Title	Secondary School Additional Mathematics Materials Compilation 6
Date	17/9/2024
Notice	This document is released into public domain under CC0. [1]

Title	Surds Manipulation
Author	-
Date	31/12/2022

Basic Surds Rules to Understand before Proceeding

Given the following expression can be written in the following form

$g\sqrt{ab^2}$

It can be rewritten as the following

$gb\sqrt{a}$

Example 1.1

$\sqrt{8}$ can be decomposed into the following

$\sqrt{2 \times 4}$

Since $4 = 2^2$, we can rewrite in the following manner: $2\sqrt{2}$

Law of Surds (Only involving square roots)
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
$a\sqrt{b} \times c\sqrt{d} = ac\left(\sqrt{bd}\right)$
$\sqrt{a^2} = a$
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
$m\sqrt{a} + n\sqrt{a} = \sqrt{a}(m+n)$
$m\sqrt{a} - n\sqrt{a} = \sqrt{a}\ (m-n)$

Rules of rationalizing the denominator in surds calculation and manipulation

Rule 1

If expression is in the following form

$$\frac{a}{g\sqrt{b}}$$

Multiply by the denominator to both numerator and denominator to get the following

$$\frac{a}{g\sqrt{b}} \times \frac{g\sqrt{b}}{g\sqrt{b}}$$

Rule 2.
If the expression is in the following form or show some near resemblance to the following form

$$\frac{a+g\sqrt{b}}{c-d\sqrt{p}}$$

Find the conjugate value of the denominator whereby the sign in between $c-d\sqrt{p}$ is flipped to positive and multiply conjugate value to both numerator and denominator to get the following

$$\frac{a+g\sqrt{b}}{c-d\sqrt{p}}\times\frac{c+d\sqrt{p}}{c+d\sqrt{p}}$$

Rule 3.
If the expression is in the following form or show some near resemblance to the following form

$$\frac{a+g\sqrt{b}}{c+d\sqrt{p}}$$

Find the conjugate value of the denominator whereby the sign in between $c+d\sqrt{p}$ is flipped to negative and multiply conjugate value to both numerator and denominator to get the following

$$\frac{a+g\sqrt{b}}{c+d\sqrt{p}}\times\frac{c-d\sqrt{p}}{c-d\sqrt{p}}$$

Question 1.
1.1 Simplify the following expression
  (a) $11\sqrt{7}+6\sqrt{28}-5\sqrt{63}$
  (b) $\left(4\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}-5\sqrt{2}\right)$

(a) $11\sqrt{7} + 6\sqrt{28} - 5\sqrt{63}$

Rewrite as $11\sqrt{7} + 6\sqrt{7 \times 4} - 5\sqrt{9 \times 7}$

Once again can be rewritten as $11\sqrt{7} + 6(2)\sqrt{7} - 5(3)\sqrt{7}$

We then proceed to simplify the expression as

$(11 + 12 - 15)\sqrt{7} = 8\sqrt{7}$

(b) $\left(4\sqrt{3} - \sqrt{2}\right)\left(\sqrt{3} - 5\sqrt{2}\right)$

Expand the expression into the following

$4\sqrt{3}\left(\sqrt{3}\right) - \sqrt{2}\left(\sqrt{3}\right) - 5\sqrt{(2)}\left(4\sqrt{3}\right) + 5\sqrt{2}\left(\sqrt{2}\right)$

$4(3) - \sqrt{6} - 20\sqrt{6} + 5(2) = 12 + 10 - \sqrt{6} - 20\sqrt{6} = 22 - 21\sqrt{6}$

1.2 Rationalize the denominator of the following

(a) $\frac{12}{\sqrt{3}}$

(b) $\frac{2-\sqrt{7}}{3+4\sqrt{7}}$

(c) $\frac{1}{3-\sqrt{5}}$

Solutions

(a) $\frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

(b) $\frac{2-\sqrt{7}}{3+4\sqrt{7}} \times \frac{3-4\sqrt{7}}{3-4\sqrt{7}} = \frac{(2-\sqrt{7})(3-4\sqrt{7})}{(3+4\sqrt{7})(3-4\sqrt{7})}$

$$= \frac{6 - 3\sqrt{7} - 8\sqrt{7} + 28}{9 - 4^2(7)}$$

$$= \frac{6 + 28 - 3\sqrt{7} - 8\sqrt{7}}{-103} = \frac{\left(34 - 11\sqrt{7}\right)}{-103} = \frac{-34 + 11\sqrt{7}}{103} = \frac{11\sqrt{7}}{103} - \frac{34}{103}$$

(c) $\frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{3+\sqrt{5}}{3^2-5}$

$$= \frac{3 + \sqrt{5}}{4} = \frac{3}{4} + \frac{\sqrt{5}}{4}$$

2.3[Equations Involving Surds]

Step by Step Instructions for solving such equations

1. Rearrange the equation (if necessary) such that all square-roots are on the left-hand-side and all non-square-roots are on the right-hand-side.
2. Square both sides and then use the usual method you learn during Elementary Mathematics to solve equation (i.e. Rearrange and then use Quadratic Formula, etc.)
3. Substitute answers back into the original questions and reject values that don't make sense (i.e. square root of negative values, etc.)
4. Use the remaining values as answers. I typically leave the answers in surd form for Additional Mathematics paper unless the paper specifies something else.

(a)$\sqrt{7x+5} = x+1$

(b) $5\sqrt{5x+9} - 4x - 3 = 0$

(a) $\left(\sqrt{7x+5}\right)^2 = (x+1)^2$

$7x + 5 = x^2 + 2x + 1$

$0 = x^2 + 2x + 1 - 7x - 5$

$0 = x^2 - 5x - 4$

After using the quadratic formula, we get the following

$x = \frac{5}{2} - \frac{\sqrt{41}}{2}\ OR\ x = \frac{5}{2} + \frac{\sqrt{41}}{2}$

(b) $5\sqrt{5x+9} - 4x - 3 = 0$

Rearrange the equation such that square roots are on one side and non-square-roots are on the other side.

$5\sqrt{5x+9} = 4x + 3$

$\left(5\sqrt{5x+9}\right)^2 = (4x+3)^2$

$25(5x+9) = (4x)^2 + 2(3)(4x) + 9$

$125x + 225 = 16x^2 + 24x + 9$

$0 = 16x^2 + 24x - 125x + 9 - 225$

$0 = 16x^2 - 101x - 216$

$x = -\frac{27}{16}\ OR\ x = 8$

After substituting the value $x = -\frac{27}{16}$ into $5\sqrt{5x+9} - 4x - 3 = 0$

$5\sqrt{5\left(-\frac{27}{16}\right)+9} - 4\left(-\frac{27}{16}\right) - 3 = 7.5$

Literally implies that $7.5 = 0$ (Reason for rejecting $x = -\frac{27}{16}$, substituting the value into the question yield illogical results.)

After substituting value $x = 8$ into $5\sqrt{5x+9} - 4x - 3 = 0$

$5\sqrt{5(8)+9} - 4(8) - 3 = 0$

When we worked out the left side of the equation, we get $0 = 0$, which is within mathematical logic, therefore, we accept this as the only valid answer and thus $x = 8$

2.4 [Equality of Surds]

(a) Given that $a + b\sqrt{2} = \left(3-\sqrt{2}\right)^2 - \frac{8}{1-\sqrt{2}}$

Find the value for a and b

$$\left(3-\sqrt{2}\right)^2 - \frac{8}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}}$$

$$= 9 - 2(3)\left(\sqrt{2}\right) + 2 - \frac{\left[8\left(1+\sqrt{2}\right)\right]}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)}$$

$$= 9 - 6\sqrt{2} + 2 - \frac{\left[8\left(1+\sqrt{2}\right)\right]}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)}$$

$$= 11 - 6\sqrt{2} - \frac{8+8\sqrt{2}}{1^2-2}$$

$$= 11 - 6\sqrt{2} + 8 + 8\sqrt{2}$$

$$= 19 + 2\sqrt{2}$$

$$19 + 2\sqrt{2} = a + b\sqrt{2}$$

Therefore $b = 2$ and $a = 19$

Title	Quadratic Functions and Quadratic Equation
Date	3/1/2023
Author	-

Finding minimum point or maximum point of a Quadratic Function by completing the square method
Step 1. Determine if the Quadratic Function in question has a minimum point or a maximum point, it can be done after equating the quadratic function to zero and rearranging in the following form $$ax^2 + bx + c = 0$$ If $a > 0$ the quadratic function in question has a minimum point If $a < 0$ the quadratic function in question has a maximum point Step 2. Apply completing the square method 1. Factor the value a out of the equation Given an equation $2x^2 - 10x - 18 = 0$ $2(x^2 - 5x) - 18 = 0$ 2. Add $\left(\frac{b}{2a}\right)^2$ into the bracket and deduct $\left(\frac{b^2}{4a}\right)$ from outside the bracket $$2\left[x^2 - 5x + \left(\frac{10}{2(2)}\right)^2\right] - 18 - \frac{10^2}{4(2)}$$ $$2\left(x - \frac{5}{2}\right)^2 - \frac{61}{2}$$ 3. In this case, x-coordinate value of the minimum point is $-\frac{b}{2a}$ which is $\frac{5}{2}$ The y coordinate value in this case is $-\frac{61}{2}$

Basic Concepts of Discriminant to Understand Before Proceeding	
Any quadratic equation being rearranged in the following form $ax^2 + bx + c = 0$ has the values has of $D = b^2 - 4ac$ as the discriminant value.	
Condition of Discriminant	Consequences and Interpretation
$D > 0$	Quadratic Equation has two real and distinct roots.
$D = 0$	Quadratic Equation has real and equal roots.
$D \geq 0$	Quadratic Equation has real roots in general.
$D < 0$	Quadratic Equation has unreal roots. (No real roots)

Discriminant Manipulation

Question 2

Find the range of value for k for which the expression $3x^2 + 6x + k$ is always positive for all real value of x.

[When you see the following keywords: Always Positive, Always Negative, No Real Roots, it implies the graph does not have any contact with the X-axis, thus having no real roots and thus $b^2 - 4ac < 0$.]

Knowing such information, we will proceed on with finding a value of k that can satisfy the requirement of no real roots at all.

$6^2 - 4(3)(k) < 0$
$36 - 12k < 0$
$-12k < -36$
$12k > 36$
$k > 3$

Question 3

The expression $(k+3)x^2 + 6x + k = 5$ has two distinct solutions for x.

(a) Show that k satisfy $k^2 - 2k - 24 < 0$
(b) Find the set of possible value of k

[When you see the following keywords, pass through graph at two distinct points, distinct roots or anything similar, it means the graph have two real and distinct roots, thus $b^2 - 4ac > 0$.]

(a)

$6^2 - 4(k+3)(k-5) > 0$

$36 - 4(k^2 + 3k - 5k - 15) > 0$
$36 - 4k^2 - 12k + 20k + 60 > 0$

$-4k^2 - 12k + 20k + 60 - 36 > 0$
$-k^2 - 3k + 5k + 24 > 0$

$-k^2 + 2k + 24 > 0$
$k^2 - 2k - 24 < 0$ [Shown]

(b)
$k^2 - 2k - 24 < 0$
Factorize left hand side to get
$(k-6)(k+4) < 0$

(Less than zero means k must be within a range of values)
Thus, we get
$k < 6$ OR $k < -4$

Which can be written as
$-4 < k < 6$

Question 3
Find the values of p for which the equation $3x^2 + 2px - p$ has
(a) Real and Equal Roots
(b) Distinct Roots
When you see the following keywords “has only 1 solution”, “real and equal roots”, “repeated roots” or anything similar the graph merely has contact with the X-axis one time, thus $b^2 - 4ac = 0$

(a)
$(2p)^2 - 4(3)(-p) = 0$
$4p^2 + 12p = 0$
$p(p+3) = 0$
$p = -3$ or $p = 0$

(b)
$(2p)^2 - 4(3)(-p) > 0$
$4p^2 + 12p > 0$
(More than zero means, value must not be within a range which means)
$p > 0$ OR $p < -3$

Title	Exponents and Logarithms
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	23/9/2018

**This topic assumes you already understand your Elementary Mathematics Concepts.

Applicable to the following courses/students.

- 'O' Level Additional Mathematics
- Nanyang Polytechnic – School of Chemical and Life Sciences – Mathematics for Life Sciences Module
- Nanyang Polytechnic – School of Engineering – Engineering Mathematics 1A Module
- Junior College/ A Levels – H1/H2 Mathematics
- Institute of Technical Education (ITE Colleges) – Technical Mathematics Module (Specific Courses and Levels Only)

Relationship Between Exponents and Logarithm as Follows	
An expression written in Exponential Form	$3^{12} = 531441$
If written as Logarithmic Format, it looks like this	$\log_3 531441 = 12$

Generally Speaking	
If	$a^b = c$ In this case, b is called the exponent or power a is called the base value.
It can be rewritten as	$\log_a c = b$ In this case b is known as the logarithm a is called the base value

Laws of Logarithm	
$\log_a(xy) = \log_a x + \log_a y$	
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	
$\log_a(x)^y = y\log_a(x)$	

$\log_a x = \frac{\log_b(x)}{\log_b(a)}$	Calculator Input [Calculator assumes the input is in base value is 10 by default] If $a^x = y$ You literally type in the following into the calculator to find x $\frac{\log(y)}{\log(a)} = x$

Relationship Between Logarithm and Exponents as follows
If $\boldsymbol{a^x = y}$
$x = \log_a(y)$

If $\boldsymbol{\log_a x = y}$
$x = a^y$

If you are thinking about calculator input, the following property will be useful if needed. (As said before, calculator assumes the $\boldsymbol{log(x)}$, as logarithm of which base is 10.)
$a^x = y$

$$x = \frac{\log(y)}{\log(a)}$$

$\log(x) = y$
$x = 10^{\log(x)} = 10^y$

Motivations for introducing logarithm and modern use of logarithm includes the following

- Historical use of logarithm includes multiplication of large numbers, which can be converted to adding of their logarithm under a certain base value and getting the final value
- Modern use of logarithm includes, astronomy (studying brightness of stars), computation (performing data encryption) and more.

Introducing concept of base e, also known as Euler's Number.

- e is approximately 2.71828 when rounded of to 6 significant figures.
- e can be approximated using the formula as shown below

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \dots \dots \frac{1}{n!}$$

$$n! = n(n-1)(n-2)(n-3) \dots \dots (3)(2)(1)$$

- The approximation gets better as more terms are added.
- Used in modelling continuous growth or continuous decay in Mathematics and Sciences.
- Modern scientific calculators have this functionality, just locate button e^x or e on your scientific calculator.

Introducing Natural Logarithm ($\ln(x)$)

- Natural Logarithm assumes that the base of a certain exponent is exactly e
- It finds the logarithm where the value has base value is e

The relationship between $\ln(x)$ and e^x goes as follows
If, $e^x = y$ and you are being asked to find the value of x
$\ln(e^x) = \ln(y)$
$x = \ln(y)$

If, $\ln(x) = y$ and you are being asked to find the value of x
$e^{\ln(x)} = e^y$
$x = e^y$

(For students studying higher level Mathematics, your lecturers may tell you Logarithm an inverse function of Exponent and vice-versa is true as well, this is because, inverse function sends a function back to where it originally started.)

Solving equations involving Logarithm and Exponents
(The following questions are provided by my friend in Nanyang Polytechnic who is studying Electrical Engineering related courses, he explained they are taken from his Engineering Mathematics Notes.)
If questions specify $\ln(x)$ or something similar, the base is exactly e. If question specify $\log(x)$ without base value information, the base value is to be assumed to be 10.

Example 1.

(a) $2^x = 16$
(b) $5^x = 0.3$
(c) $e^{-x} = 17.54$

(d) $\log(x) = 2.3$
(e) $\log(2x) + \log(x) = 9$

Solutions

1(a)

$2^x = 16$

$\log_2 16 = x = 4$

1(b)

$5^x = 0.3$

$\log_{0.3} 5 = -0.748$

1(c)

$e^{-x} = 17.54$

$\ln(e^{-x}) = \ln(17.54)$

$-x = \ln(17.54) = 2.86$

$x = -2.86$

1(d)

$\log(x) = 2.3$

$10^{\log(x)} = 10^{2.3}$

$x = 10^{2.3} = 200$

1(e)

$\log(2x) + \log(x) = 9$

Applying law of logarithm, we get

$\log(2x) + \log(x) = \log(2x^2) = 9$

$10^{\log(2x^2)} = 2x^2 = 10^9$

$2x^2 = 10^9$

$$x^2 = \frac{10^9}{2}$$

$$x = \sqrt{\frac{10^9}{2}}$$

Example 2

(a) $2^{2x} - 8(2^x) + 15 = 0$
(b) $e^{2x} - 7e^x + 12 = 0$

(c) $e^{e^x} = 3$

Depending on the situation, you sometimes may have to reject values, you should always substitute back into the original equation or check with the original question if the values make sense in the question's context.

[I always use u to perform substitution, you can use any letter you like but make sure you don't get yourself confused afterwards.]

Solutions

2(a)

$$2^{2x} - 8(2^x) + 15 = 0$$

Substitute $u^x = 2^x$ (In this case, I pick the smallest power component within the equation to substitute.)

The equation effectively becomes the following after I substituted the component values.

$u^2 - 8u + 15 = 0$

Using Quadratic Formula, which I won't elaborate further on.

I get the following roots for u

$u = 3\ OR\ u = 5$

The roots can be written as the following values since $u = 2^x$ as defined earlier

$2^x = 3\ OR\ 2^x = 5$

$x = \log_2(3)\ \ OR\ x = \log_2(5)$

$x = 1.584962\ OR\ x = 2.321928$

2(b) $e^{2x} - 7e^{x} + 12 = 0$
Substitute $u = e^{x}$
Equation becomes the following
$u^2 - 7u + 12 = 0$

Using quadratic formula, we can get the following answers
$u = 3\ OR\ u = 4$

Which can be rewritten as
$e^{x} = 3\ OR\ e^{x} = 4$

Taking natural logarithm on both sides of each solution to arrive at value of x
$\ln(e^{x}) = \ln(3)\ \ OR \ln(e^{x}) = \ln(4)$
$x = \ln(3) = 1.098612\ OR\ x = \ln(4) = 1.386294$

2(c) $e^{e^{x}} = 3$
Take natural logarithm on both sides,
$\ln\left(e^{e^{x}}\right) = \ln(3)$

We get the following equation
$e^{x} = \ln(3)$

Take natural logarithm on both sides once again the get the following,
$\ln(e^{x}) = \ln(\ln(3))$
$x = \ln(\ln(3))$
$x = 0.094048$

Title	Binomial Expansion and Binomial Theorem
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	1/6/2018

Our learning objective

- To learn how to read the notation of binomial expansion
- To learn how to use binomial theorem for various calculations

Application of Binomial Expansion includes the following:

Serve as a good foundation for studying

- H2 Mathematics – Statistics
- Engineering – Engineering Mathematics (Statistics)
- Mathematics for Life Sciences – Biostatistics
- Information Technology – Computing Mathematics (Statistics)
- Business – Business Statistics

As Additional Mathematics itself isn't a compulsory subject or prerequisite for Polytechnic courses, those going for polytechnic can still study statistics module at higher level and the below will be mentioned in your lecture. However, those going for Junior College, it is important to take this seriously as the Mathematics in H2 is considerably harder than the Polytechnic's Statistical Analysis module, without such foundation, catching up is rather difficult if not impossible.

While the notation for binomial expansion varies in different textbooks, they literally, mean the same thing. I will try my best to explain the various ways binomial expansion formula is being written and explain how they correlate the each other.

Standard Universal Binomial Expansion Notation

$(a+b)^n$	Multiply $(a+b)$ by itself n number of times
$\binom{n}{r}$	From n number of objects, you choose r of them. The value $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, is the number of combinations or the number of ways you can arrange r objects, assuming the order doesn't matter. To calculate $\binom{n}{r}$ on your scientific calculator, you press the value of n, followed by "C", followed by value of r (Ask teachers if in doubt) *** Various online resources may write this notation in a different way. Notations like $n\ C\ r$ mean the same thing, it meant from n number of objects, choose r of them and calculate the number of combinations.
$n!$	Known as n factorial. Multiply the integer value of $n, (n-1), (n-2)$ all the way to $(3), (2)(1)$ Note: n must be an integer and $n \geq 0$ ($n < 0$ is undefined) Example: $5! = 5 \times 4 \times 3 \times 2 \times 1$
	If you get an error message while computing factorials in your calculator, check the following and correct them accordingly. (Casio FX-95 SG Plus and Casio FX-96 GS PLUS Calculator) **Syntax Error** ➢ Check if you keyed in something wrongly into the calculator, anything from, missing values, forget to close bracket, etc. **Math Error** ➢ Check if you accidentally keyed in negative numbers, followed by the factorial button. (Factorials of negative numbers are undefined) ➢ Check if you accidentally keyed in a value with decimal places, followed by the factorial button (Factorials of decimal digits are invalid) ➢ Check if you accidentally keyed in a value ≥ 70, followed by the factorial button (Calculators are unable to compute Factorials of 70 or greater)

Local Standard Notation

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 \ldots \binom{n}{r} a^{n-r} b^r \ldots + b^n$$

Expression to be evaluated

1st term

2nd term

3rd term

$(r+1)th$ term

(n) th term

Binomial Expansion General Term Formula as Follows
$$\boldsymbol{T_{r+1}} = \binom{n}{r} a^{n-r} b^r$$

Examples as Follows

Example 1:
Given the following binomial expansion, write down the first 3 terms of the expansion, in the simplest form possible.

$(x+5)^{12}$

Approaching the question using General Term Formula:

$$T_1 = \binom{12}{0} x^{12}(5^0) = x^{12}$$

$$T_2 = \binom{12}{1} x^{12-1}(5^1) = 60x^{11}$$

$$T_3 = \binom{12}{2} x^{12-2}(5^2) = 66(25)x^{10} = 1650x^{10}$$

First three terms as written as below is:
$x^{12}, 60x^{11}, 1650x^{10}$

Example 2:
Expand $(4p+q)^5$ and write down only the first 4 terms of the expansion in descending powers of p.

$(4p+q)^5 =$

$\binom{5}{0}4p^5 + \binom{5}{1}(4p)^4(q^1) + \binom{5}{2}(4p)^3(q^2) + \binom{5}{3}(4p)^2(q^3) \ldots =$

After some calculator input, we get the following 4 terms in descending powers of p

$1024p^5 + 1280p^4q + 640p^3q^2 + 160p^2q^3 \ldots$

Example 3:
Expand $(3-x)^8$ and write down only the first 4 terms of the expansion in ascending powers of x and the term independent of x.

$(3-x)^8 = \binom{8}{0}3^8 + \binom{8}{1}3^7(-x)^1 + \binom{8}{2}3^6(-x)^2 - \binom{8}{3}3^5(-x)^3 \ldots =$

After some calculator input, we get the following 4 terms

$6561 - 17496x + 20412x^2 - 13608x^3 \ldots$

Since the question also demanded that we should find the term independent of x, we should also explain what it means as well, by breaking it down.
In the case of the above binomial expansion, the phrase "term independent of x" means a term that doesn't include the letter x at all, and in this case, we are referring to the constant term which is 6561

Title	Coordinate Geometry of Circles
Date	28/11/2023
Note	Questions all taken from Additional Mathematics Textbook

Warnings and Assumptions: This article assumes that you already have a good foundation in the following:

- Basic Coordinate Geometry (Such as the ability to use formula to calculate the length of a line given coordinate of both ends of the line)
- Use and understand substitution method to find solutions of simultaneous equations in Elementary Mathematics.
- Able to solve quadratic equations, use completing the square method extensively, understand how to solve quadratic inequalities efficiently and understand how discriminant affect the behavior of a quadratic function

Standard Form of a Circle
$(x-a)^2+(y-b)^2=r^2$ Coordinates $C(a,b)$ is located at the center of the circle. r refers to the radius of the circle
General Form of a Circle
$x^2+y^2+2gx+2fy+c=0$ In this case, the coordinates of the center of the circle are $C(-g,-f)$, the radius of the circle will be $r=\sqrt{g^2+f^2-c}$
Note: Make sure the coefficient of both x^2 and y^2 are 1 before proceeding, if the coefficient is not 1, you need to divide the whole equation by the coefficient until you get 1 as the coefficient of both x^2 and y^2.

Conditions Relating to Points Being on the circle, inside the circle or outside of the circle (Let d be the distance between the center of the circle and the point in question)	
Condition	Interpretation & Consequences
$d>r$	Point is outside of the circle
$d=r$	Point is on the circle
$d<r$	Point is inside the circle

Given the following equation of circles, derive the coordinates of the center of the circle along with the radius of the circle.

1(a). $(x-4)^2+(y+5)^2=36$
1(b). $3(x-7)^2+3y^2-48=0$

1(a) Since the equation of the circle is already written in the standard form, we can deduce the center of the circle is $C(4,-5)$ and the radius $r=6$ units

1(b).
Step 1: Divide the entire equation by 3
Which would get you $(x-7)^2+y^2-\frac{48}{3}=0$
$(x-7)^2+(y-0)^2=16$
Coordinates of center of the circle = $C(7{,}0)$ and the radius $r=4$ units

Given the following equation of circles, derive the coordinates of the center of the circle along with the radius of the circle.

2(a). $x^2+y^2+4x-6y-3=0$
Method 1: Comparing with Generic Equation and Apply Formula Directly
Comparing the above equation with the generic equation for a circle,
$x^2+y^2+2gx+2fy+c=0$

Therefore, $2gx=4x$ and $2fy=6y$
$g=-2$ and $f=3$
Coordinates of center of the circle is $C(-2{,}3)$

Radius of circle = $\sqrt{g^2+f^2-c}=\sqrt{(-2)^2+3^2-(-3)}=\sqrt{16}$
Radius of circle = $r=4$

Method 2: Complete the Square
$x^2+y^2+4x-6y-3=0$
Step 1. Rearrange the equation in a way it looks like the following
$x^2+4x+y^2-6y=3$

Step 2. Complete the square for both x^2 and y^2

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 + y^2 - 6y + \left(\frac{6}{2}\right)^2 = 3 + \left(\frac{4}{2}\right)^2 + \left(\frac{6}{2}\right)^2$$

$$(x + 2)^2 + (y - 3)^2 = 16$$

Comparing with the standard form of a circle which is
$(x - a)^2 + (y - b)^2 = r^2$

Center $C(-2,3)$, $r = 4$

3(a)

(i) Find the equation of the circle which passes through point (1,3) and has its center at $C(2,5)$.

(ii) Determine whether $A(3,4)$ and $B(0,6)$ lies outside, inside or on the circle.

(i) Set (x_1, y_1) as the point at center and set (x_2, y_2) as the point which the circle passes through

Distance from center to the point which circle pass through is precisely the radius of the circle, therefore, we can use the formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to deduce the radius of the circle.

$r = \sqrt{(1-2)^2 + (3-5)^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$
$r^2 = 5$

Center of Circle = (2,5), therefore the equation for the circle in question is
$(x - 2)^2 + (y - 5)^2 = 5$

After expanding the terms, we get the following
$x^2 - 2(2x) + 4 + y^2 - 2(5y) + 25 = 5$
$x^2 - 4x + 4 + y^2 - 10y + 25 = 5$
$x^2 + y^2 - 4x - 10y + 24 = 0$

(ii) Length of Line AC is $d = \sqrt{(3-2)^2 + (4-5)^2} = \sqrt{2}$
Since $\sqrt{2} < \sqrt{5}$, Point A lies inside the circle

Length of Line BC is d = $\sqrt{(0-2)^2 + (6-5)^2} = \sqrt{4+1} = \sqrt{5}$
Since $\sqrt{5} = \sqrt{5}$ Point B Lies on the circle.

Solving simultaneous equations involving a straight-line intersecting a circle.

4. Solve the following simultaneous equation.
$x - 4y + 16 = 0$ (Equation 1)
$x^2 + y^2 - 4x - 6y + 5 = 0$ (Equation 2)

$x = 4y - 16$ (Equation 1A)

Substitute Equation 1A into Equation 2

$(4y - 16)^2 + y^2 - 4x - 6y + 5 = 0$
$16y^2 - 2(16)(4y) + 16^2 \mp y^2 - 4(4y - 16) - 6y + 5 = 0$
$16y^2 - 128y + 256 + y^2 - 16y + 64 - 6y + 5 = 0$
$17y^2 - 150y + 325 = 0$

Using quadratic formula, we get the following values for y

$y = \frac{65}{17}$ OR $y = 5$

Substitute both values into Equation 1A, we get the following values for x

$x = 4\left(\frac{65}{17}\right) - 16$ OR $x = 4(5) - 16$

$x = -\frac{12}{17}$ OR $x = 4$

5. Finding a range of value for which a line intersect, is a tangent to or doesn't intersect a circle.

The line $y = 2x + k$, where k is a constant, is a tangent to the circle $x^2 + y^2 + 4x - 6y + 8 = 0$. Find the possible values of k and the corresponding points of contact for each value of k.

Equation 1: $y = 2x + k$
Equation 2: $x^2 + y^2 + 4x - 6y + 8 = 0$

Substitute Equation 1 into Equation 2
$x^2 + (2x + k)^2 + 4x - 6(2x + k) + 8 = 0$
$x^2 + 4x^2 + 2(2x)(k) + k^2 + 4x - 12x - 6k + 8 = 0$

Rearrange the equation and we get the following
$5x^2 + 4kx + k^2 + 4x - 12 - 6k + 8 = 0$
$5x^2 + 4kx - 8x + k^2 - 6k + 8 = 0$

Note that the general form of a quadratic equation is $ax^2 + bx + c = 0$

$5x^2 + (4k - 8)x + (k^2 - 6k + 8) = 0$

Since the question demands a set of lines that is tangent to the circle, we can use the rule of discriminant we learnt previously in the quadratic equation topic where a line tangent to a circle would also produce a discriminant of $b^2 - 4ac = 0$

$b^2 - 4ac = (4k - 8)^2 - 4(5)(k^2 - 6k + 8)$
$= 16k^2 - 2(4)(8)k + 64 - 20(k^2 - 6k + 8)$
$= 16k^2 - 64k + 64 - 20k^2 + 120k - 160$
$= -4k^2 + 56k - 96$

$-4k^2 + 56k - 96 = 0$

Using the quadratic formula, we get the following values for k

$k = 2$ OR $k = 12$

Equation for both lines that are tangent to the circle are therefore
$y = 2x + 12$ and $y = 2x + 2$

Title	Trigonometric Identities & Formula
Author	
Date	12/9/2023

Assumptions: This article assumes you have already studied basic trigonometry in Elementary Mathematics.

Converting Between Degrees and Radians
Degree → Radian: Multiply Angle by $\frac{\pi}{180}$ Radian → Degree: Multiply Angle by $\frac{180}{\pi}$

Let r be the hypotenuse side of the right-angled triangle, x be the adjacent side and y be the opposite side.

Basic Trigonometric Functions and Ratio	Reciprocal Trigonometric Functions
$\sin(\theta) = \frac{Opposite}{Hypotenuse} = \frac{y}{r}$ $\cos(\theta) = \frac{Adjacent}{Hypotenuse} = \frac{x}{r}$ $\tan(\theta) = \frac{Opposite}{Adjacent} = \frac{y}{x}$	$cosec(\theta) = \frac{1}{\sin(\theta)} = \frac{r}{y}$ $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{r}{x}$ $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y}$

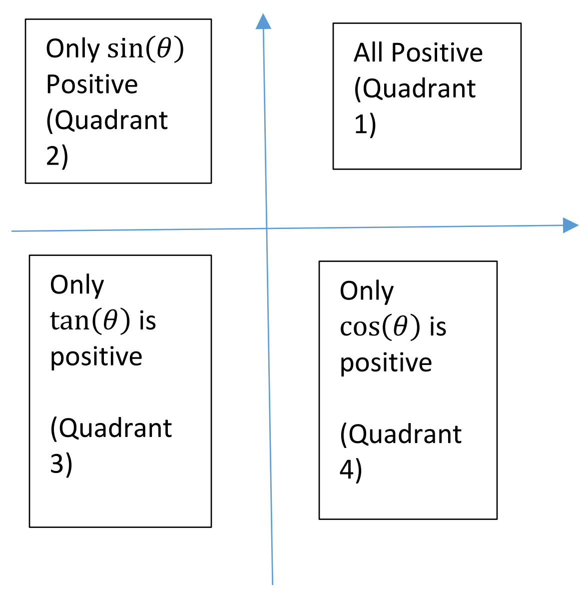
Trigonometric Identities Involving division of one trigonometric function by another function
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities
$\sin^2(\theta) + \cos^2(\theta) = 1$
$\tan^2(\theta) + 1 = \sec^2(\theta)$
$\cot^2(\theta) + 1 = cosec^2\,(\theta)$

Compound Angle Formulae
$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
$\tan(A \pm B) = \dfrac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$

Double Angle Formulae
$\sin(2A) = 2\sin(A)\cos(A)$
$\cos(2A) = 1 - 2\sin^2(A)$
$\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

R-Formulae
$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$
$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$
$R = \sqrt{a^2 + b^2}$ $\alpha = \tan^{-1}\frac{b}{a}$ where $a, b > 0$ α is an acute angle



The above diagram is also called the ASTC Diagram.

Concept of Reference Angle
Reference angle refers to trigonometric functional value of any arbitrary angle expressed as an acute angle, The reference angle of a given angle θ is the positive acute angle between the x-axis and the terminal side of θ.

Solutions involving trigonometric functions can be found using the following steps:
1. Use ASTC diagram to determine the quadrant in which angle θ is in
2. Find the reference angle θ_{ref}
3. The value of θ is as follows:

Quadrant	In Degrees	In Radians
1	$\theta = \theta_{ref}$	$\theta = \theta_{ref}$
2	$\theta = 180° - \theta_{ref}$	$\theta = \pi - \theta_{ref}$
3	$\theta = 180° + \theta_{ref}$	$\theta = \pi + \theta_{ref}$
4	$\theta = 360° - \theta_{ref}$	$\theta = 2\pi - \theta_{ref}$

Typical approach to dealing with trigonometric equations.
Step 1. (If necessary) Rewrite the equation so to ensure the trigonometric equation is expressed in terms of a single trigonometric functions using trigonometric identities.

Step 2. (If necessary): If there is a need, substitute the trigonometric function using a letter and obtain the value of trigonometric functions through algebraic methods. i.e. (Quadratic Formula)

Step 3. Use inverse trigonometric function to obtain value of θ after all trigonometric functions are combined together into a single function and use ASTC diagram to determine other possible value of θ.
Question 1.
Solve $2\sin^2 x + \sin x - 1 = 0$ for $0° \leq x \leq 360°$

Substitute $u = \sin x$

And we get the following equation

$$2u^2 + u - 1 = 0$$

Using quadratic formula, which we will not elaborate here,
We get the following

$u = \frac{1}{2}$ OR $u = -1$
$x = 30°$ or $x = -90°$

Using ASTC, we apply the formula for finding angle in quadrant 2

$x = 180° - 30 = 150$ or $x = 180 - (-90) = 270°$

Question 2.
Solve $2\cos^2 x + \cos x = 3$ for between $0°$ to $360°$ (both inclusive)

Substitute $u = \cos x$
$2u^2 + u = 3$
$2u^2 + u - 3 = 0$

$u = 1$ Or $u = -1.5$ (Rejected, invalid input)

$x = 0°$

Using ASTC, we deduce cosine is positive at the 4^{th} quadrant and therefore
$x = 360° - 0° = 360°$

Question 3.
$\sec^2 x + 13 = 9\tan x$

Using identity: $1 + \tan^2 x = \sec^2 x$

$1 + \tan^2 x + 13 = 9\tan x$
$14 + \tan^2 x = 9\tan x$

Rearranging we get the following
$\tan^2 x - 9\tan x + 14 = 0.$

Substitute $u = \tan x$
$u^2 - 9u + 14 = 0$
$u = 7$ or $u = 2$

$x = 81.8699°$ or $x = 63.4349°$

Since $\tan\theta$ is positive on the 3rd quadrant, we apply the formula for quadrant 3 and get the following:
$x = 180° + 81.8699° = 261.8699°$ OR $x = 180 + 63.435° = 243.4349°$

Question 4
Solve $12\cos^2 x + 5\sin x - 10 = 0$ for $0° \leq x \leq 360°$

Using identity: $\sin^2\theta + \cos^2\theta = 1$
The above identity can be rewritten as: $1 - \sin^2\theta = \cos^2\theta$

Substitute the rewritten identity into the equation above, we get:
$12(1 - \sin^2 x) + 5\sin x - 10 = 0$
$12 - 12\sin^2 x + 5\sin x - 10 = 0$
$2 - 12\sin^2 x + 5\sin x = 0$
$-12\sin^2 x + 5\sin x + 2 = 0$

Substitute $u = \sin x$ to get the following equation
$-12u^2 + 5u + 2 = 0$
$u = \frac{2}{3}\ OR\ u = -\frac{1}{4}$

$x = 41.8103°$ OR $x = -14.4775°$

$x = -14.478$ is located at the fourth quadrant and is to be rewritten as $360 - 14.4775° = 345.5225°$

Since $\sin\theta$ is positive on the 2nd quadrant, we use the formula for the second quadrant to deduce the value of x

$x = 180° - 41.810° = 138.190°$

$x = 180° - (-14.478°) = 194.478°$

Proving of trigonometric identities
In this section, we will be talking about how to prove a trigonometric identity.
Typically, students will be given an identity and as a student, you have to prove the left-hand-side of a trigonometric identity is exactly equal to the right-hand-side of that same identity, while the idea is rather simple, there is no standard procedures that applies to all and each trigonometric identity is to be dealt with on a case-by-case basis.

Question 5

Prove that the following trigonometric identity is true

$\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$

Using identity: $1 + \tan^2 \theta = \sec^2 \theta$ and substituting in the left-hand side of the identity, we get the following:

$(1 + \tan^2 A)^2 - (1 + \tan^2 A) = \tan^4 A + \tan^2 A$

$(1^2 + 2(1)\tan^2 A + \tan^4 A) - (1 + \tan^2 A) = \tan^4 A + \tan^2 A$

$1 + 2\tan^2 A + \tan^4 A - 1 - \tan^2 A = \tan^4 A + \tan^2 A$

$2\tan^2 A - \tan^2 A + \tan^4 A = \tan^4 A + \tan^2 A$

$\tan^2 A + \tan^4 A = \tan^4 A + \tan^2 A$

(Since LHS = RHS, the identity is proven)

Question 6

Prove the following trigonometric identity

$$1 + \frac{1}{\tan^2 \theta} = cosec^2\theta$$

Using identity: $1 + \cot^2 \theta = cosec^2\theta$

To be rewritten as $1 + \frac{1}{\tan^2 \theta} = cosec^2\theta$

$$1 + \frac{1}{(\tan^2 \theta)} = 1 + \frac{1}{(\tan^2 \theta)}$$

Since LHS = RHS, the identity is proven

Use of Compound Angle Formula

Question 7

It is given that A and B are acute angles such that $\tan A = \frac{3}{4}$ and $\tan B = \frac{1}{7}$. Without using a calculator, find the values of angle $A + B$.

Using formula as follows:

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

Substituting $\tan A$ *and* $\tan B$ with specified values in the question, we get the following

$$\tan(A + B) = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}\left(\frac{1}{7}\right)} = 1$$

$A + B = 45°$

Question 8

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$ and that A and B are in the first quadrant.
Find, without solving for the angles

(a) $\sin(A + B)$ (b) $\cos(A + B)$

8(a)
Using formula
$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$ while we substitute the values in the formula with values mentioned in the question, we get the following:

$$\sin(A + B) = \frac{3}{5}\left(\frac{12}{13}\right) + \cos A \sin B$$

To find the value of $\cos A$, we can apply Pythagoras Theorem and obtain the following value, which is $\frac{4}{5}$

To find the value of $\sin B$, we can apply Pythagoras Theorem and obtain the following value, which is $\frac{5}{13}$

Substituting the above value into $\cos A \ \sin B$, we get the following

$$\sin(A + B) = \frac{3}{5}\left(\frac{12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right) = \frac{56}{65}$$

8(b)
Using formula
$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

$$\cos(A + B) = \frac{4}{5}\left(\frac{12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right) = \frac{33}{65}$$

Question 9

If $\sin A = \frac{7}{25}$ and $\cos B = -\frac{3}{5}$, where A and B are obtuse angles, find the value of $\sin(A+B)$

Illustration of A, as obtuse angle in second quadrant

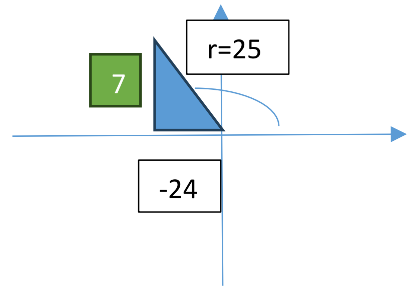
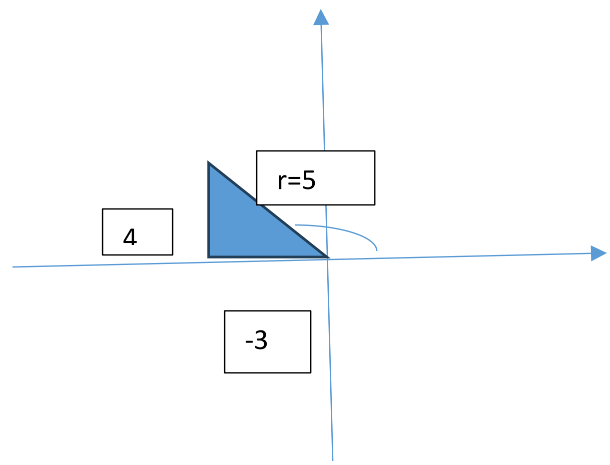


Illustration of B, as obtuse angle in second quadrant



Using formula

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\sin A = \frac{7}{25}$$

$$\cos B = -\frac{3}{5}$$

$$\cos A = -\frac{24}{25}$$

$$\sin B = \frac{4}{5}$$

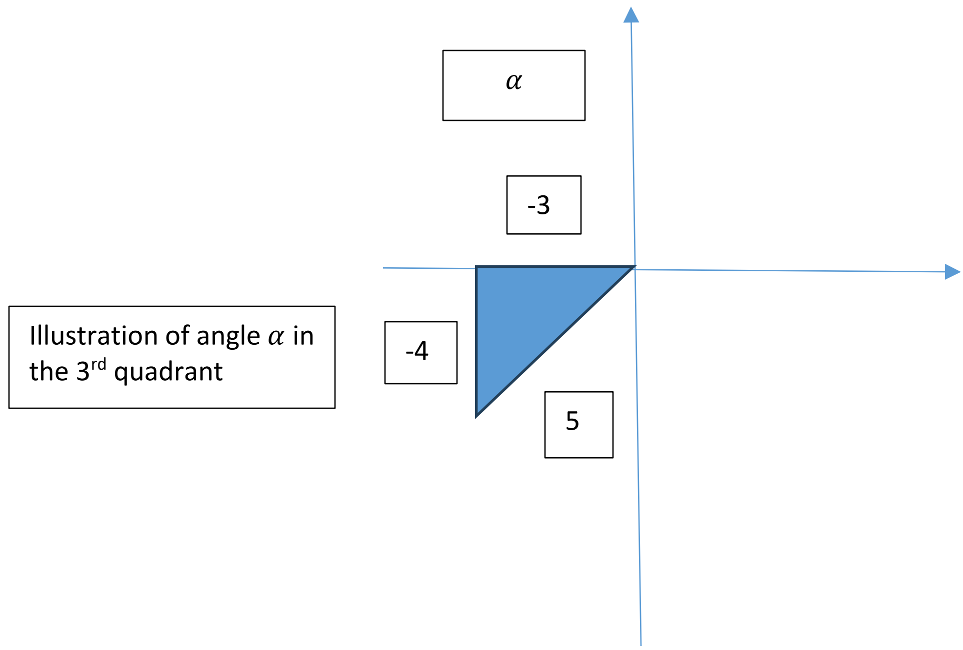
Multiply according to the formula, we get the following

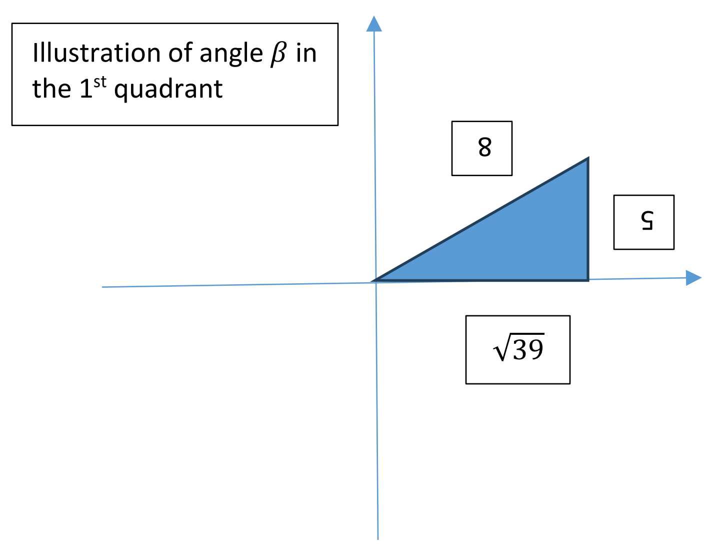
$$\frac{7}{25}\left(-\frac{3}{5}\right) + \left(-\frac{24}{25}\right)\left(\frac{4}{5}\right) = -\frac{117}{125}$$

Question 10

If $\cos\alpha = -\frac{3}{5}$ and $\sin\beta = \frac{5}{8}$ and α is a third quadrant angle and β is a first quadrant angle. Find, without solving for the angles:

(a) $\sin(\alpha - \beta)$
(b) $\cos(\alpha + \beta)$
(c) $\tan(\alpha + \beta)$





10(a)
Using identity
$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
Which rewritten in terms of α and β looks like the following in context of this question
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \left(-\frac{4}{5}\right)\left(\frac{\sqrt{39}}{8}\right) - \left(-\frac{3}{5}\right)\left(\frac{5}{8}\right) = -0.249$$

10(b)
Using identity $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
Which rewritten in terms of α and β looks like the following in context of this question
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{3}{5}\right)\left(\frac{\sqrt{39}}{8}\right) - \left(-\frac{4}{5}\right)\left(\frac{5}{8}\right) = -0.0316$$

10(c)
Using identity

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

Which rewritten in terms of α and β looks like the following in context of this question

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

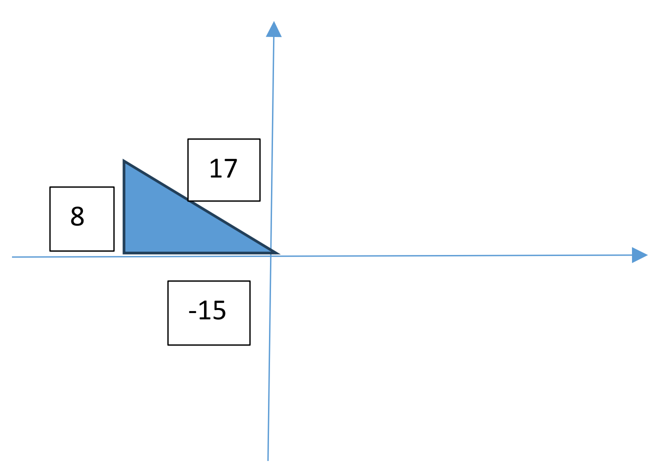
$$= \frac{\left(\frac{-4}{-3}\right) + \left(\frac{5}{\sqrt{39}}\right)}{1 - \left(\frac{-4}{-3}\right)\left(\frac{5}{\sqrt{39}}\right)} = -31.6$$

Use of double angle formula
Question 11

If $sin\ \theta = \frac{8}{17}$ and θ is an obtuse angle, find the values of the following using the double angle formula.

(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$

θ is an obtuse angle and therefore shown in the diagram sketch on the right to be in quadrant 2.



11(a)

Using identity $\sin(2A) = 2\sin(A)\cos(A)$

$$\sin 2\theta = 2\sin\theta\, cos\theta$$

$$= 2\left(\frac{8}{17}\right)\left(-\frac{15}{17}\right) = -\frac{240}{289}$$

11(b)

Using identity $\cos(2A) = 1 - 2\sin^2(A)$

$\cos 2\theta = 1 - 2\sin^2\theta$

$$1 - 2\left(\frac{8}{17}\right)^2 = \frac{161}{289}$$

11(c)

Using identity

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2(\theta)}$$

$$= \frac{2\left(\frac{8}{-15}\right)}{1 - \left(\frac{8}{-15}\right)^2}$$

$$= -\frac{240}{161}$$

Use of R-Formula
Question 12
Solve $7\cos x - 2\sin x = 1$ for between $0°$ and $360°$
Using identity $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$

$R = \sqrt{7^2 + 2^2} = \sqrt{53}$ $\quad\tan\alpha = \frac{2}{7}$
$\alpha = \tan^{-1}\left(\frac{2}{7}\right) = 15.9454$

$7\cos x - 2\sin x = 1$
$\sqrt{53}\cos(x + 15.9454°) = 1$
$\cos(x + 15.9454°) = \dfrac{1}{\sqrt{53}}$

Condition: $0° \leq x \leq 360°$

$\therefore 15.9454° \leq x \leq 375.9454°$

Since $\cos(x) > 0$, it exists in quadrant 1 as well as quadrant 4.

Let $\cos\alpha = \frac{1}{\sqrt{53}}$

$\alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{53}}\right) = 82.1049°$

$x + \tan^{-1}\left(\frac{2}{7}\right) = \cos^{-1}\left(\frac{1}{\sqrt{53}}\right)$ OR $x + \tan^{-1}\left(\frac{2}{7}\right) = 360 - \cos^{-1}\left(\frac{1}{\sqrt{53}}\right)$

$x + 15.9454° = 82.1049$ OR $x + 15.9454° = 277.8951°$

$x = 66.1595°$ OR $x = 261.9497°$

Title	Differentiation of Algebraic Functions
Editor	
Date	1/4/2023

This topic is about basic differentiation on algebraic functions specifically. The following concepts are covered in this topic:

Basic Overview of Differentiation

- Reason to study this topic
- Notation guide

Basic Rules of Differentiation on Algebraic Functions

- Power Rule
- Sum Rule
- Difference Rule
- Product Rule
- Quotient Rule
- Chain Rule

Brief Overview of Differentiation (And Calculus in General)

Without going into too much detail about the history of calculus, calculus is a branch of Mathematics introduced to study about change. Differentiation is a sub-branch that deals with the gradient of a given function at every point on a curve.

Calculus is very important in the following field for the following reasons

- Computing – To study time complexity of programming algorithms and do various data analytics and estimation
- Physics – Majority of physical laws are derived with help of calculus (Newton used calculus to research how physics works and used that in his later works)
- Engineering – Engineering is closely related to physics as Engineering students study physics as their basic module before they carry on with intermediate modules
- Chemistry – Study speed of chemical reactions

And the list can go on and on

Notation	Meaning	Explanation	Notes
$F(x)$	Function of x		I also use $f(x), g(x)$ in certain situations
$F'(x)$	First Derivative of $f(x)$		I also use $f'(x)$, $g'(x)$ in certain situations
$F''(x)$	Second Derivative of $f(x)$	First derivative of $f'(x)$	I also use $f''(x), g''(x)$ in certain situations
$\frac{dy}{dx}$	First derivative of y with respect to x		
$\frac{d^2y}{dx^2}$	Second derivative of y with respect to x	First derivative of $\frac{dy}{dx}$	

Follow the question with regards to notation matters:

If the question mentions a function in the following form:

$$f(x) = ax^n + ax^{n-1} \dots + c$$

When answering the question, use $f'(x)$ for first derivative and $f''(x)$ for second derivative.

If the question mentions an equation in the following form:

$$y = ax^n + ax^{n-1} \dots + c$$

When answering the question use $\frac{dy}{dx}$ for first derivative and $\frac{d^2y}{dx^2}$ for second derivative

Power Rule
Given Function $f(x) = ax^n$
The derivative of the function is $f'(x) = anx^{n-1}$

Question 1.
Differentiate the following with respect to x

(a) $y = x^5$
(b) $y = \frac{1}{x^3}$
(c) $y = 3x^4$
(d) $y = \pi$

1(a)
$y = x^5$
Differentiate using power rule $\frac{dy}{dx} x^5 = 5x^4$

1(b)
$$y = \frac{1}{x^3}$$

Convert to index notation: $y = x^{-3}$
Differentiate using power rule: $\frac{dy}{dx}\ x^{-3} = \ -3x^{-4}$

1(c)
$y = 3x^4$
Differentiate using power rule: $\frac{dy}{dx}\ 3x^4 = 12x^3$

1(d)
$y = \pi$
Derivative of any constant value is 0: $\frac{dy}{dx}\ \pi = 0$

Sum Rule

Given Function $F(x) = f(x) + g(x)$

The derivative is $F'(x) = f'(x) + g'(x)$

I know in notation form it is rather complicated so I am going to explain it in relatively simple terms.

Imagine you have the following function

$f(x) = ax^n + bx^m$

To find the derivative, you must first split the function into two sections:

First Section	Second Section
ax^n	bx^m

Differentiate both separately to get

Derivative of first section	Derivative of second section
anx^{n-1}	bmx^{m-1}

Combine them back together to get your "overall" derivative

$f'(x) = anx^{n-1} + bmx^{m-1}$

If you still don't get it, take note of the following examples, often seeing how a real question work out clears most of your doubts.

Question 2

Differentiate the following with respect to x

(a) $y = 6x^7 + 2x^3$

(b) $y = 15x^2 + 4x^{-2} + \frac{1}{x}$

2(a)

Split the equation into two sections:

$$\frac{dy}{dx} 6x^7 + 2x^3 = \frac{d}{dx} 6x^7 + \frac{d}{dx} 2x^3$$

Differentiate the components one by one to get

$\frac{d}{dx} 6x^7 = 42x^6$	$\frac{d}{dx} 2x^3 = 6x^2$

Combine them back together to get the following

$$\frac{dy}{dx} = 42x^6 + 6x^2$$

2(b)

Convert all to index notation

$15x^2 + 4x^{-2} + x^{-1}$

Split equation into three sections

$$\frac{dy}{dx} 15x^2 + 4x^{-2} + x^{-1} = \frac{d}{dx} 15x^2 + \frac{d}{dx} 4x^{-2} + \frac{d}{dx} x^{-1}$$

Differentiate the components one by one to get

$\frac{d}{dx} 15x^2 = 30x$	$\frac{d}{dx} 4x^{-2} = -8x^{-3}$	$\frac{d}{dx} x^{-1} = -1(x)^{-2} = -x^{-2}$

Combine them to get

$$\frac{dy}{dx} = 30x - 8x^{-3} - x^{-2}$$

Difference Rule
Given Function $F(x) = f(x) - g(x)$
The derivative is $F'(x) = f'(x) - g'(x)$

Imagine the following function:
$$f(x) = ax^n - bx^m$$
To find the derivative split the function into 2 sections

First Section	Second Section
ax^n	$-bx^m$

Differentiate both section separately to get

anx^{n-1}	$-bmx^{m-1}$

Combine them back together to get your "overall" derivative

$f'(x) = anx^{n-1} - bmx^{m-1}$

Question 3:
Differentiate the following with respect to x
(a) $y = 5x^7 - 2x^3 - 7$
(b) $y = 3x^4 - 4x^2 + 5x^{-4}$

3(a)
Split equation into 3 sections to get
$\frac{dy}{dx}5x^7 - 2x^3 - 7 = \frac{d}{dx}5x^7 - \frac{d}{dx}2x^3 - \frac{d}{dx}7$

Differentiate the components one by one to get

$\frac{d}{dx}5x^7 = 35x^6$	$\frac{d}{dx} - 2x^3 = -6x^2$	$\frac{d}{dx}7 = 0$

Combine them back and you should get the following
$\frac{dy}{dx} = 35x^6 - 6x^2$

3(b)

Split equation into 3 sections to get

$\frac{dy}{dx} 3x^4 - 4x^2 + 5x^{-4} = \frac{d}{dx} 3x^4 - \frac{d}{dx} 4x^2 + \frac{d}{dx} 5x^{-4}$

Differentiate components one by one to get

$\frac{d}{dx} 3x^4 = 12x^3$	$\frac{d}{dx} - 4x^2 = -8x$	$\frac{d}{dx} 5x^{-4} = -20x^{-5}$

Combine them back to get the following:

$$\frac{dy}{dx} = 12x^3 - 8x - 20x^{-5}$$

Product Rule
Given Function $F(x) = f(x) \cdot g(x)$
The derivative is $F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Imagine the following function
$f(x) = (ax^n)(bx^m)$
$f'(x) = anx^{n-1}(bx^m) + ax^n(bmx^{m-1})$

Question 4
Differentiate the following with respect to x

(a) $y = (x^2 + 15)(6x^4 + 9)$
(b) $y = (5x^9 + 6x - 5)(2x^2 + 6)$

4(a)
$y = (x^2 + 15)(6x^4 + 9)$
$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 15) \cdot (6x^4 + 9) + \frac{d}{dx}(6x^4 + 9) \cdot (x^2 + 15)$$

$$\frac{dy}{dx} = 2x(6x^4 + 9) + 24x^3(x^2 + 15)$$
$$= 12x^5 + 18x + 24x^5 + 360x^3$$
$$= 36x^5 + 360x^3 + 18x$$

4(b)
$y = (5x^9 + 6x - 5)(2x^2 + 6)$
$$\frac{dy}{dx} = \frac{d}{dx}(5x^9 + 6x - 5) \cdot (2x^2 + 6) + \frac{d}{dx}(2x^2 + 6) \cdot (5x^9 + 6x - 5)$$

$$\frac{dy}{dx} = (45x^8 + 6)(2x^2 + 6) + (4x)(5x^9 + 6x - 5)$$

$$= (90x^{10} + 12x^2 + 270x^8 + 36) + (20x^{10} + 24x^2 - 20x)$$
$$= 110x^{10} + 270x^8 + 36x^2 - 20x + 36$$

Quotient Rule

Given function $F(x) = \frac{f(x)}{g(x)}$

The derivative is $F'(x) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2}$

Imagine you have the following function

$f(x) = \frac{ax^n}{bx^n}$

$$f'(x) = \frac{anx^{n-1}\ (bx^n) - bnx^{n-1}(ax^n)}{(bx^n)^2}$$

Question 5

Differentiate the following with respect to x

(a) $y = \frac{2x^5+3x^2-7}{(2x+6)}$

$$y = \frac{2x^5 + 3x^2 - 7}{(2x + 6)}$$

$$\frac{dy}{dx} = \frac{\left[\frac{d}{dx}\left(2x^5+3x^2-7\right) \cdot (2x+6) - \frac{d}{dx}(2x+6) \cdot \left(2x^5+3x^2-7\right)\right]}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{\left[(10x^4+6x)(2x+6) - 2(2x^5+3x^2-7)\right]}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{20x^5+12x^2+60x^4+36x-4x^5-6x^2+14}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{16x^5+6x^2+60x^4+36x+14}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{16x^5+60x^4+6x^2+36x+14}{\left(2(x+3)\right)^2}$$

$$\frac{dy}{dx} = \frac{16x^5+60x^4+6x^2+36x+14}{4(x+3)^2}$$

$$\frac{dy}{dx} = \frac{8x^5+30x^4+3x^2+18x+7}{2(x+3)^2}$$

Chain Rule
Given function
$F(x) = [f(x) + g(x)]^c$ {where c is a constant.}

The derivative is
$F'(x) = c[f(x) + g(x)]^{c-1} \cdot [f'(x) + g'(x)]$

Imagine you have the following function
$f(x) = (ax^n + bx^m)^c$
The derivative is
$f'(x) = c(ax^n + bx^m)^{c-1}\ [anx^{n-1} + bmx^{n-1}]$

Question 6
Differentiate the following with respect to x
(a) $y = (2x^3 - 5x^2 + 2x)^7$

$y = (2x^3 - 5x^2 + 2x)^7$

$\frac{dy}{dx} = 7(2x^3 - 5x^2 + 2x)^{7-1} \cdot (6x^2 - 10x + 2)$

$\frac{dy}{dx} = 7(2x^3 - 5x^2 + 2x)^6\ \ (6x^2 - 10x + 2)$

Tips and Tricks Below
Question 7
Differentiate the following with respect to x

$$y = \frac{1}{(x+9)^5}$$

Question: Should I use chain rule or quotient rule for this question?
Answer: Chain rule, it is easier to use chain rule as demonstrated below.

Reason Below:
By Chain Rule (3 Steps)

$$y = \frac{1}{(x+9)^5} = (x+9)^{-5}$$

$$\frac{dy}{dx} = -5(x+9)^{-6}(1)$$

$$\frac{dy}{dx} = -5(x+9)^{-6}$$

By Quotient Rule (4 Steps):

$$y = \frac{1}{(x+9)^5}$$

$$\frac{dy}{dx} = \frac{[0(x+9)^5 - 5(x+9)^{5-1}(1)]}{(x+9)^{5(2)}}$$

$$\frac{dy}{dx} = \frac{-5(x+9)^4}{(x+9)^{10}}$$

$$\frac{dy}{dx} = -5(x+9)^{4-10}$$

$$\frac{dy}{dx} = -5(x+9)^{-6}$$

Conclusion: If the numerator $= 1$ and the denominator is in the form of $(ax^n + bx^m \ldots + c)$, you just have to convert the expression or equation into index notation and use chain rule to differentiate. (This only applies when the numerator = 1.)

Title	Differentiation of Exponential and Logarithmic Function
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	20/12/2018

This article assumes you know and understand the notation for differentiation.
Applicable to

- Singapore-Cambridge Secondary School GCE 'O' Level Additional Mathematics
- Nanyang Polytechnic, School of Chemical and Life Sciences – Mathematics for Life Science Module
- Nanyang Polytechnic, School of Engineering – Engineering Mathematics 1A Module
- Singapore-Cambridge Junior College GCE 'A' Level H1 Mathematics
- Singapore-Cambridge Junior College GCE 'A' Level H2 Mathematics (Bridging Only)

Chain Rule (In General)
Given $f(x) = g(h(x))$ The derivative is given as $f'(x) = [g'(x)][h'(x)]$

General Cases for Derivatives of Exponential and Logarithmic Functions [Derived by Applying Chain Rule to the Specific Cases Mentioned Further Below.]	
Function	Derivatives
$f(x) = a^{g(x)}$	$f'(x) = [a^{g(x)} \ln(a)]g'(x)$
$f(x) = ce^{g(x)}$ Where c is a coefficient of $e^{g(x)}$, with e representing Euler's Number.	$f'(x) = \left[ce^{g(x)}\right]g'(x)$
$f(x) = \ln\big(g(x)\big)$	$f'(x) = \frac{1}{g(x)}[g'(x)] = \frac{g'(x)}{g(x)}$

Specific Case	Derivatives
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = ce^x$	$f'(x) = ce^x$
$f(x) = a^x$	$f'(x) = a^x \ln(a)$
$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$

Question 1.

Differentiate the following with respect to x

(a) e^{6x}
(b) $7e^{2x+6}$
(c) 5^{7x}
(d) 3^{2x+6}
(e) $\ln(6x^2 + 5x)$

Question 1

(a) $\frac{d}{dx}e^{6x}$

Rewrite as

$f(x) = e^{6x}$

$g(x) = e^{6x}$	$g'(x) = e^{6x}$
$h(x) = 6x$	$h'(x) = 6$
$f'(x) = 6e^{6x}$ $\frac{d}{dx} = 6e^{6x}$	

(b) $\frac{d}{dx}7e^{2x+6}$

Rewrite as

$f(x) = 7e^{2x+6}$

$g(x) = 7e^{2x+6}$	$g'(x) = 7e^{2x+6}$
$h(x) = 2x + 6$	$h'(x) = 2$
$f'(x) = 14e^{2x+6}$ $\frac{d}{dx} = 14e^{2x+6}$	

(c) $\frac{d}{dx} 5^{7x}$

Rewrite as
$f(x) = 5^{7x}$

$g(x) = 5^{7x}$	$g'(x) = 5^{7x} \ln(5)$
$h(x) = 7x$	$h'(x) = 7$
$f'(x) = 7\,(5^{7x}) \ln(5)$ $\frac{d}{dx} = 7(5^{7x}) \ln(5)$	

(d)
$\frac{d}{dx} 3^{2x+6}$

Rewritten as $f(x) = 3^{2x+6}$	
$g(x) = 3^{2x+6}$	$g'(x) = 3^{2x+6} \ln(3)$
$h(x) = 2x + 6$	$h'(x) = 2$

$\frac{d}{dx} = 2(3^{2x+6} \ln(3))$

(e)

$$\frac{\mathrm{d}}{\mathrm{dx}}\ln(6x^2 + 5x)$$

For students who can remember the formula for differentiation of $\ln[g(x)]$, go ahead and use the formula, unfortunately, this might not be the case for all students and I still want to demonstrate the more "tedious" approach. However, please note that the more tedious method requires less memory work.

Rewritten as:

$$f(x) = \ln(6x^2 + 5x)$$

$$f'(x) = \frac{1}{6x^2+5x}(12x + 5)$$

$$f'(x) = \frac{(12x+5)}{6x^2+5x} \qquad \frac{d}{dx} = \frac{12x+5}{6x^2+5x}$$

Title	Differentiation of Trigonometric Functions
Author	AprilDolphin
Date	15/9/2024
Notice	Questions all taken from books

Chain Rule in General
Given a function $F(x) = f(g(x))$ The derivative is given by $F'(x) = f'(x)\ g'(x)$

Rules of differentiation of trigonometric functions	
Function	Derivatives
$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$f(x) = \tan(x)$	$f'(x) = \sec^2(x)$

Application of Chain Rule to the above trigonometric functions	
Function	Derivatives
$f(x) = \sin[g(x)]$	$f'(x) = \cos[f(x))]\ [g'(x)]$
$f(x) = \cos[g(x)]$	$f'(x) = -\sin[f(x)]\ [g'(x)]$
$f(x) = \tan\ [g(x)]$	$f'(x) = \sec^2[f(x)][g'(x)]$

Example

Given the following functions, write down the derivative

1. $f(x) = x\tan(3x)$

   Using product rule of differentiation, given function of $f(x) = g(x)h(x)$, the derivative is given by $f'(x) = g(x)h'(x) + h(x)g'(x)$

   Using chain rule of differentiation, given function of $f(x) = g[h(x)]$, the derivative is given by $f'(x) = g'(x)\ h'(x)$

   Given $g(x) = \tan(3x)$
   The derivative is $g'(x) = 3\sec^2(3x)$

   Given $f(x) = x\tan(3x)$
   $f'(x) = \tan(3x)\ [1] + x[3\sec^2(3x)]$
   $f'(x) = \tan(3x) + 3x\sec^2(3x)$

2. $f(x) = \frac{1+\cos(x)}{\sin(x)}$

Using quotient rule of differentiation

$$F(x) = \frac{f(x)}{g(x)}$$

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[(g(x)]^2}$$

$$f'(x) = \frac{\sin(x)\,[-\sin(x)] - [1 + \cos(x)][\cos(x)]}{\sin^2(x)}$$

$$f'(x) = \frac{[-\sin^2(x) - \cos^2(x) - \cos(x)]}{\sin^2(x)}$$

Using trigonometric identity $\sin^2 x + \cos^2 x = 1$,
Therefore $-\sin^2 x - \cos^2 x = -1$

$$f'(x) = \frac{-1 - \cos(x)}{\sin^2(x)}$$

3. $y = \sin\,(2x^2 + 5)$

Using chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{du} = \cos(2x^2 + 5)$$

$$\frac{du}{dx} = 4x$$

$$\frac{dy}{dx} = 4x\,[(\cos(2x^2 + 5)]$$

Title	Additional Mathematics ['N' Levels] and Computing Mathematics Calculus – Basic Integration and Indefinite Integrals
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	7/4/2018

This document is suitable for students of the following categories

- Nanyang Polytechnic School of Information Technology Students [G1]
- 'N' Levels Additional Mathematics Students [G2]

This document is however not useful for students studying in the following fields for the following reasons

- 'O' Level and 'A' Level (Trigonometric, Exponential and Logarithmic Function are not covered in this document)
- Engineering (Trigonometric Function Not Covered)
- Chemistry (Logarithmic and Exponential Function Not Covered)

However, if you haven't studied additional mathematics before and want exposure to the topic before entering your course or before you enter Secondary 4, this should give you the basic concepts of how it works.

The following are covered in this release:

Basic Overview of Integration

- Integration as the reverse of differentiation
- Notation guide
- Indefinite integral

Rules of integration

- Constant Multiple Rule
- Sum Rule
- Difference Rule

***Remember to print page 3 if you need something to refer to while doing your assignments. This material is not designed in the same way as the differentiation topic notes I created previously.

Integration as the reverse process of differentiation
As covered in earlier guide, we discussed how differentiation works, the reason why we study integration is the following reasons:

- There will be situations where finding the "anti-derivative" (AKA integral) is useful, if you are only given the gradient and ask to find out what is the equation of the function.
- Further study of integration will also be applied to finding the area under the graph of the function, using definite integral, which is useful for solving certain problems in various fields like Physics, Engineering, Chemistry and Information Technology.

Notation of Indefinite Integral

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

"Indefinite Integral of Function" Symbol

Arbitrary Constant

Explanation
Arbitrary Constant: The reason why we must have a "+C" to any indefinite integral is

- Reversing the integration process by differentiating the integral will result in many possible values of the constants that satisfy the integral, thus a "$+C$" is added, to represent "Constant"

Rules of Integration:

Integration of Constant Values

$$\int k\,dx = kx + C$$

Constant Multiple (Power Function Integration) Rule (Provided $n \neq -1$):

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int kx^n\,dx = k\int x^n = k\left(\frac{x^{n+1}}{n+1}\right) + C = \frac{kx^{n+1}}{n+1} + C$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$ (Provided $a \neq 0$ AND $n \neq -1$)

Generalized Sum Rule

$$\int k \cdot f(x) + h \cdot g(x) = k\int f(x) + h\int g(x) + C$$

Generalized Difference Rule

$$\int k \cdot f(x) - h \cdot g(x) = k\int f(x) + h\int g(x) + C$$

Generalization of Constant Multiple, Sum and Difference Rule (Provided $n \neq -1$):

$$\int kx^n \pm gx^m = k\left(\frac{x^{n+1}}{n+1}\right) \pm g\left(\frac{x^{n+1}}{n+1}\right) + C = \frac{kx^{n+1}}{n+1} \pm \frac{gx^{n+1}}{n+1} + C$$

Indefinite Integral

Example 1:

Find the following integrals [Power Function Integration Rule]

(a) $\int x^6 \ dx$

(b) $\int \frac{1}{x^8} \ dx$

1(a) $\int x^6 \ dx = \frac{x^{6+1}}{6+1} = \frac{x^7}{7} + C$

1(b) $\int \frac{1}{x^8} \ dx = \int x^{-8} = \frac{x^{-8+1}}{-8+1} = \frac{x^{-7}}{-7} + C = -\frac{1}{7}x^{-7} + C$

Example 2:

Find the following integrals [Integration of Constant Values]

(a) $\int 15 \, dx$

(b) $\int 25 \, dx$

2(a) $\int 15 \, dx \ = 15x + C$

2(b) $\int 25 \, dx = 25x + C$

Example 3:

Find the following integrals [Constant-Multiple (Power Function Integration) Rule]

(a) $\int 2x^5 \, dx$

(b) $\int 16x^7 \, dx$

3(a) $\int 2x^5 \, dx$

$$\int 2x^5 \, dx = 2 \int x^5 \, dx + C = 2\left(\frac{x^{5+1}}{5+1}\right) + C = \frac{2x^{5+1}}{6} = \frac{2x^6}{6} + C = \frac{1}{3}x^6 + C$$

3(b) $\int 16x^7 \, dx$

$$\int 16x^7 \, dx = 16 \int x^7 \, dx = 16\left(\frac{x^{7+1}}{7+1}\right) + C = \frac{16x^8}{8} + C = 2x^8 + C$$

Example 4:

Find the following integrals [Generalized Sum Rule]

(a) $\int 2x^9 + 5x^3\, dx$

(b) $\int 9x^7 + 6x^{-8} + 12x + 7\, dx$

4(a) $\int 2x^9 + 5x^3\, dx$

$$= 2\int x^9 + 5\int x^3$$

$$= 2\left(\frac{x^{9+1}}{9+1}\right) + 5\left(\frac{x^{3+1}}{3+1}\right) + C$$

$$= 2\left(\frac{x^{10}}{10}\right) + 5\left(\frac{x^4}{4}\right) + C$$

$$= \frac{1}{5}x^{10} + \frac{5}{4}x^4 + C$$

4(b) $\int 9x^7 + 6x^{-8} + 12x + 7\, dx$

$$= 9\int x^7 + 6\int x^{-8} + 12\int x + \int 7$$

$$= 9\left(\frac{x^{7+1}}{7+1}\right) + 6\left[\frac{x^{(-8)+1}}{(-8)+1}\right] + 12\left(\frac{x^{1+1}}{1+1}\right) + 7x + C$$

$$= \frac{9}{8}x^8 + 6\left(\frac{x^{-7}}{-7}\right) + \frac{12x^2}{2} + 7x + C$$

$$= \frac{9}{8}x^8 - \frac{6}{7}x^{-7} + 6x^2 + 7x + C$$

Example 5

Find the following integrals [Generalized Difference Rule]

(a) $\int 2x^2 - 9x - \frac{2}{x^9} - 8 \ dx$

5(a)

$$\int 2x^2 - 9x - \frac{2}{x^9} - 8\,dx$$

$$= 2\int x^2 - 9\int x - 2\int x^{-9} - \int 8$$

$$= 2\left(\frac{x^{2+1}}{2+1}\right) - 9\left(\frac{x^{1+1}}{1+1}\right) - 2\left(\frac{x^{-9+1}}{-9+1}\right) - 8(x) + C$$

$$= 2\left(\frac{x^3}{3}\right) - 9\left(\frac{x^2}{2}\right) - 2\left(\frac{x^{-8}}{-8}\right) - 8(x) + C$$

$$= 2\left(\frac{x^3}{3}\right) - 9\left(\frac{x^2}{2}\right) + 2\left(\frac{x^{-8}}{8}\right) - 8x + c$$

$$= \tfrac{2}{3}x^3 - \tfrac{9}{2}x^2 + \tfrac{2}{8}x^{-8} - 8x + C$$

$$\frac{2}{3}x^3 - \frac{9}{2}x^2 + \frac{1}{4}x^{-8} - 8x + C$$

Example 6

Find the following integrals [Generalization of all Rules Mentioned Combined]

(a) $\int 3x^5 - 6x^4 + 11x - 16 + 9x^{-7}\ dx$

$$3\int x^5 - 6\int x^4 + 11\int x - \int 16 + 9\int x^{-7}$$

$$3\left(\frac{x^{5+1}}{5+1}\right) - 6\left(\frac{x^{4+1}}{4+1}\right) + 11\left(\frac{x^{1+1}}{1+1}\right) - 16(x) + 9\left(\frac{x^{-7+1}}{-7+1}\right) + C =$$

$$3\left(\frac{x^6}{6}\right) - 6\left(\frac{x^5}{5}\right) + 11\left(\frac{x^2}{2}\right) - 16x + 9\left(\frac{x^{-7+1}}{-7+1}\right) + C =$$

$$\frac{3}{6}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x + \frac{9x^{-6}}{(-6)} + C =$$

$$\frac{1}{2}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x - \frac{9}{6}x^{-6} + C =$$

$$\frac{1}{2}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x - \frac{3}{2}x^{-6} + C$$

Title	Integration Leading to Logarithmic Functions and Integration of Exponential Functions
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	21/12/2018

Applicable to

- JC H2 'A' Level Mathematics (Bridging only) and H1 'A' Level Mathematics
- Nanyang Polytechnic School of Chemical and Life Sciences – Mathematics for Life Sciences
- Nanyang Polytechnic School of Engineering – Engineering Mathematics 1B
- Institute of Technical Education – Calculus/Mathematics Modules
- Secondary School GCE 'O' Level Additional Mathematics

Function	Corresponding Integrals
$f(x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$f(x) = e^x$	$\int e^x\,dx = e^x + C$
$f(x) = ae^x$	$\int ae^x\,dx = ae^x + C$

(Generic Cases) Functions	Corresponding Integrals
$f(x) = \frac{1}{ax+b}$	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$
$f(x) = \frac{a}{x}$	$\int \frac{a}{x} = a(\ln(x)) + C$
$f(x) = e^{ax+b}$	$\int e^{ax+b}\,dx = \frac{1}{a}\,(e^{ax+b}) + C$

Find the following integrals

a) $\int \frac{3}{x}\ dx$

b) $\int e^{6x+5}\ dx$

c) $\int 5e^{x}\ dx$

d) $\int \frac{1}{9x+5} dx$

e) $\int e^{8x+6}$

Solutions

a) $\int \frac{3}{x}\ dx = 3(\ln(x)) = 3\ln(x) + C$

b) $\int e^{6x+5}\ dx = \frac{1}{6}(e^{6x+5}) + C$

c) $\int 5e^{x}\ dx = 5e^{x} + C$

d) $\int \frac{1}{9x+5} dx\ = \frac{1}{9}(\ln(9x+5)) + C$

e) $\int e^{8x+6}\ dx = \frac{1}{8}(e^{8x+6}) + C$

Title	Integration of Trigonometric Functions
Author	AprilDolphin
Date	16/9/2024
Notice	Questions all taken from books

Functions	Corresponding Integrals
$F(x) = \sin(x)$	$f(x) = -\cos(x) + c$
$F(x) = \cos(x)$	$f(x) = \sin(x) + c$
$F(x) = \sec^2(x)$	$f(x) = \tan(x) + c$

Functions	Corresponding Integrals
$F(x) = \sin(ax + b)$	$f(x) = \frac{-\cos(ax+b)}{a} + c$
$F(x) = \cos(ax + b)$	$f(x) = \frac{\sin(ax+b)}{a} + c$
$F(x) = \sec^2(ax + b)$	$f(x) = \frac{\tan(ax+b)}{a} + c$

Example:

Given the following derivatives, find the corresponding integrals

1. $\cos(2x) + \sin(5x - 1)$

$$\int [\cos(2x) + \sin(5x - 1)]\, dx =$$

$$\frac{\sin(2x)}{2} + \frac{-\cos(5x-1)}{5} =$$

$$\frac{\sin(2x)}{2} - \frac{\cos(5x-1)}{5} + c$$

2. $\sec^2(3x + 5) - \cos(5x + 5)$

$$\int [\sec^2(3x + 5) - \cos(5x + 5)]\, dx =$$

$$\frac{\tan(3x+5)}{3} - \frac{\sin(5x+5)}{5} + c$$

Title	Additional Mathematics ['N' Levels] and Computing Mathematics – Finding the Definite Integral of a Function
Editor	Lee Jian Lian
Date	30/4/2018
Updates	(Removed Unnecessary Text From Original Version)

Disclaimer: This file does not cover logarithmic functions, exponential function and trigonometric function.

You need the following prerequisite knowledge before proceeding

- Able to find indefinite integral of a function
- Understand the notation of indefinite integral

Notation used in finding definite integral	
$$\int_h^g f'(x) = \lvert f(g) - f(h) \rvert$$	
g *and* h	Refers to the boundary between the two points on the x-axis you are finding the area under graph for
$\lvert n \rvert$	This is sometimes referred to as the absolute value of n, any negative value that is between the two vertical strokes becomes positive while positive value within the vertical strokes remains positive.

Example 1
Find the following definite integral

$$\int_0^2 x^2 + 1\, dx =$$

Indefinite integral of the mentioned function is:	$\int_0^2 x^2 + 1\, dx = \frac{x^3}{3} + x$
Since we are finding the definite integral, we rewrite the integral in the following notation	$\left[\frac{x^3}{3} + x\right]\begin{matrix} 2 \\ 0 \end{matrix}$
Substitute 0 and 2 into the value x.	$\left\lvert\frac{2^3}{3} + 2 - \left(\frac{0^3}{3} + 0\right)\right\rvert =$
Final Answer	$\frac{14}{3} units$

Example 2

Find the following definite integral

$$\int_0^1 4 + 3x^2 \, dx$$

The indefinite integral of the function is	$\int_0^1 4 + 3x^2 = 4x + \frac{3x^3}{3}$
Rewrite using the following notation	$\left[4x + \frac{3x^3}{3}\right]_0^1$
Substitute 0 and 1 into the value of x	$\left\|4(1) + \frac{3(1)^3}{3} - \left(4(0) + \frac{3(0)^3}{3}\right)\right\|$
Final Answer	$5\ units$

Example 3

Find the following definite integral

$$\int_{7}^{0} 6x^7 + 2x - 6x^2 + 5$$

The indefinite integral of the function is	$\int_{7}^{0} 6x^7 + 2x - 6x^2 + 5 = \frac{6x^8}{8} + \frac{2x^2}{2} - 6\left(\frac{x^3}{3}\right) + 5x$
Rewrite using the following notation	$\left[\frac{6x^8}{8} + \frac{2x^2}{2} - 6\left(\frac{x^3}{3}\right) + 5x\right]_{7}^{0}$
Substitute 0 and 7 into the value of x	$\left\|\frac{6(0)^8}{8} + \frac{2(0)^2}{2} - 6\left(\frac{0^3}{3}\right) + 5(0) - \left[\frac{6(7)^8}{8} + \frac{2(7)^2}{2} - \frac{6(7^3)}{3} + 5(7)\right]\right\|$
Final Answer	$= 4322998.75\ units$

Example 4

Find the following definite integral

$$\int_{10}^{2} \sqrt{x^3} + 2x - 9$$

Indefinite integral of the function is	$\frac{3}{4}x^{\frac{4}{3}} + x^2 - 9x$
Rewrite using the notation as shown	$\left[\frac{3}{4}x^{\frac{4}{3}} + x^2 - 9x\right]\begin{matrix}2\\10\end{matrix}$
Substitute 2 and 10 to find the definite integral	$\left\|\frac{3}{4}(2)^{\frac{4}{3}} + 2^2 - 9(2) - \left[\frac{3}{4}(10)^{\frac{4}{3}} + 10^2 - 9(10)\right]\right\|$
Answer is	$38.268\ units$