}x$

$= \frac{9}{2}$ units^2

Learning Point:

Step 1: Find the intersection points between the 2 curves.

Step 2: Use the idea of $\int_a^b \left[\mathrm{f}(x) - \mathrm{g}(x) \right] \mathrm{d}x$

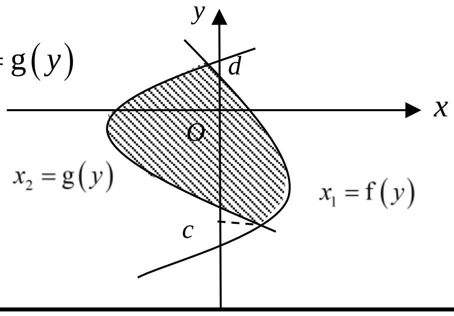
Bigger Area **Smaller Area**

(II) **<u>Area bounded by the 2 graphs such that one graph x_1=f(y) is entirely to the right of another graph x_2=g(y) over the interval $c \le y \le d$</u>**

Area bounded by the curves $x_1 = \mathrm{f}(y)$ and $x_2 = \mathrm{g}(y)$

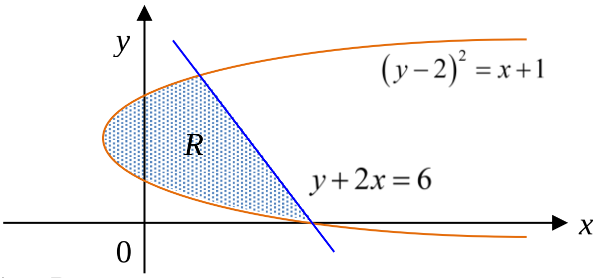
$$= \int_c^d \left[\mathrm{f}(y) - \mathrm{g}(y)\right] \, \mathrm{d}y$$

$$= \int_c^d (x_1 - x_2) \, \mathrm{d}y$$



Example 7 [2013/AJC/Promo/8b modified]

The diagram shows a shaded region R bounded by the curve $(y-2)^2 = x+1$ and the line $y + 2x = 6$.



Find the area of region R.

Using GC, the intersection points are $(3, 0)$ and $\left(\frac{5}{4}, \frac{7}{2}\right)$.

Area of $R = \int_0^{\frac{7}{2}} \left[\frac{6-y}{2} - \left((y-2)^2 - 1\right)\right] \mathrm{d}y = 7.15$ (3 s.f)

$y + 2x = 6$

$2x = 6 - y$

$x = \dfrac{6-y}{2}$

Learning Point:

Step 1: Find the intersection points between the 2 curves.

Step 2: Use the idea of $= \int_c^d (x_1 - x_2) \, \mathrm{d}y$

Bigger Area **Smaller Area**

2.4 <u>Integration of Modulus Function</u>

Recall the definition:

$$|x| = \begin{cases} x & ,\text{if } x \geq 0 \\ -x & ,\text{if } x < 0 \end{cases}$$

Thus

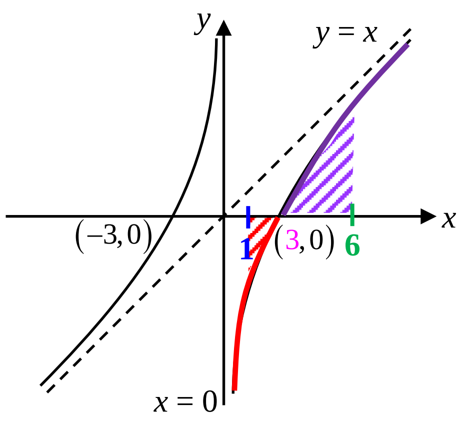
$$\boxed{y = |\,\mathrm{f}(x)\,| = \begin{cases} \mathrm{f}(x) & ,\ \text{if } \mathrm{f}(x) \geq 0 \\ -\mathrm{f}(x) & ,\ \text{if } \mathrm{f}(x) < 0 \end{cases}} \quad \ldots\ldots\ldots\ldots\ldots (*)$$

Therefore, if given a function f(x) such that f(x) ≥ 0 for $a \leq x \leq b$ and $\mathrm{f}(x) \leq 0$ for $b \leq x \leq c$, then

$$\boxed{\int_a^c |\,\mathrm{f}(x)\,|\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c -\,\mathrm{f}(x)\,\mathrm{d}x}$$

Note: Aim is to remove the modulus by finding the **appropriate expressions.**

Example 8 Evaluate $\int_1^6 \left|x - \frac{9}{x}\right| \mathrm{d}x$ without the use of the graphing calculator.



$\int_1^6 \left|x - \frac{9}{x}\right| \mathrm{d}x$

$= \int_1^3 -\left(x - \frac{9}{x}\right) \mathrm{d}x + \int_3^6 x - \frac{9}{x}\,\mathrm{d}x$

$= -\left[\frac{x^2}{2} - 9\ln|x|\right]_1^3 + \left[\frac{x^2}{2} - 9\ln|x|\right]_3^6$

$= \frac{19}{2} + 9\ln\frac{3}{2}$

Step-By-Step Guide:

1) Draw the graph $y = \mathrm{f}(x)$ (without the modulus), i.e. $y = x - \frac{9}{x}$.
2) Find the x-intercepts.

   $x - \frac{9}{x} = 0$

   $x^2 = 9$

   $x = 3$ or $x = -3$
3) Note the interval of x where $\mathrm{f}(x) \geq 0$ and $\mathrm{f}(x) \leq 0$. In this case, $\mathrm{f}(x) \leq 0$ from $x = 1$ to $x = 3$ and $\mathrm{f}(x) \geq 0$ from $x = 3$ to $x = 3$.
4) Split the limits accordingly based by point (3) and include negative sign to the region where $\mathrm{f}(x) \leq 0$

2.5 <u>Integration of Piecewise Periodic Functions</u>

Example 9 [2009/A Level (9740)/P1/4]

It is given that

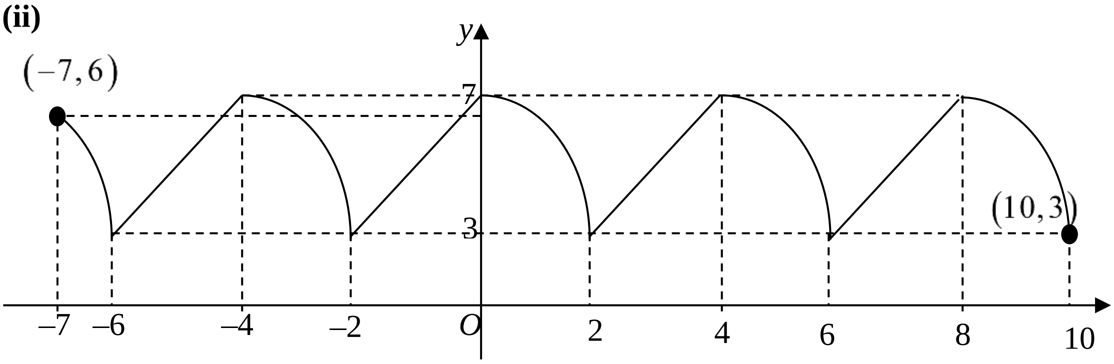
$$f(x)=\begin{cases}7-x^2 & \text{for } 0<x\le 2,\\ 2x-1 & \text{for } 2<x\le 4,\end{cases}$$

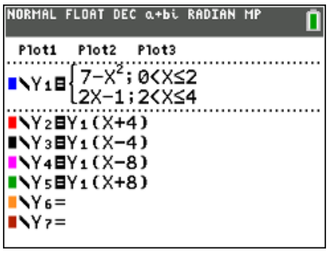
and that f (x) = f (x + 4) for all real values of x.

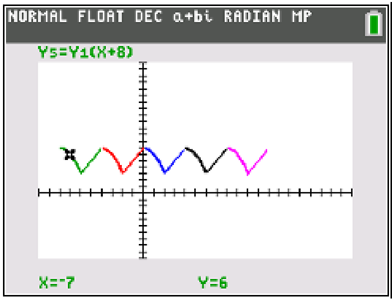
(i) Evaluate $f(27)+f(45)$. [2]

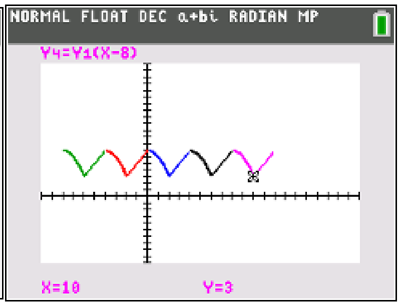
(ii) Sketch the graph of $y=f(x)$ for $-7\le x\le 10$. [3]

(iii) Find $\int_{-4}^{3} f(x)\,dx$. [3]

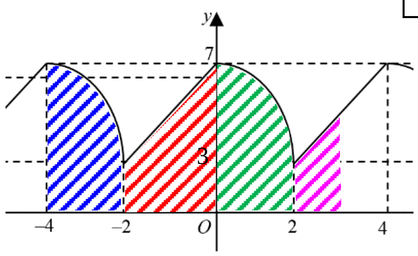
7(i)	$f(x)=f(x+4)$ $f(45)$ $=f(41+4)=f(41)$ $=f(37+4)=f(37)$ $=\ldots$ $=f(5)$ $=f(1)=7-(1)^2=6$ Similarly, $f(27)=f(23)=f(19)=\ldots=f(3)=2(3)-1=5$ $\therefore f(27)+f(45)$ $=f((3)+f(1)$ $=5+6$ $=11$
(ii)	







(iii) $\int_{-4}^{3} \mathrm{f}(x)\,\mathrm{d}x$ ← Find the area bounded by $y=\mathrm{f}(x)$ and x-axis from $x=-4$ to $x=3$.



$$\mathrm{f}(x)=\begin{cases}7-x^2 & \text{for } 0<x\le 2,\\ 2x-1 & \text{for } 2<x\le 4,\end{cases}$$

$$=\int_{-4}^{-2} \mathrm{f}(x)\,\mathrm{d}x+\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x+\int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x+\int_{2}^{3} \mathrm{f}(x)\,\mathrm{d}x$$

Observe that $\mathrm{f}(x)=7-x^2$ is only defined on $0<x\le 2$. Hence
$\int_{-4}^{-2} \mathrm{f}(x)\,\mathrm{d}x=\int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x=\int_{0}^{2}\left(7-x^2\right)\mathrm{d}x$

Observe that $\mathrm{f}(x)=2x-1$ is only defined on $2<x\le 4$. Hence
$\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x=\int_{2}^{4} \mathrm{f}(x)\,\mathrm{d}x=\int_{2}^{4}(2x-1)\,\mathrm{d}x$

$$=\int_{0}^{2} 7-x^2\,\mathrm{d}x+\int_{2}^{4} 2x-1\,\mathrm{d}x+\int_{0}^{2} 7-x^2\,\mathrm{d}x+\int_{2}^{3} 2x-1\,\mathrm{d}x$$

$$=2\left[7x-\frac{x^3}{3}\right]_0^2+\left[x^2-x\right]_2^4+\left[x^2-x\right]_2^3$$

$$=2\left(14-\frac{8}{3}\right)+(12-2)+[6-2]$$

$$=36\frac{2}{3}$$

Remark: It would be **wrong** to find $\int_{-4}^{-2}\left(7-x^2\right)\mathrm{d}x$ and $\int_{-2}^{0}(2x-1)\,\mathrm{d}x$ as the function is not defined at the respective domains.

Alternatively, it is possible to use the **area of trapezium** to calculate

1. $\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x=\frac{1}{2}(3+7)(2)=10$
2. $\int_{2}^{3} \mathrm{f}(x)\,\mathrm{d}x=\frac{1}{2}(3+5)(1)=4$

§3 Volumes of Solids of Revolution

Please watch the video in SLS before this segment:

If a bounded region is rotated about a straight line, it generates a **full solid of revolution.**

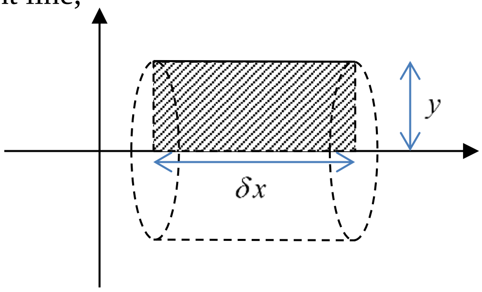
Note: The solid is always symmetrical about the axis of rotation.

Imagine a rectangle as shown below is rotated about a straight line,

(a) What is the shape of the solid formed? A cylinder

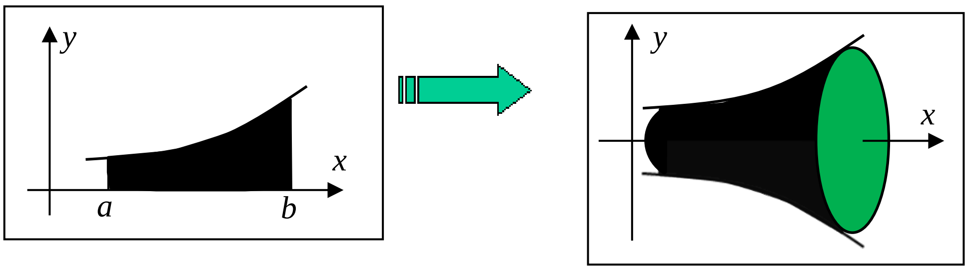
(b) What are the dimensions? Radius = y, height = y

(c) What is the volume generated? Vol = $\pi r^2 h = \pi\left(y^2\right)\delta x$

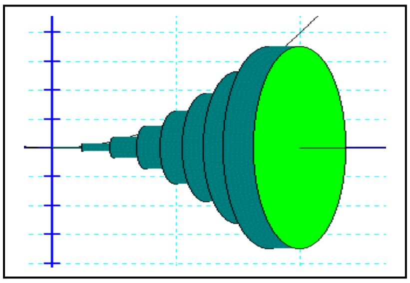


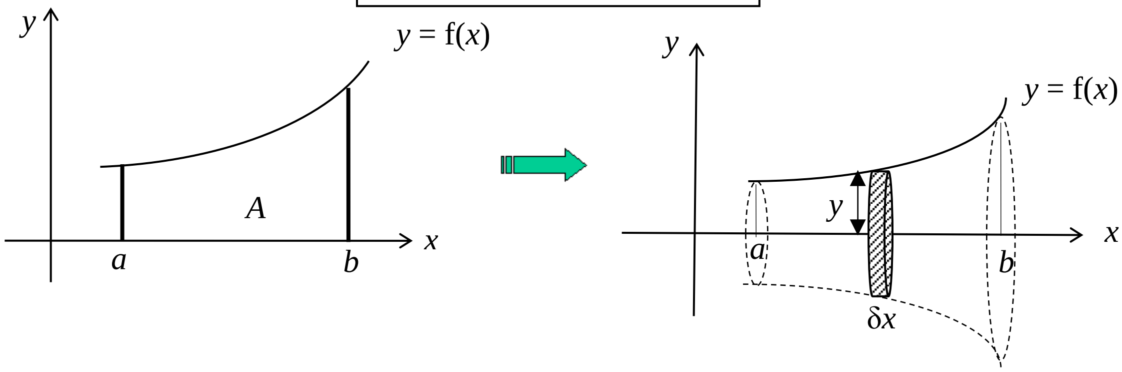
§3.1 Rotation about *x*-axis

Consider the region bounded by the graph of f, the *x*-axis and the vertical lines $x = a$ and $x = b$. Rotate this region ***A*** about *x*-axis through 2π radians. The solid generated is as follows:



To estimate the volume of the solid of revolution, we can split the volume into *n* thin circular discs with equal thickness.





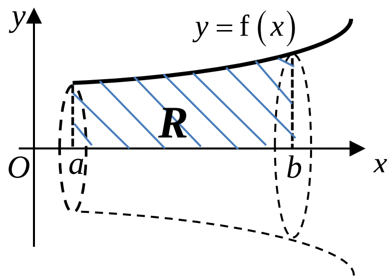
The thinner the disc (width δx gets smaller), the better the approximation for the volume of solid of revolution. Therefore, as the number of discs, n, increases, the sum of volume of the all the discs will reach a limiting value which is the volume of the full solid of revolution.

In other words, the volume of solid, $V = \pi\int_a^b y^2 \, dx$ is the limit of the sum of volume of infinitely many discs for the interval $a \le x \le b$. [Recall : **Definite Integral is a limit of sum.]**

Important Conclusion:

Volume of solid of revolution formed by rotating the region R bounded by the curve $y = f(x)$ from $x = a$ and $x = b$ about x-axis.

$= \pi\int_a^b y^2 \, dx$

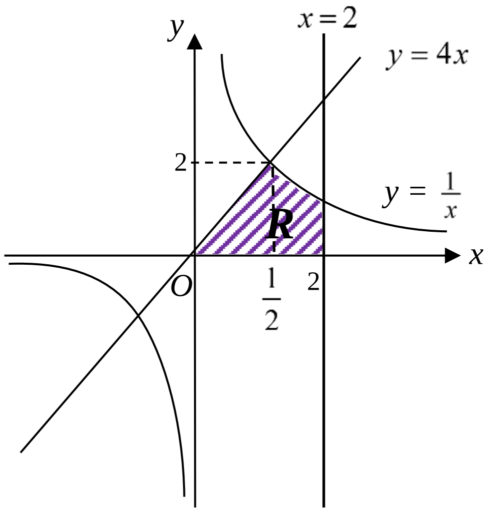


Example 10

The region R is bounded by the line $y = 4x$, the curve $y = \dfrac{1}{x}$, the line $x = 2$ and the positive x-axis. Find the exact volume of the solid of revolution obtained by rotating region R about the x-axis through four right angles.

Observation 1:
Sketch the graphs to identify the region R accurately and solve for the point of intersection first.
At the point of intersection,

$$\Rightarrow 4x = \frac{1}{x} \quad \Rightarrow 4x^2 = 1 \quad \Rightarrow x = \frac{1}{2} \quad \text{(since } x > 0\text{)}$$



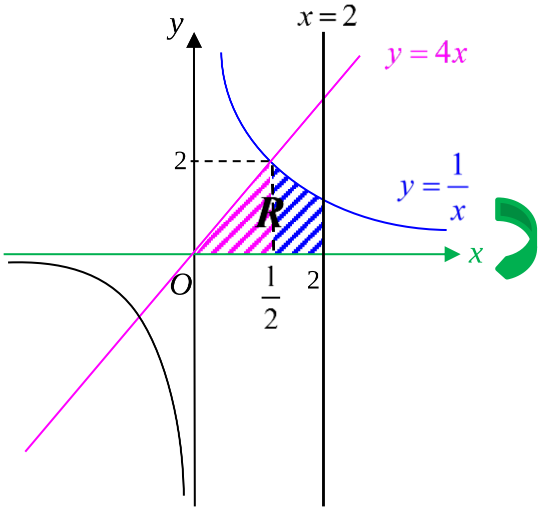
Observation 2:
To find the volume of the solid formed when the region R is rotated through four right angles about the x-axis, we need apply the formula $\pi\int_a^b y^2 \, dx$, so we need to make y the subject (which is already given in this case).

Observation 3:
In order to obtain the required volume generated by the region R, we need to consider the addition of volume generated by 2 different regions.

Volume required

$$= \pi \int_{\frac{1}{2}}^{2} \left(\frac{1}{x}\right)^2 \mathrm{d}x + \pi \int_0^{\frac{1}{2}} (4x)^2 \mathrm{d}x$$

Region bounded by $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 2$ rotated about the x-axis.



$$= \pi \left[-\frac{1}{x}\right]_{\frac{1}{2}}^{2} + \pi \left[\frac{16x^3}{3}\right]_0^{\frac{1}{2}}$$

$$= \frac{3\pi}{2} + \frac{2\pi}{3}$$

$$= \frac{13\pi}{6} \text{ unit}^3$$

Region bounded by $y = 4x$ from $x = 0$ to $x = \frac{1}{2}$ rotated about the x-axis.

Alternatively, can use the formula to find

Vol of cone $= \frac{1}{3}\pi(2)^2\left(\frac{1}{2}\right) = \frac{2}{3}\pi$

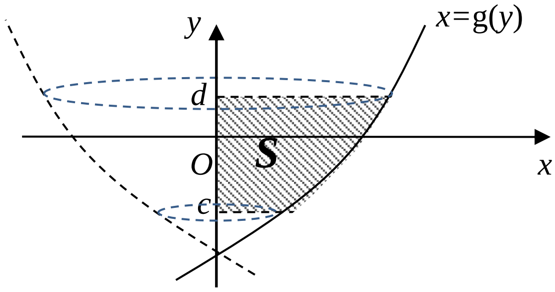
3.2 <u>Rotation about the *y*-axis</u>

Consider the region bounded by the graph of g, the y-axis and the horizontal lines $y = c$ and $y = d$. Rotate this region about y-axis through 2π radians.

Important Conclusion:

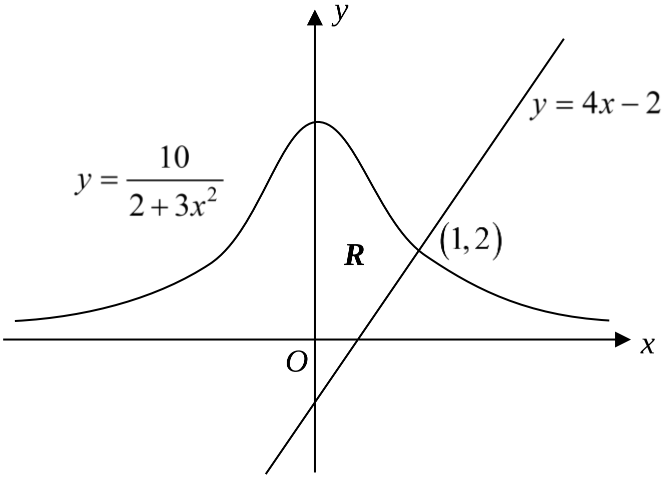
Volume of solid of revolution formed by rotating the region S bounded by the curve $x = \mathrm{g}(y)$ from $y = c$ and $y = d$ about y-axis.

$$= \pi \int_c^d x^2 \, \mathrm{d}y$$



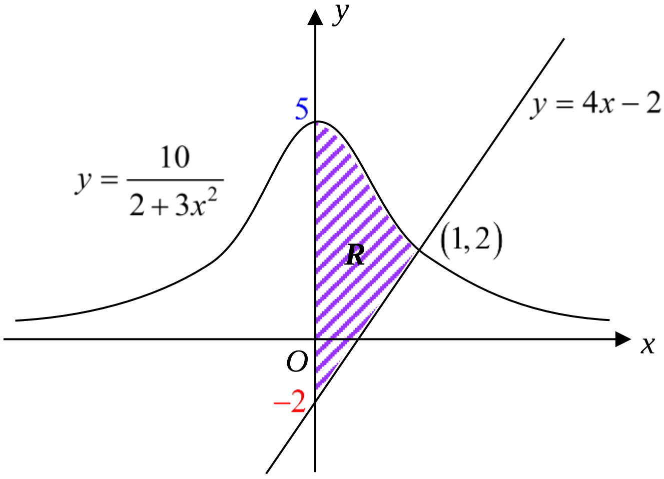
Example 11: MI Prelim 9758/2020/01/Q4ii

The diagram below shows the region R which is bounded by the curve $y=\dfrac{10}{2+3x^2}$, the line $y=4x-2$ and the y-axis.



Find the exact volume of the solid of revolution formed when R is rotated through 2π radians about the y-axis. [5]

Observation 1:
Identify the required region and find the y-intercept of each curve.
For $y=\dfrac{10}{2+3x^2}$, when $x = 0$, $y = 5$.
For $y=4x-2$, when $x = 0$, $y=-2$.



Observation 2:

To find the volume of the solid formed when R is rotated through 2π radians about the y-axis, we need apply the formula $\pi\int_c^d x^2 \, dy$, so we need to make x the subject.

$$y=\frac{10}{2+3x^2}$$

$$\Rightarrow 2+3x^2=\frac{10}{y^2}$$

$$\Rightarrow x^2=\frac{1}{3}\left(\frac{10}{y}-2\right)$$

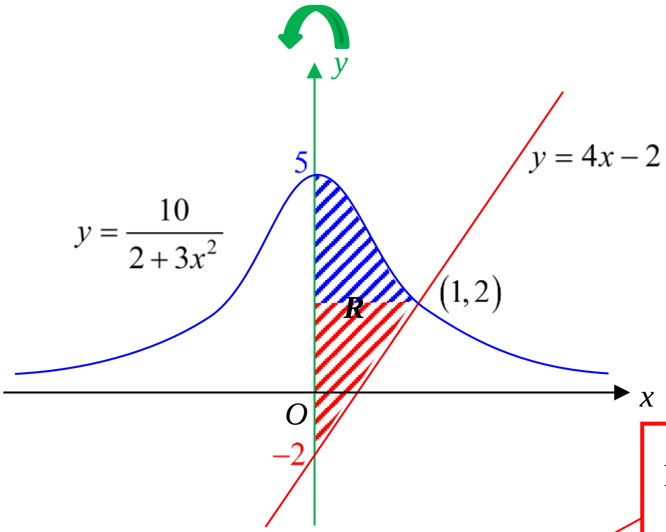
$$=\frac{2}{3}\left(\frac{5}{y}-1\right).$$

$$y=4x-2$$

$$x=\frac{y+2}{4}$$

Observation 3:

In order to obtain the required volume generated by Region R, we need to consider the addition of volume generated by 2 different regions.



Region bounded by $x=\frac{y+2}{4}$ from $x=-2$ to $x=2$ rotated about the y-axis.
Alternatively, can use the formula to find Vol of cone $=\frac{1}{3}\pi(1)^2\left[2-(-2)\right]=\frac{4}{3}\pi$

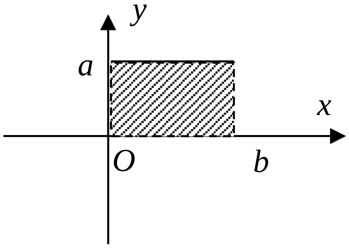
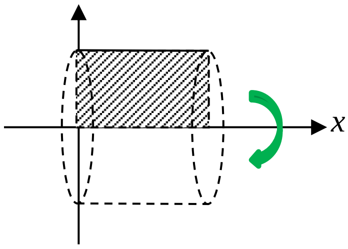
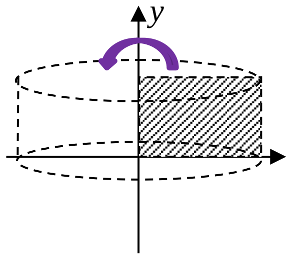
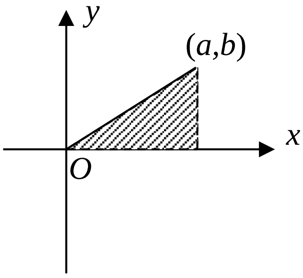
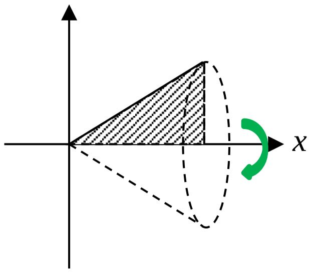
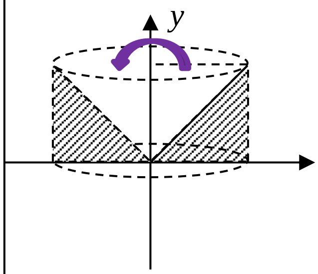
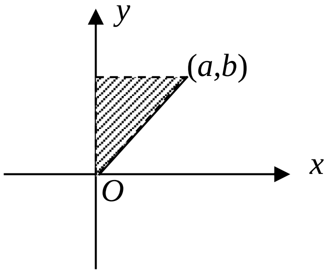
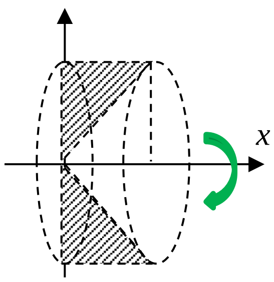
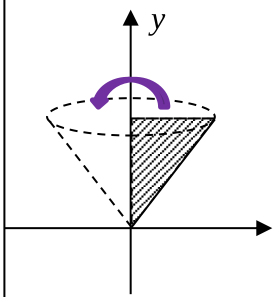
$$\text{Required volume}=\pi\int_2^5\frac{2}{3}\left(\frac{5}{y}-1\right)dy+\pi\int_{-2}^{2}\left(\frac{y+2}{4}\right)^2 dy$$

$$=\frac{2}{3}\pi\left[5\ln|y|-y\right]_2^5+\frac{\pi}{16}\left[\frac{(y+2)^3}{3}\right]_{-2}^{2}$$

$$=\frac{2}{3}\pi\left[(5\ln 5-5)-(5\ln 2-2)\right]+\frac{\pi}{48}\left(4^3\right)$$

$$=\frac{2}{3}\pi\left(5\ln\frac{5}{2}-1\right)\text{ units}^3.$$

Exercise: Sketch the solid of revolution formed when the shaded region is rotated through 360° about the x-axis and y –axis respectively.

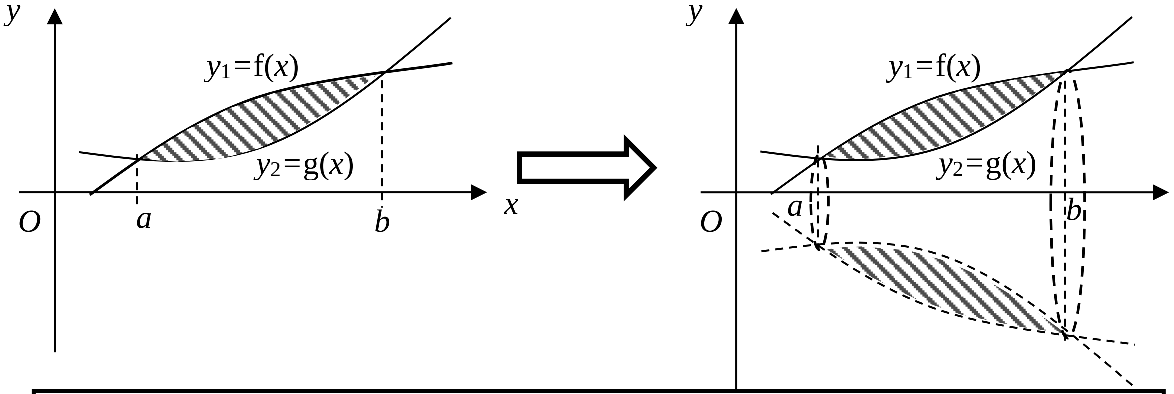
		Rotated about x-axis	Rotated about y-axis
(a)			
(b)			
(c)			

Important Note:

The volume generated when the shaded region is rotated about the x-axis **is different from** the volume generated when the shaded region is rotated about the y-axis. (Refer to Example 12)

3.3 Region Bounded by 2 curves rotated through 360º about *x*-axis

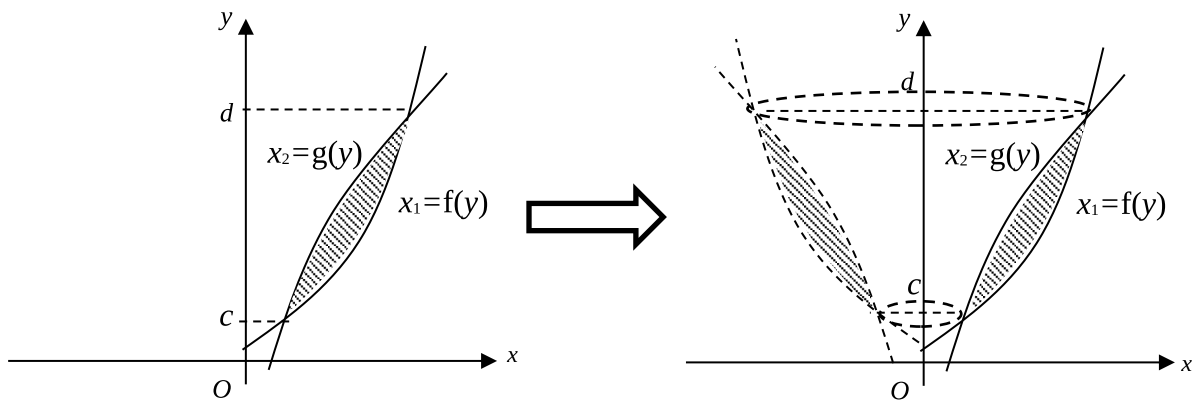
3D view after rotation about *x*-axis



> Volume of revolution when region bounded by 2 curves is rotated about *x*-axis is
>
> $$V = \pi\int_a^b \left[\mathrm{f}\left(x\right)\right]^2 \mathrm{d}x - \pi\int_a^b \left[\mathrm{g}\left(x\right)\right]^2 \mathrm{d}x$$
> $$= \pi\int_a^b \left(y_1\right)^2 \mathrm{d}x - \pi\int_a^b \left(y_2\right)^2 \mathrm{d}x$$
> $$= \pi\int_a^b \left(y_1{}^2 - y_2{}^2\right)\,\mathrm{d}x$$

3.4 Region Bounded by 2 curves rotated through 360º about *y*-axis

3D view after rotation about *y*-axis



> Volume of revolution when region bounded by 2 curves is rotated about *y*-axis is
>
> $$V = \pi\int_c^d \left[\mathrm{f}\left(y\right)\right]^2 \mathrm{d}y - \pi\int_c^d \left[\mathrm{g}\left(y\right)\right]^2 \mathrm{d}y$$
> $$= \pi\int_c^d \left(x_1\right)^2 \mathrm{d}y - \pi\int_c^d \left(x_2\right)^2 \mathrm{d}y$$
> $$= \pi\int_c^d \left(x_1{}^2 - x_2{}^2\right)\,\mathrm{d}y$$

Quick check : Is $\pi\int_c^d \,(x_1^2 - x_2^2)\,\mathrm{d}y = \pi\int_c^d \,(x_1 - x_2)^2\,\mathrm{d}y$?

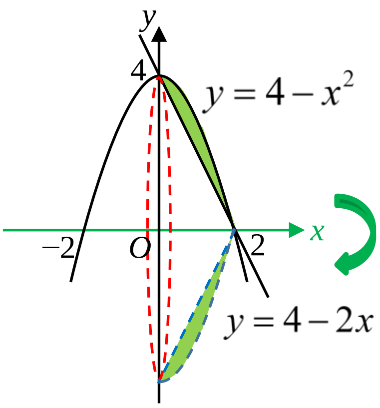
No as $a^2 - b^2 = \left(a-b\right)\left(a+b\right) \neq \left(a-b\right)^2$

Example 12

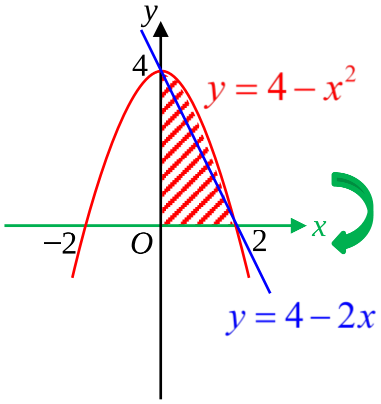
The region R is bounded by the curve $y = 4 - x^2$ and the line $y = 4 - 2x$. Find the volume of the solid formed when this region is rotated through 4 right angles about

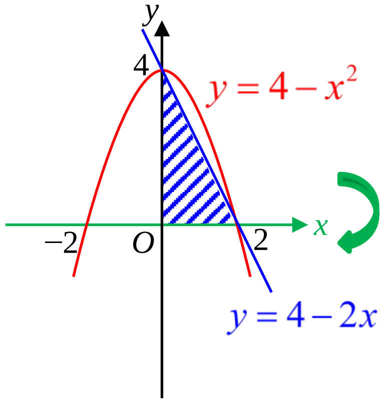
(i) the x-axis (ii) the y-axis.

(i)



In order to to obtain the required volume, we need to use
Bigger Volume – Smaller Volume





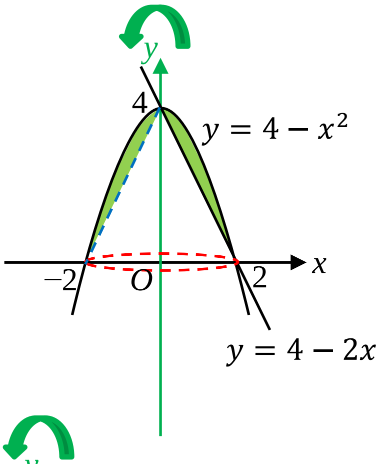
Required volume

$= \pi\int_0^2 \left(4 - x^2\right)^2 \, dx - \text{volume of cone}$

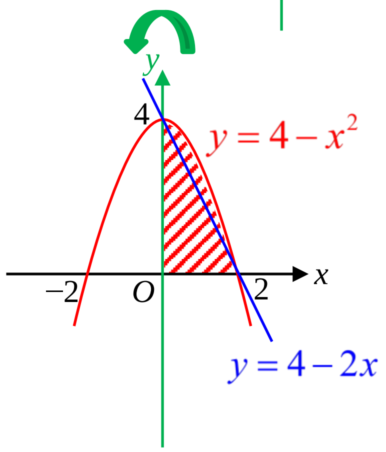
$= \pi\int_0^2 \left(4 - x^2\right)^2 \, dx - \frac{1}{3}\pi(4)^2(2)$

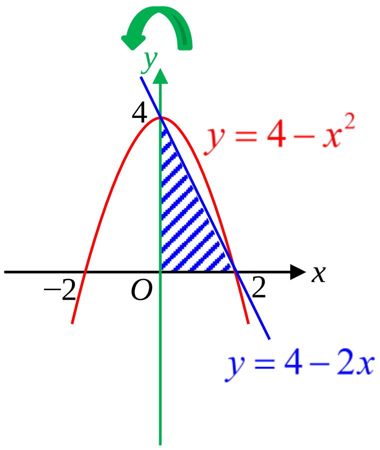
$= 20.1 \text{ units}^3$ (3 s.f)

(ii)



In order to to obtain the required volume, we need to use
Bigger Volume – Smaller Volume





Required volume

$= \pi\int_0^4 (4-y)\,dy$ – volume of cone

$= \pi\int_0^4 (4-y)\,dy - \frac{1}{3}\pi(2)^2(4)$

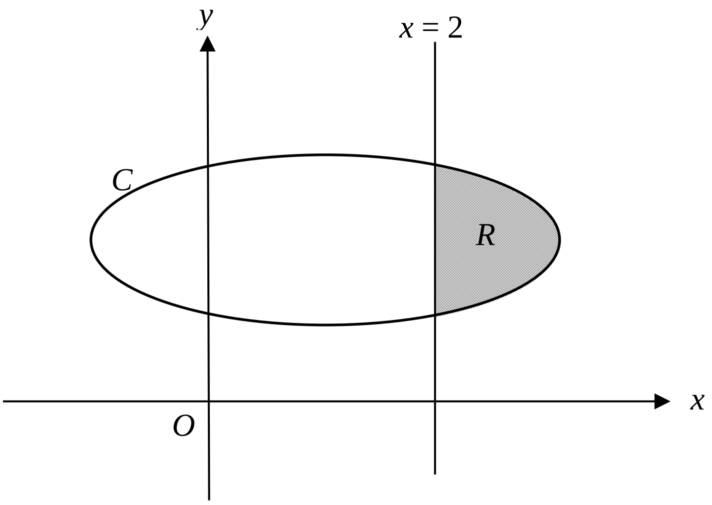
$= 8.38$ units^3 (3 s.f)

Conclusion:

The volume generated when the shaded region is rotated about the *x*-axis **is different from** the volume generated when the shaded region is rotated about the *y*-axis.

Example 13 [2014 DHS Prelim/I/2b]

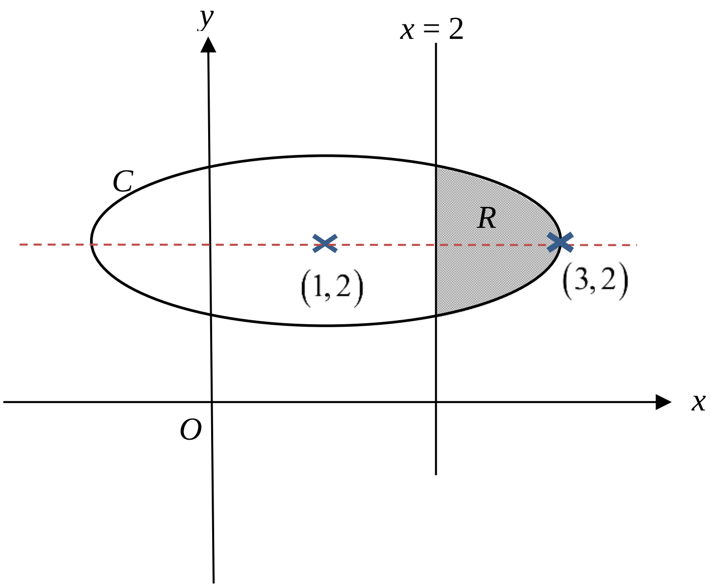
The diagram shows the region *R* bounded by the curve *C* with equation $\frac{(x-1)^2}{4}+(y-2)^2=1$ and the line $x=2$. Find the volume of the solid formed when *R* is rotated through 2π radians about the *x*-axis. [4]



Observation 1:

$\frac{(x-1)^2}{4}+(y-2)^2=1$ is an ellipse with centre $(1,2)$ with vertical semi-minor axis of length 1 unit and horizontal semi-major axis of length 2 units.

The lines of symmetry are $x=1$ and $y=2$.



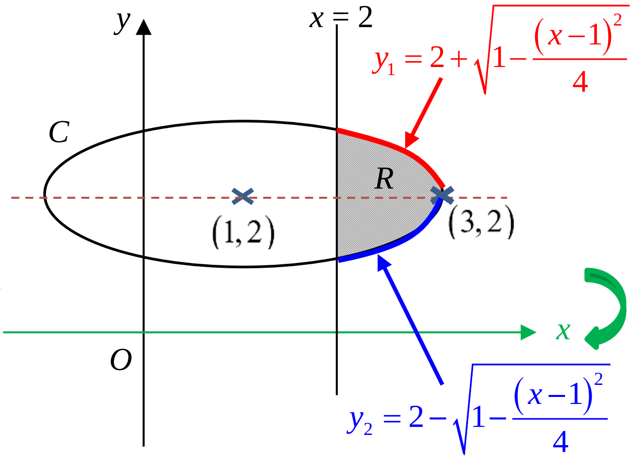
Observation 2:

To find the volume of the solid formed when R is rotated through 2π radians about the x-axis, we need apply the formula $\pi\int_a^b y^2\,dx$, so we need to make y the subject.

$$\frac{(x-1)^2}{4}+(y-2)^2=1$$

$$y=2\pm\sqrt{1-\frac{(x-1)^2}{4}}$$

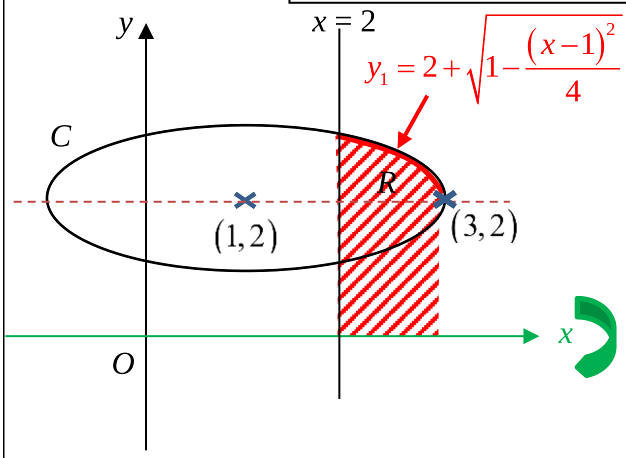
$$y_1=2+\sqrt{1-\frac{(x-1)^2}{4}} \text{ or } y_2=2-\sqrt{1-\frac{(x-1)^2}{4}}$$

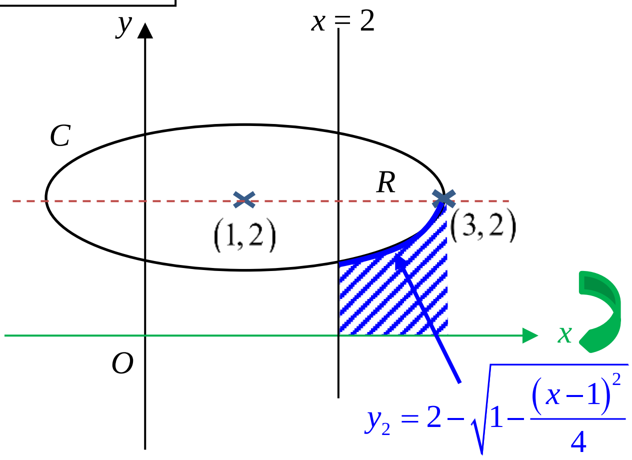


Observation 3:

In order to obtain the required volume generated by Region R, we need to use

Bigger Volume – Smaller Volume





Required volume

$$=\pi\int_2^3 (y_1)^2-(y_2)^2\,dx$$

$$=\pi\int_2^3 \left(2+\sqrt{1-\frac{(x-1)^2}{4}}\right)^2-\left(2-\sqrt{1-\frac{(x-1)^2}{4}}\right)^2 dx$$

Region bounded by $y=2+\sqrt{1-\frac{(x-1)^2}{4}}$ from $x=2$ to $x=3$ rotated about the x-axis.

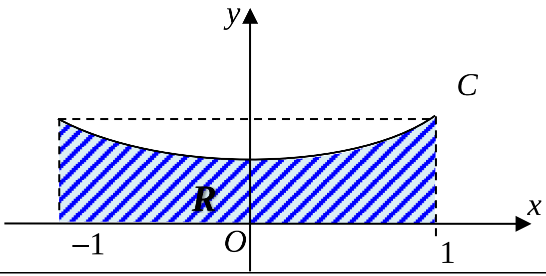
Region bounded by $y=2-\sqrt{1-\frac{(x-1)^2}{4}}$ from $x=2$ to $x=3$ rotated about the x-axis.

$=15.4$ (3 s.f)

Question did not ask for exact form, so GC is allowed here.

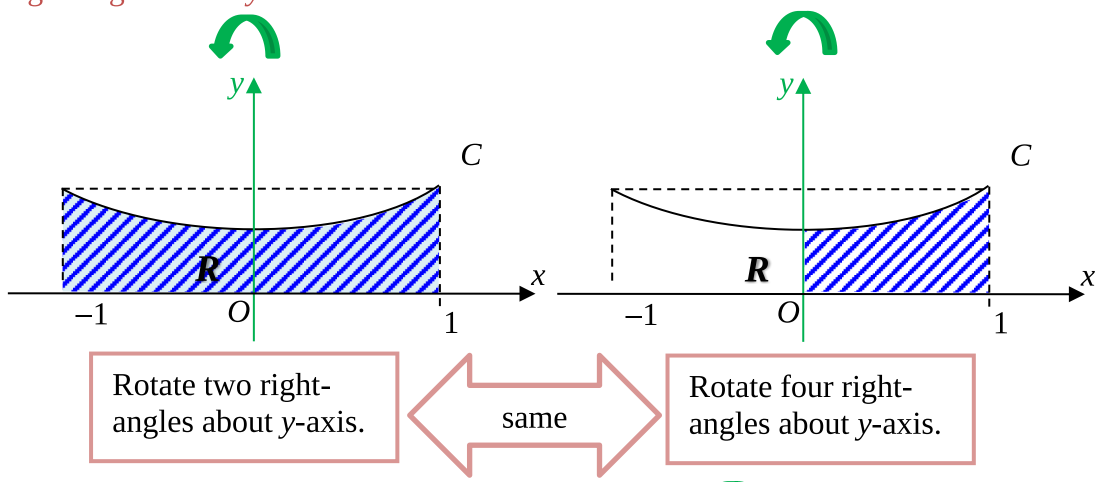
Example 14

A curve C has equation $y=\left(4-x^2\right)^{-\frac{1}{2}}$ for $-1\le x\le 1$. The region R is enclosed by C, the x-axis and the lines $x=-1$ and $x=1$. Find the exact value of the volume generated when R is rotated through two right-angles about y-axis.



Observation 1:

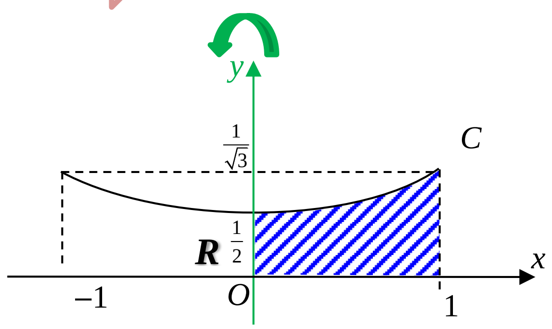
$y=\left(4-x^2\right)^{-\frac{1}{2}}$ is symmetrical about the y-axis. Thus, when R is rotated through two right-angles about y-axis results in a full revolution.



Observation 2:

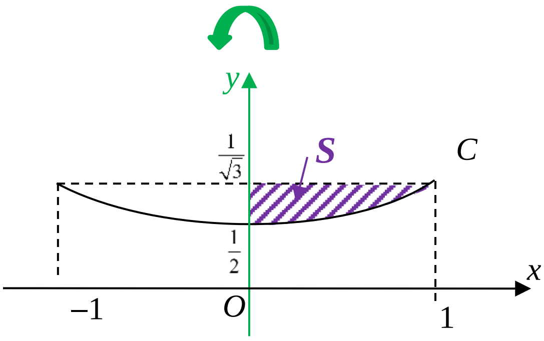
When $x=1, y=\left(4-1^2\right)^{-\frac{1}{2}}=\frac{1}{\sqrt{3}}$

When $x=0, y=\left(4-0^2\right)^{-\frac{1}{2}}=\frac{1}{2}$



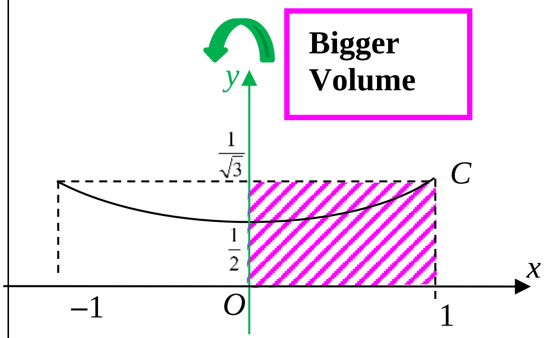
Observation 3:

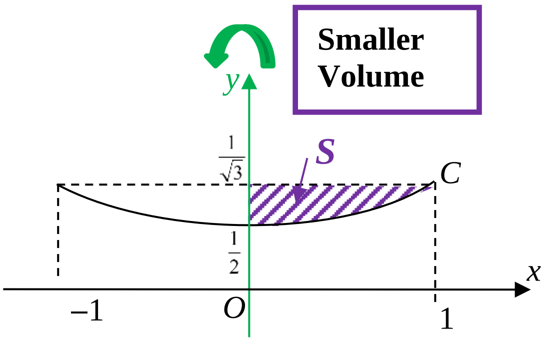
When the region bounded by the curve $y=\left(4-x^2\right)^{-\frac{1}{2}}$ and the y-axis from $y=\frac{1}{2}$ to $y=\frac{1}{\sqrt{3}}$ is rotated about y-axis, it is given by the formula $\pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} x^2\ \mathrm{d}y$. However, this does not give the volume of region *R* and we denote this region as region *S*.



Observation 4:
In order to to obtain the required volume generated by Region *R*, we need to use

Bigger Volume – Smaller Volume





Volume of cylinder – Volume generated by **Region *S*.**

Required volume

$$=\pi(1)^2\left(\frac{1}{\sqrt{3}}\right)-\pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}\left(4-\frac{1}{y^2}\right)\mathrm{d}y$$

$$=\frac{\pi}{\sqrt{3}}-\pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}\left(4-y^{-2}\right)\mathrm{d}y$$

$$=\frac{\pi}{\sqrt{3}}-\pi\left[4y+\frac{1}{y}\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}}$$

$$=\frac{\pi}{\sqrt{3}}-\pi\left[\frac{7}{3}\sqrt{3}-4\right]$$

$$=\pi\left[4-2\sqrt{3}\right]\text{ units}^3$$

Making x^2 the subject:

$$y=\left(4-x^2\right)^{-\frac{1}{2}}\Rightarrow\frac{1}{y^2}=4-x^2$$

$$\Rightarrow x^2=4-\frac{1}{y^2}$$

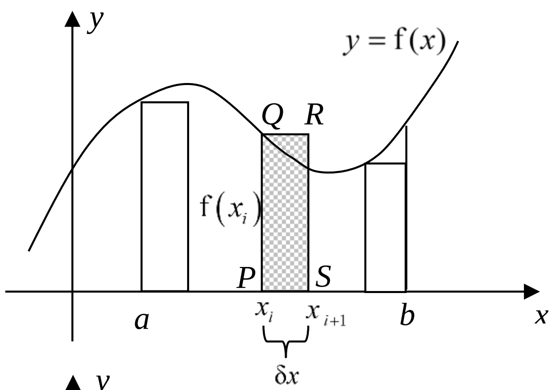
Annex A: Definite Integral as the Limit of a Sum

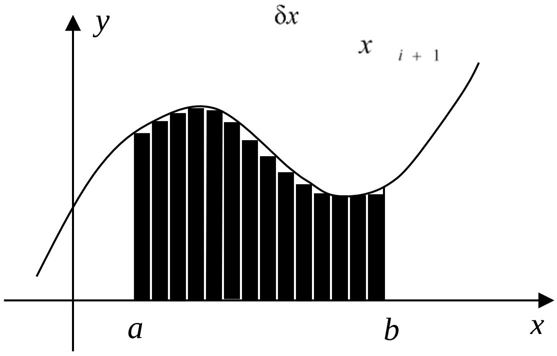
Consider the curve $y = \mathrm{f}(x)$ that is continuous on the interval $a \le x \le b$.

The region A bounded by curve $y = \mathrm{f}(x)$, the lines $x = a$, $x = b$ and the x-axis is referred to as the **area under the curve** $\boldsymbol{y = \mathbf{f}(x)}$ **on the interval** $[\boldsymbol{a}, \boldsymbol{b}]$.

We approximate the area under the curve using thin rectangles with equal width.

Split A into n rectangular strips of **equal width** δx.





Consider one **typical** rectangular strip $PQRS$ where Q is a point $(x_i, \mathrm{f}(x_i))$ on the curve $y = \mathrm{f}(x)$.

The length of strip, $PQ = \mathrm{f}(x_i)$

The width of strip, $PS = \delta x$.

The area of the strip $PQRS$, $\delta A = \mathrm{f}(x_i)\,\delta x$.

Thus,

The area of 1st rectangle = $\mathrm{f}(x_1)\,\delta x$

The area of 2nd rectangle = $\mathrm{f}(x_2)\,\delta x$

The area of 3rd rectangle = $\mathrm{f}(x_3)\,\delta x$

$\vdots$

The area of nth rectangle = $\mathrm{f}(x_n)\,\delta x$

The total area of n rectangles

$= \mathrm{f}(x_1)\,\delta x + \mathrm{f}(x_2)\,\delta x + \mathrm{f}(x_3)\,\delta x + \ldots + \mathrm{f}(x_n)\,\delta x$

$$= \sum_{i=1}^{n} \mathrm{f}(x_i)\,\delta x$$

The thinner the strips are (δx gets smaller), the sum of areas of all the rectangles would be a better approximation for A. Therefore, as the number of rectangular strips, n, increases, the total area of n rectangles will reach a limiting value which is the area under the curve, A.

In other words, area under the curve, $A = \int_a^b y \, \mathrm{d}x$ is the limit of the sum of areas of infinitely many rectangles for the interval $[a, b]$.

Therefore, Area bounded by the curve x_{i+1} and the x-axis for the interval $[a, b]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{f}(x_i)\,\delta x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x = \int_a^b y \,\mathrm{d}x$$

Area bounded by the curve $y = \mathrm{f}(x)$ and the x-axis for the interval $[a, b]$ = $\int_a^b y \,\mathrm{d}x$

Remark: Definite Integral is the limit of a sum.

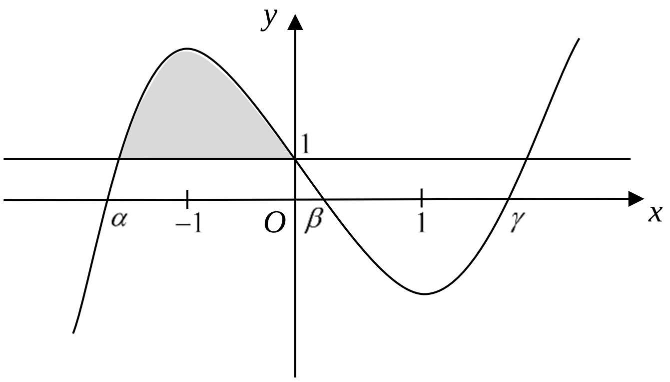
H2 Mathematics (9758)
Chapter 11 Definite Integrals
Discussion Questions

Level 1

1 **(a)** Find the exact area bounded by the line $y=2x+5$ and the curve $y=x^2+2$.

(b) A region is bounded by the parabola $y=2-x^2$ and the line $y=-x$. Show this region clearly on a sketch and find the area of this region.

2 **2010/A Level/P1/6**



The diagram shows the curve with equation $y=x^3-3x+1$ and the line with equation $y=1$. The curve crosses the x-axis at $x=\alpha$, $x=\beta$ and $x=\gamma$ and has turning points at $x=-1$ and $x=1$.

(i) Find the values of β and γ, giving your answers correct to 3 decimal places. [2]

(ii) Find the area of the region bounded by the curve and the x-axis between $x=\beta$ and $x=\gamma$. [2]

(iii) Use a non-calculator method to find the area of the shaded region between the curve and the line. [4]

(iv) Find the set of values of k for which the equation $x^3-3x+1=k$ has three real distinct roots. [2]

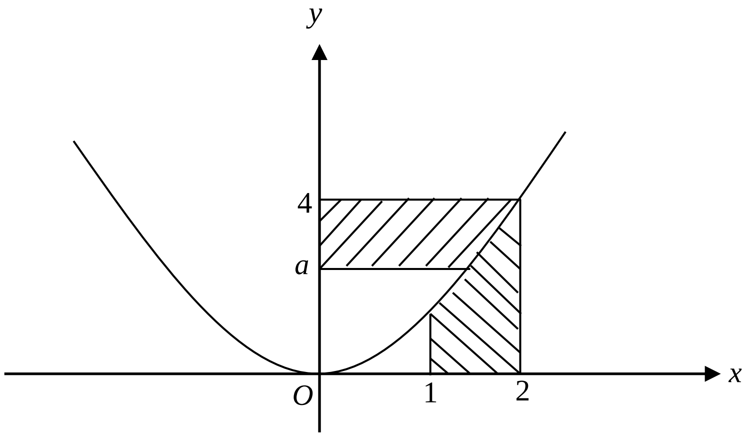
3 **2009/DHS Prelim/I/8a**

Evaluate $\int_0^2 \left|x^2-4x+3\right| \mathrm{d}x$ without the use of the graphic calculator. [3]

4 Find the exact value of the volume generated when the region between the curves $y=x^2$ and $y=\sqrt{x}$ is rotated about the x-axis.

Level 2

5 2008(9740)/I/1



The diagram shows the curve with equation $y = x^2$. The area of the region bounded by the curve, the lines $x = 1$, $x = 2$ and the x – axis is equal to the area of the region bounded by the curve, the lines $y = a$, $y = 4$ and the y – axis, where $a < 4$. Find the value of a. [4]

6 2020/SAJC/Prelim/P1/Q7 part

(i) Find $\int x \tan^{-1}\left(x^2\right) \, dx$. [3]

The finite region R is bounded by the curve $y = \frac{\pi(3x-2)}{4}$, $y = x\tan^{-1}\left(x^2\right)$ and the x-axis.

(ii) Find the exact value of the area of R. [3]

7 The curve C has equation $y = x\cos 2x$, where $0 \le x \le \pi$.

(i) Find the exact x-coordinates of the points where C crosses the x-axis. [3]

(ii) Sketch C, stating the coordinates of any points where the curve crosses the x- and y-axes. [2]

(iii) Find the exact value of $\int_{\frac{\pi}{4}}^{\pi} \left|x\cos 2x\right| \, dx$. [4]

8 **N2011/1/5**

It is given that $\mathrm{f}(x) = 2 - x$

(i) On separate diagrams, sketch the graphs of $y = \mathrm{f}(|x|)$ and $y = |\mathrm{f}(x)|$, giving the coordinates of any points where the graphs meet the *x*- and *y*-axes. You should label the graphs clearly. [3]

(ii) State the set of values of *x* for which $\mathrm{f}(|x|) = |\mathrm{f}(x)|$. [1]

(iii) Find the exact value of the constant *a* for which $\int_{-1}^{1} \mathrm{f}(|x|)\,\mathrm{d}x = \int_{1}^{a} |\mathrm{f}(x)|\,\mathrm{d}x$. [3]

9 **2020/TMJC/Prelim/P2/Q3**

It is given that

$$\mathrm{f}(x) = \begin{cases} 1, & \text{for } -1 < x \le 0, \\ \dfrac{1}{1+4x^2}, & \text{for } 0 < x \le 1, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+2)$ for all real values of *x*.

(i) Find the exact value of $\mathrm{f}(23)$. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \le x \le 3$. [3]

(iii) Find $\int_{-2}^{1} \mathrm{f}(x)\,\mathrm{d}x$, leaving your answer in exact form. [4]

(iv) The region bounded by the curve $y = \mathrm{f}(x)$, the line $y = \frac{1}{5}$ and the *y*-axis is rotated through 2π radians about the *y*-axis. Find the volume of the solid generated, giving your answer to three decimal places. [2]

10 **N2011 (9740)/2/4b**

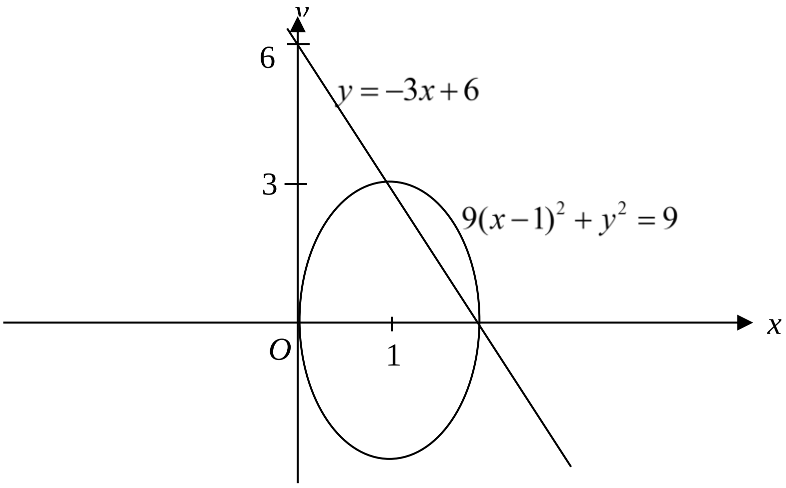
The region bounded by the curve $y = \dfrac{4x}{x^2+1}$, the *x*-axis and the lines $x = 0$ and $x = 1$ is rotated through 2π radians about the *x*-axis. Use the substitution $x = \tan\theta$ to show that the volume of the solid obtained is given by $16\pi\int_0^{\frac{\pi}{4}} \sin^2\theta\,\mathrm{d}\theta$, and evaluate this integral exactly. [6]

11 2009/NJC/Prelim/2/5b

The region R is bounded by the curve $9(x-1)^2 + y^2 = 9$, the line $y = -3x + 6$ and the y-axis shown in the diagram below.

Find the coordinates of the points of intersection between $9(x-1)^2 + y^2 = 9$ and $y = -3x + 6$. [2]

Find the numerical value of the volume of revolution formed when R is rotated completely through 4 right angles about the y-axis. [3]



12 N2002/P1/14 OR

O is the origin and A is the point on the curve $y = \tan x$ where $x = \frac{1}{3}\pi$.

(i) Calculate the area of the region R enclosed by the arc OA, the x-axis and the line $x = \frac{1}{3}\pi$, giving your answer in an exact form.

(ii) The region S is enclosed by the arc OA, the y-axis and the line $y = \sqrt{3}$. Find the volume of the solid of revolution formed when S is rotated through 360° about the x-axis, giving your answer in an exact form.

(iii) Find $\int_0^{\sqrt{3}} \tan^{-1} y \, dy$.

<u>Level 3</u>

13 N2015/P1/3

(i) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\mathrm{f}\left(\frac{n}{n}\right)\right\}$$

is $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$. [2]

(ii) Hence evaluate $\lim_{n\to\infty}\frac{1}{n}\left\{\frac{\sqrt[3]{1}+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n}}\right\}$. [3]

14 2020/VJC/ Prelim/Q4c modified

(i) On the same diagram, sketch the graphs of $y=\left|x^2-2\right|$ and $y=x$ for $0\leqslant x\leqslant 2$. [2]

(ii) Find the exact area of the region bounded by the curve $y=\left|x^2-2\right|$, the line $y=x$ and the *x*-axis. [4]

(iii) The region *R* is bounded by the curve $y=\left|x^2-2\right|$, the line $y=x$ and the *y*-axis. Find the exact volume of the solid obtained when region *R* is rotated through 4 right angles about the *y*-axis. [4]

15 2019/NJC/Prelim/P2/Q1

It is given that

$$\mathrm{f}(x)=\begin{cases} ax, & \text{for } 0\le x<2,\\ \dfrac{12a}{x}-4a, & \text{for } 2\le x<3,\end{cases}$$

where *a* is a positive constant and that $\mathrm{f}(x)=\mathrm{f}(x+3)$ for all real values of *x*.

(i) Find the exact value of $\mathrm{f}\left(\frac{44}{3}\right)$ in terms of *a*. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-3\le x\le 3$, indicating clearly the coordinates of the maximum points and axial intercepts. [3]

(iii) Find $\int_{-2}^{2}\mathrm{f}(x)\,\mathrm{d}x$ in terms of *a*, leaving your answer in the exact form. [3]

16 2019/NYJC/JC2 MYE/P1/Q8

(i) By using the substitution $x = a\sin\theta$, show $\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x = \frac{a\pi}{4}$. [3]

(ii) The finite region, *R*, in the first quadrant is bounded by the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$, where $a > b > 0$. Find the exact value of the area of *R*. [3]

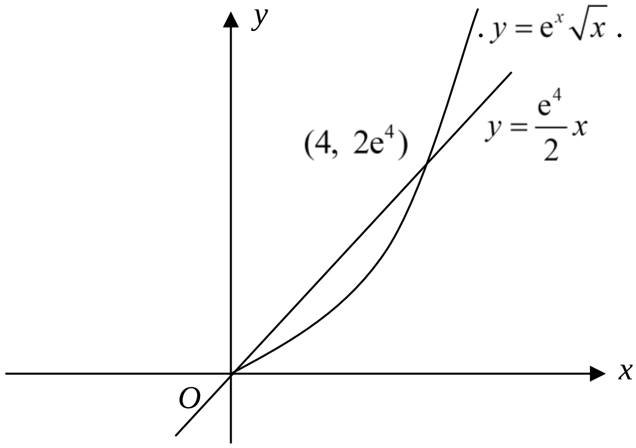
(iii) By writing down the equation of the resulting curve when $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is translated *b* units in the direction of the negative *y*-axis, show that the volume of the region *R* when it is rotated through 2π radians about the line $y = b$ is given by $\frac{\pi ab^2}{6}(3\pi - 8)$. [4]

Answer Key:	
1	(a) $\frac{32}{3}$ (b) $\frac{9}{2}$
2	(i) $\beta = 0.347; \gamma = 1.532$ (ii) 0.781 (iii) $\frac{9}{4}$ (iv) $\{k \in \mathbb{R} : -1 < k < 3\}$
3	2
4	$\frac{3\pi}{10}$
5	2.73
6	(i) $\frac{x^2}{2}\tan^{-1}(x^2) - \frac{1}{4}\ln(1+x^4) + C$ (ii) $\frac{\pi}{12} - \frac{1}{4}\ln 2$
7	(i) $x = 0, \quad x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}$ (iii) $\frac{7\pi}{8} + \frac{1}{4}$
8	(ii) $\{x \in \mathbb{R} : 0 \le x \le 2\}$ (iii) $a = 2 + \sqrt{5}$
9	(i) $\frac{1}{5}$ (iii) $\tan^{-1}(2) + 1$ (iv) 0.636
10	$2\pi(\pi - 2)$
11	(1, 3) and (2, 0); 4.05
12	(i) $\ln 2$ (ii)_ $\frac{4}{3}\pi^2 - \sqrt{3}\pi$ (iii) $\frac{\sqrt{3}\pi}{3} - \ln 2$
13	(ii) $\frac{3}{4}$
14	(ii) $\frac{4\sqrt{2}}{3} - \frac{7}{6}$ (iii) $\frac{5\pi}{6}$
15	(i) $\frac{a}{2}$ (iii) $12a\ln\left(\frac{3}{2}\right) - \frac{1}{2}a$
16	(ii) $\frac{ab}{4}(\pi - 2)$

H2 Mathematics (9758)
Chapter 11 Definite Integrals
Extra Practice Questions

1. 2009/VJC Prelim/1/10

Use integration by parts to show that $\int xe^{2x}\,dx = \frac{e^{2x}}{4}(2x-1)+c$, where c is an arbitrary constant. [2]



The diagram above shows the graph of a straight line $y = \frac{e^4}{2}x$ and part of the graph of a curve C given by $y = e^x\sqrt{x}$.

(a) The region R is bounded by the line $y = \frac{e^4}{2}x$ and C between $x = 0$ and $x = 8$. Find, correct to 1 decimal place, the area of region R. [2]

(b) The region S is bounded by C and the line $y = \frac{e^4}{2}x$, between $x = 0$ and $x = 4$. Show that the volume V of the solid formed when S is rotated 2π radians about the x-axis is given by $V = \pi\left(Ae^8 - B\right)$, where A and B are exact constants to be determined. [3]

2. 2011/TJC Prelim/1/10

On a diagram, shade the region bounded by the curve $y = \frac{6}{x}$ and the lines $y = 2x$ and $y = 3x$ for $x \geq 0$. Label all the intersection points clearly. [2]

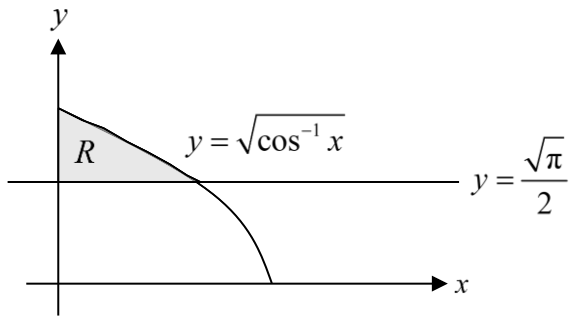
Find the exact area of the shaded region. [3]

Find the volume of the solid formed by rotating the circle $x^2 + y^2 - 4y + 3 = 0$ about the x-axis. Give your answer correct to one decimal place. [4]

3 2013/VJC Prelim/1/10 modified

(i) Show that $\int_0^{\frac{1}{\sqrt{2}}} \cos^{-1} x \, dx = \frac{\pi}{4\sqrt{2}} + 1 - \frac{1}{\sqrt{2}}$. [5]

(ii) The diagram below shows the curve with equation $y = \sqrt{\cos^{-1} x}$. The region bounded by the curve, the line $y = \frac{\sqrt{\pi}}{2}$ and the y-axis is denoted by R.



Find the exact volume of revolution when R is rotated completely about the x-axis. [4]

(iii) Write down the equation of the curve obtained when C is translated by $\frac{\sqrt{\pi}}{2}$ units in the negative y-direction. Hence, or otherwise, find the volume of the solid generated when R is rotated through 2π radians about the line $y = \frac{\sqrt{\pi}}{2}$.

4 2013/TJC Prelim/1/9

(i) Find $\int e^{2x} \cos 4x \, dx$. [4]

(ii) Sketch the curves $y = e^x \sin 2x$ and $y = e^x$ for $0 \le x \le \frac{\pi}{2}$ on a single diagram. Find the exact x-coordinate of the point of intersection between the two curves. [3]

(iii) Find the exact volume of the solid generated when the region bounded by the two curves and the y-axis is rotated through 2π radians about the x-axis. [4]

5 2017/MJC Promo/4a

Find the exact value of $\int_0^3 \left| x^3 - 5x^2 + 4x \right| dx$. [4]

6 2018/RVHS Promo/9

(i) Find $\int (\ln x)^2 \, dx$. [3]

(ii) The region bounded by the curve $y = e^x$, the line $x = 4$, and the axes is rotated 2π radian about the y-axis. Find the exact volume of the solid formed. [5]

7 2018/YJC Promo/11

Find the volume of the solid formed when the region bounded by the curve $y = xe^{x^3}$ and the line $y = 3x$ is rotated through 2π radians about the x-axis. [4]

8 **N2009 (9740)/1/11**

The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = x\mathrm{e}^{-x^2}$.

(i) Sketch the curve C. [2]

(ii) Find the exact coordinates of the turning points on the curve. [4]

(iii) Use the substitution $u = x^2$ to find $\int_0^n \mathrm{f}(x)\,\mathrm{d}x$, for $n > 0$. Hence find the area of the region between the curve and the positive x-axis. [4]

(iv) Find the exact value of $\int_{-2}^{2} |\mathrm{f}(x)|\,\mathrm{d}x$. [2]

(v) Find the volume of the revolution when the region bounded by the curve, the lines $x = 0$, $x = 1$ and the x-axis is rotated completely about the x-axis. Give your answer correct to 3 significant figures. [2]

9 **2016/RVHS Promo/11**

(a) By using the substitution $x = 3\tan\theta$, find $\dfrac{x^2}{(x^2+9)^2}\,\mathrm{d}x$. [5]

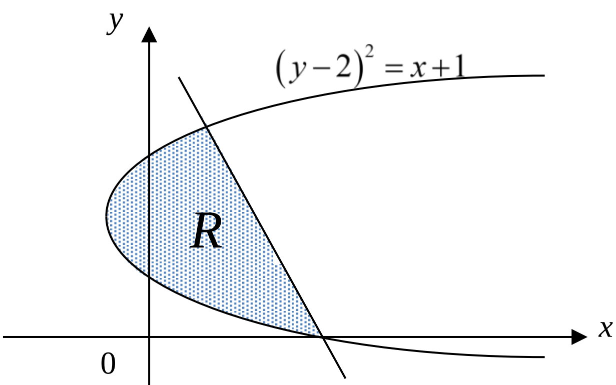
(b) An hourglass is a mechanical device used to measure the passage of time. It comprises two glass bulbs connected vertically by a narrow neck that allows a regulated trickle of sand from the upper bulb to the lower one. The size of the hourglass can be generated when the curve $y = \dfrac{x}{x^2+9}$ is rotated through 2π about the x-axis, from $x = -a$ to $x = a$. Two hourglasses, A and B, are generated.

(i) Find the exact volume of the hourglass A generated when $a = 3$. [3]

(ii) It is found that the actual volume of the hourglass A produced is larger than the theoretical volume found in part **(i)**. State a possible reason. [1]

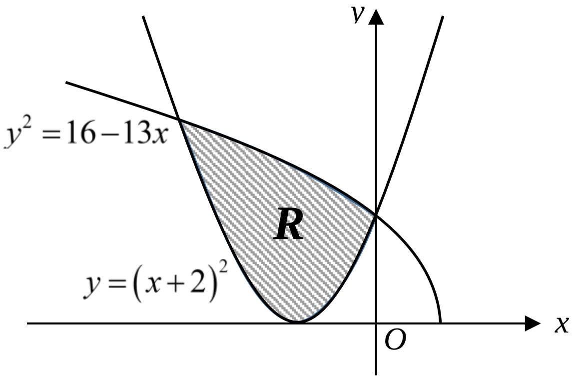
(iii) A new hourglass B is to be generated such that its volume is double that of A. Find the value of a. [3]

10 **(a)** The diagram shows a shaded region R bounded by the curve $(y-2)^2 = x+1$ and the line $y + 2x = 6$.



Find the volume generated when R is rotated through 2π radians about the x-axis, leaving your answer correct to 3 significant figures. $y + 2x = 6$ [4]

(b) The diagram shows the finite region ***R*** bounded by the curves $y^2 = 16 - 13x$ and $y = (x+2)^2$. Find the volume generated when ***R*** is rotated completely about the y-axis, giving your answer correct to 3 significant figures. [4]



Answer Key

1	$\frac{e^{2x}}{4}(2x-1)+C$, (a) 7239.2 (b) $V = \pi\left(\frac{43}{12}e^8 - \frac{\pi}{4}\right)$
2	$3(\ln 3 - \ln 2)$; 39.5
3	(i) $-x\left(1-x^2\right)^{-\frac{1}{2}}$ (ii) $\frac{\pi}{4\sqrt{2}}+1-\frac{1}{\sqrt{2}}$; $\pi-\frac{\pi}{\sqrt{2}}$ (iii) 0.116
4	(i) $\frac{1}{10}e^{2x}(\cos 4x + 2\sin 4x) + C$ (ii) $\frac{\pi}{4}$ (iii) $\frac{1}{10}\pi\left(2e^{\frac{\pi}{2}}-3\right)$
5	$\frac{95}{12}$
6	(i) $x(\ln x)^2 - 2x\ln x + 2x + C$ (ii) $6\pi e^4 + 2\pi$
7	6.17
8	(ii) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ (iii) $\frac{1}{2}\left(1-e^{-n^2}\right)$; 0.5 (iv) $1-e^{-4}$ (v) 0.363
9	(a) $\frac{1}{6}\tan^{-1}\frac{x}{3} - \frac{x}{2(x^2+9)} + c$ (b)(i) $\frac{1}{12}\pi^2 - \frac{1}{6}\pi$ (iii) 4.86
10	**(a)** 78.6 **(b)** 350