(TOPICAL REVISION – SUGGESTED SOLUTIONS)
NUMERICAL METHODS (PART 1)
APPROXIMATION OF ROOTS OF EQUATIONS

1 Show that the equation $x^3 - 5x + 1 = 0$, has exactly one root in (0,1). Use two iterations of linear interpolation between $x = 0$ and $x = 1$ to yield a fractional approximation of this root. [4]

Three possible rearrangements of the given equation in the form $x = F(x)$ are

$$x = \sqrt[3]{5x-1},$$

$$x = \frac{1}{5}(x^3+1),$$

$$x = x^3 - 4x + 1.$$

Only one of these rearrangements will provide an iterative method, of the form $x_{n+1} = F(x_n)$, $x_0 = 2$, which converges to the **root between 2 and 3**. Use this rearrangement to find this root correct to 3 significant figures. [4]

It is known the correct rearrangement above provides a convergent iterative method by analysing the derivative of F . For this purpose, show that $0 < F'(x) < 1$, whenever $x \geq 2$. [2]

[EJC/FM/2018/P1/Q9]

[Solution]

With $\mathrm{g}(x) = x^3 - 5x + 1,\ x \in \mathbb{R}$.
Note that $\mathrm{g}(0) = 1 > 0, \mathrm{g}(1) = -3 < 0$, thus $\mathrm{g}(0)\cdot\mathrm{g}(1) < 0$
Since g is continuous on $[0,1]$, the equation $\mathrm{g}(x) = 0$ has at least 1 root in [0,1].
$\mathrm{g}^{(1)}(x) = 3x^2 - 5 < 0, x \in [0,1]$.
Thus $\mathrm{g}(x) = 0$ has at most 1 root in [0,1].
Therefore, $\mathrm{g}(x) = 0$ has exactly 1 real root in [0,1].

Using linear interpolation on $[0,1]$, we have

$$x_1 = \frac{(0)\mathrm{g}(1) - (1)\mathrm{g}(0)}{\mathrm{g}(1) - \mathrm{g}(0)} = \frac{-1}{-3-1} = \frac{1}{4}$$

Since $\mathrm{g}\left(\frac{1}{4}\right) = \frac{1}{64} - 5\left(\frac{1}{4}\right) + 1 = \frac{1}{64} - \frac{5\times16}{64} + \frac{64}{64} = -\frac{15}{64}, < 0$ and $\mathrm{g}(0) = 1 > 0$

$\therefore \alpha$ lies in (0, 1/4).

$$x_2 = \frac{(0)g\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)g(0)}{g\left(\frac{1}{4}\right) - g(0)} = \frac{-\left(\frac{1}{4}\right)}{-\frac{15}{64} - 1} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{79}{64}\right)} = \frac{16}{79}$$

$x = \sqrt[3]{5x-1}$, (This fixed point iterations based on this rearrangement converges to 2.13)

$x_0 = 2$

$x_1 \approx 2.08$

$x_2 \approx 2.11$

$x_3 \approx 2.12$

$x_4 \approx 2.13$

$x_5 \approx 2.13$

Using GC, we can see for the other 2 schemes based on:

$x = \frac{1}{5}(x^3 + 1)$, the fixed point iteration does not converge to a value between 2 and 3, but to 0.205.

$x = x^3 - 4x + 1$, the fixed point iteration diverges.

$$F_1(x) = \sqrt[3]{5x-1}$$

$$F^{(1)}(x) = \frac{5}{3}(5x-1)^{-\frac{2}{3}} > 0, x > \frac{1}{5}$$

$$F^{(1)}(x) = \frac{5}{3}(5x-1)^{-\frac{2}{3}} < 1,$$

whenever $5x - 1 > \left(\frac{5}{3}\right)^{3/2}$,

i.e., $x > \frac{1}{5}\left[1 + \left(\frac{5}{3}\right)^{3/2}\right] \approx 0.630,$

which is satisfied as $F(x)$ increases, whenever $x \geq 2 > 0.630$.

2 **(a)** The curve with equation $y = 2e^{x^2} + x - 3$ has exactly one stationary point in the interval $[-1, 0]$. Use the Newton-Raphson method to find the *x*-coordinate of the stationary point, correct to 4 decimal places. [5]

(b) **(i)** Show that the equation $x^3 - 7x + 2 = 0$ has a root, α, in the interval [0, 1]. [1]

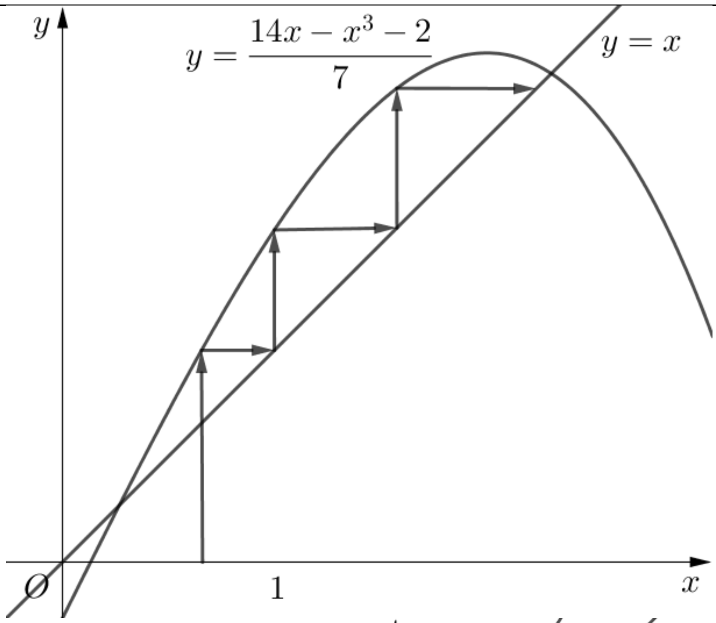
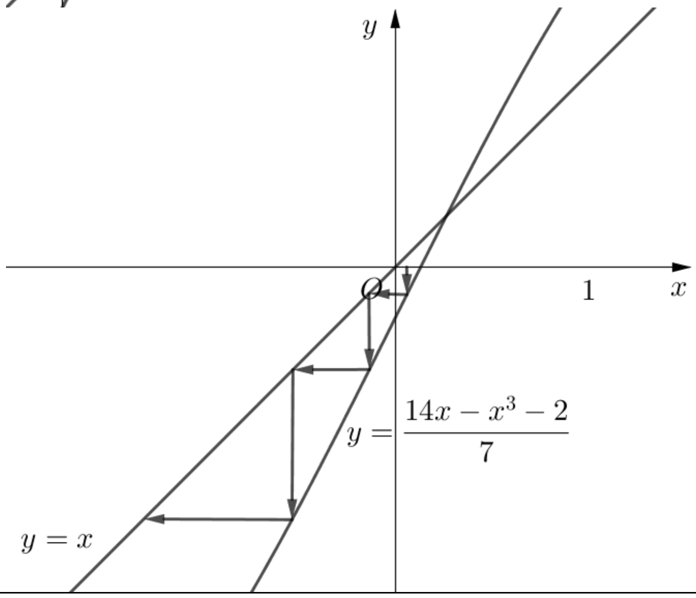
(ii) Student A uses the recurrence relation $x_{n+1} = \frac{1}{7}\left(14x_n - x_n^{\ 3} - 2\right)$ for finding α. Explain why Student A will fail to find α. [2]

(iii) Student B uses the recurrence relation $x_{n+1} = \frac{1}{7}\left(x_n^{\ 3} + 2\right)$ for finding α. Find the approximate value of α to 5 decimal places. [2]

NJC/FM/2019/P1/8

[Solution]

2(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 4xe^{x^2} + 1 = 0$ The Newton-Raphson formula as follows: $x_{n+1} = x_n - \frac{4x_n e^{x_n^{\ 2}} + 1}{8x_n^{\ 2} e^{x_n^{\ 2}} + 4e^{x_n^{\ 2}}}$ $x_0 = -0.5$ $x_1 = -0.29647$ $x_2 = -0.23906$ $x_3 = -0.23642$ $x_4 = -0.23641$ Hence, $x = -0.2364$. Checking, $4\left(-0.23635\right)e^{(-0.23635)^2} + 1 = 2.9\times10^{-4} > 0$ $4\left(-0.23645\right)e^{(-0.23645)^2} + 1 = -1.8\times10^{-4} < 0$
(b) (i)	$\mathrm{f}\left(x\right) = x^3 - 7x + 2$ $\mathrm{f}\left(0\right) = 2$ $\mathrm{f}\left(1\right) = -4$ Since $\mathrm{f}\left(0\right)\mathrm{f}\left(1\right) < 0$ and f is continuous, there is a root in [0, 1].
(b) (ii)	Either use graph or display a series of values to show convergence fail.

	 
(b) (iii)	$x_0 = 1$ $x_1 = 0.428571429$ $x_2 = 0.2969596$ $x_3 = 0.289455341$ $x_4 = 0.289178834$ $x_5 = 0.289168915$ $x_6 = 0.28916856$ $\alpha = 0.28917$ Checking, $0.289165^3 - 7(0.289165) + 2 = 2.4 \times 10^{-5} > 0$ $0.289175^3 - 7(0.289175) + 2 = -4.4 \times 10^{-5} < 0$

3 Let $\mathrm{f}(x)=\sec^2 2x-\mathrm{e}^x$. The equation $\mathrm{f}(x)=0$ has a root α in the interval [0.1, 0.3].

(a) Use Newton-Raphson method with initial approximation $x_0=0.1$ to find the first four approximations $x_1,\ x_2,\ x_3,\ x_4$ to α. Deduce the behaviour of x_n for large n and use a graph of the function f to help you explain why the sequence is not converging to α. [5]

(b) Determine, with explanation, which one of these iterative formulae is more suitable in approximating the root α.

(I) $x_{n+1}=\ln\left(\sec^2\left(2x_n\right)\right)$

(II) $x_{n+1}=\dfrac{1}{2}\cos^{-1}\left(\mathrm{e}^{-\frac{x_n}{2}}\right)$

Using an initial approximation of $x_0=0.3$ and the chosen iterative formula, find the approximation to the root α, correct to three decimal places. [4]

[RVHS/FM/2018/P1/Q10]

[Solution]

2

$\mathrm{f}\,'(x)=4\sec^2 2x\tan 2x-\mathrm{e}^x$

$x_1=0.1-\dfrac{\mathrm{f}(0.1)}{\mathrm{f}\,'(0.1)}$

$=-0.1455040283$

≈ -0.146

$x_2=-0.0417668268$

≈ -0.0418

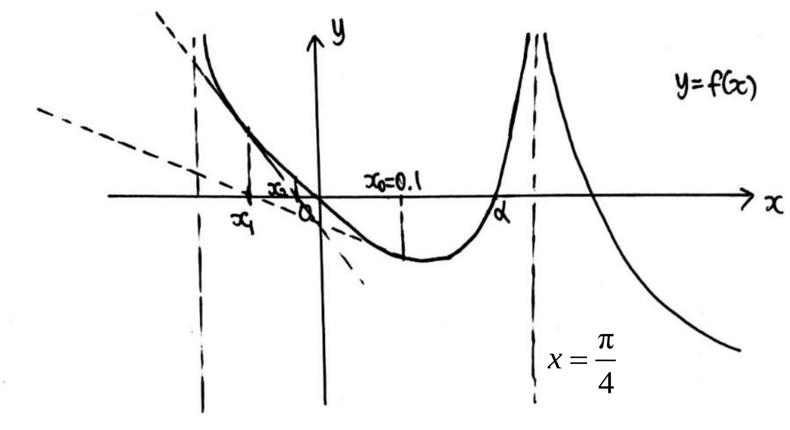
$x_3=-0.0048039104$

≈ -0.00480

$x_4=-0.0000781946$

≈ -0.00008

As $n\to\infty,\ x_n\to 0$



The initial approximation x_0 is too far from α. It is nearer to the root $x=0$. Thus the sequence converges to this root instead of α.

(b)

Let $\mathrm{F}(x)=\ln\left(\sec^2(2x)\right)$ and $\mathrm{G}(x)=\frac{1}{2}\cos^{-1}\left(\mathrm{e}^{-\frac{x}{2}}\right)$

$$\mathrm{F}'(x)=\frac{4\sec^2 2x\tan 2x}{\sec^2 2x}$$
$$=4\tan 2x$$

$$\mathrm{G}'(x)=-\frac{1}{2}\frac{\left(-\frac{1}{2}e^{-\frac{x}{2}}\right)}{\sqrt{1-\left(e^{-\frac{x}{2}}\right)^2}}$$

$$=\frac{e^{-\frac{x}{2}}}{4\sqrt{1-e^{-x}}}$$

$|F'(0.3)|=2.737>1$

$|G'(0.3)|=0.4227<1$

Thus $x_{n+1}=\frac{1}{2}\cos^{-1}\left(\mathrm{e}^{-\frac{x_n}{2}}\right)$ is a more suitable iteration to use.

$x_0=0.3$
$x_1=0.2670688194$
$x_2=0.2526829817$
$x_3=0.2460804385$
$x_4=0.2415055517$
$x_5=0.2408020568$
$x_6=0.2404652776$
$x_7=0.2403038578$

$\therefore \alpha\approx 0.240$ (to 3 d.p.)

4 **(i)** The function f is such that $f(a)f(b) < 0$, where $a < b$. A student concludes that the equation $f(x) = 0$ has exactly one root in the interval (a, b).
Illustrate with a sketch, two possible scenarios in which the student could be wrong. [2]

GIVEN NOW THAT THE EQUATION $\sqrt{x-3}-\frac{2}{x^2}=0$ HAS EXACTLY ONE ROOT α IN THE INTERVAL **(3,4).**

(ii) Derive the iterative formula $x_{n+1}=3+\frac{4}{x_n^4}$ using the fixed point iteration method.

Using 3 as the initial value, apply this iterative formula to find an approximation for α, correct to 3 decimal places. You are required to check the accuracy of your answer in this question. [3]

(iii) By using linear interpolation once, obtain, correct to 3 decimal places, a first approximation α_1 to α . Using α_1 as the initial value, apply the Newton-Raphson method once to obtain α_2 , leaving your answer to 3 decimal places. [3]

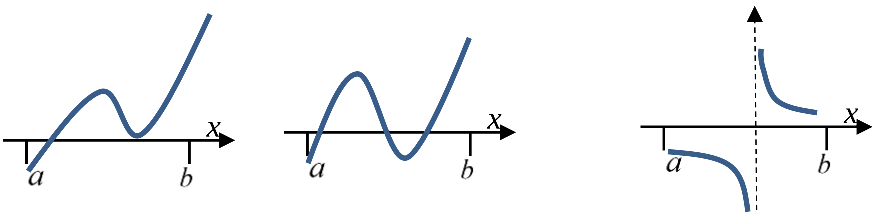
(iv) Explain why the Newton-Raphson method in this case fails to give an approximation to α. [1]

(v) Illustrate, on a single diagram, how α_1 and α_2 are obtained. [2]

[CJC/FM/2018/P1/Q5]

[Solution]

(i) Two possible scenarios with more than one root:



The student failed to check if f is strictly increasing or decreasing function in the interval (a, b) or if the curve is continuous over the interval (a,b).

(ii)

$$\sqrt{x-3}-\frac{2}{x^2} \Rightarrow \sqrt{x-3}=\frac{2}{x^2}$$

Squaring both sides, $x-3=\frac{4}{x^4} \Rightarrow x_{n+1}=3+\frac{4}{x_n{}^4}$

Let f(x) = $\sqrt{x-3}-\frac{2}{x^2}$

Method 1:

$$x_{n+1}=3+\frac{4}{x_n^4}$$

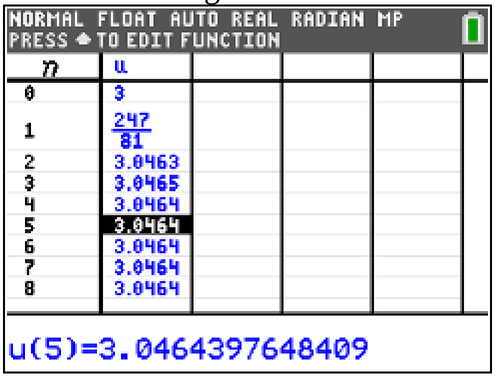
$\alpha_o = 3$

$\alpha_1 = 3 + \dfrac{4}{3^4} = 3.04938$

$\alpha_2 = 3 + \dfrac{4}{3.04938^4} = 3.04626$

$\alpha_3 = 3 + \dfrac{4}{3.04626^4} = 3.046451$

So correct to 3 decimal places, $\alpha = 3.046$
Checking f(3.0455) = -0.0023 < 0 and f(3.0465) = 1.483 x 10^{-4} > 0
Method 2: Using GC



Since $\alpha_4 = \alpha_5 = 3.0464$, so correct to 3 decimal places, $\alpha = 3.046$
Checking f(3.0455) = -0.0023 < 0 and f(3.0465) = 1.483 x 10^{-4} > 0

(iii) Since α lies between 3 and 4, so Linear Interpolation gives

$$\alpha_1 = \frac{3|f(4)|+4|f(3)|}{|f(3)|+|f(4)|} = 3.203$$

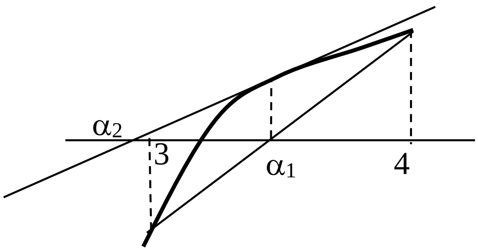
$$f'(x) = \frac{1}{2\sqrt{x-3}}+\frac{4}{x^3}$$

By N-R method, $\alpha_2 = \alpha_1 - \dfrac{f(\alpha_1)}{f'(\alpha_1)} = 3.203 - \dfrac{0.255608}{1.23147} = 2.995$ (correct to 3 decimal places)

(iv) Since $\alpha_2 = 2.995 < 3$ and f is not defined when $x < 3$, so the N-R method fails.

(v) $f'(x) = \dfrac{1}{2\sqrt{x-3}}+\dfrac{4}{x^3} > 0$ for $3 < x < 4$ and $f''(x) = -\dfrac{1}{4\sqrt{(x-3)^3}} - \dfrac{12}{x^4} < 0$

So f is increasing and concave downwards in (3, 4)



5

(i) By sketching the graphs of $y = \tan^{-1}\left(\frac{x}{5}\right)$ and $y = \sin^{-1}\left(\frac{x}{6}\right)$ on the same diagram, show that the equation $\tan^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{x}{6}\right)$ has a positive real root, α. [2]

(ii) Find the integer N such that $N < \alpha < N+1$. [2]

(iii) Illustrate graphically, using $x_0 = N$, why the iterative procedure

$$x_{n+1} = 5\tan\left(\sin^{-1}\left(\frac{x_n}{6}\right)\right)$$

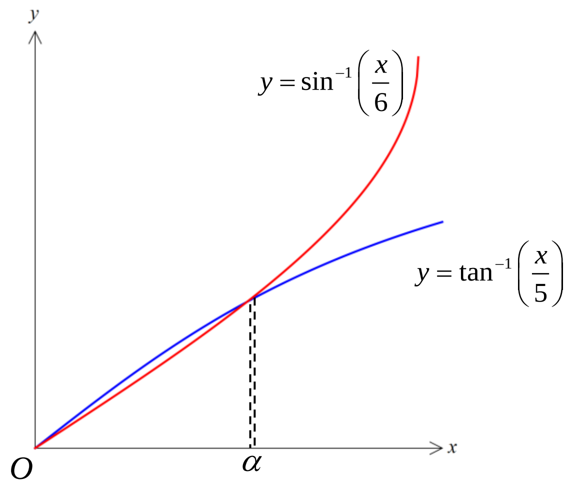
will not give a good approximate value of α. [2]

(iv) Taking the initial approximation to be N, use the Newton-Raphson method to find α to 2 decimal places. [3]

[RI/FM/2017/P2/Q1]

Solution:

(i)



The graphs of $y = \tan^{-1}\left(\frac{x}{5}\right)$ and $y = \sin^{-1}\left(\frac{x}{6}\right)$ intersect at a point for $x > 0$.

Hence, the equation $\tan^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{x}{6}\right)$ has a positive real root.

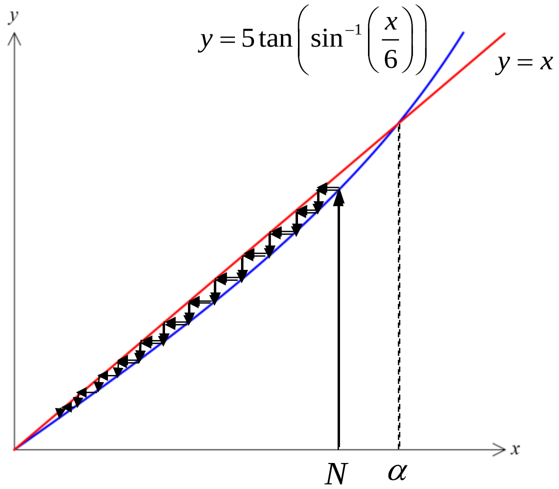
(ii) Let $\mathrm{f}(x) = \tan^{-1}\left(\frac{x}{5}\right) - \sin^{-1}\left(\frac{x}{6}\right)$

$\mathrm{f}(3) = 0.016821 > 0$; $\quad \mathrm{f}(4) = -0.054987 < 0$;

So, α lies between 3 and 4.

And so, $N = 3$

(iii)



The iteration will approach the root 0, instead of α .

(iv)

$$x_{n+1}=x_n-\frac{\tan^{-1}\left(\frac{x_n}{5}\right)-\sin^{-1}\left(\frac{x_n}{6}\right)}{\frac{5}{25+x_n^2}-\frac{1}{\sqrt{36-x_n^2}}}$$

n	x_n
0	3
1	3.3706
2	3.3178
3	3.3166
4	3.3166

Check $\mathrm{f}(3.315)=0.0000992>0$; $\quad \mathrm{f}(3.325)=-0.0000514<0$

The root is 3.32 (to 2 d.p.)

6

(i) Show, with the aid of a sketch graph, that the equation

$$x + k\ln x = 0$$

has exactly one real root, α, in the interval $\frac{1}{2} < x < 1$ if $k > \frac{1}{2\ln 2}$. [3]

(ii) In the case $k = 1$, a student tries to use fixed point iteration in the form of $x = \mathrm{F}(x)$ to find the value of α.

(a) In his first attempt, the student uses $\mathrm{F}(x) = -\ln x$ and $x_0 = 0.5$. Calculate the value of x_1 and x_2, correct to 4 decimal places. Explain why this method will fail to find the value of α. [2]

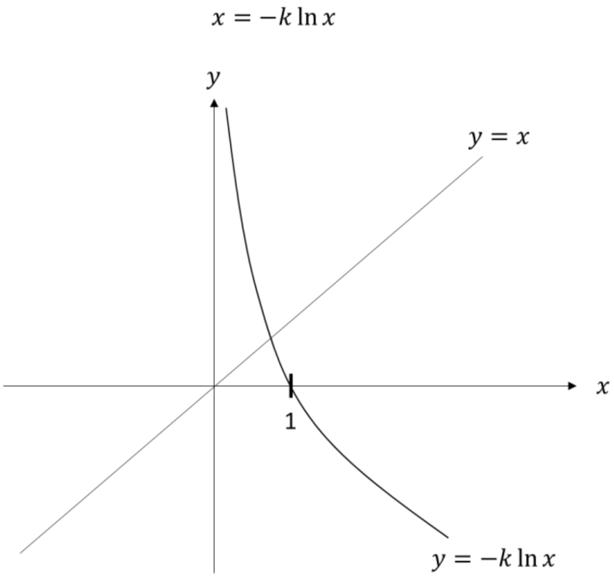
(b) Suggest a possible $\mathrm{F}(x)$ for the student and using $x_0 = 0.5$, find the value of α, correct to 3 decimal places. [2]

(c) Use a diagram to explain how the iteration in **(b)** converges to α, showing clearly the position of x_0, x_1 and x_2. [2]

[HCI/FM/2017/P2/Q2]

[Solution]

i From the graph, the graphs $y = x$ and $y = -k\ln x$ intersect only once. Hence there is exactly one real root.



Let $\mathrm{f}(x) = x + k\ln x$

$\mathrm{f}\left(\frac{1}{2}\right) = \frac{1}{2} + k\ln\frac{1}{2} < 0 \quad (\because k > \frac{1}{2\ln 2})$

$\mathrm{f}(1) = 1 + k\ln 1 > 0$

Since f (x) is continuous on $\left[\frac{1}{2}, 1\right]$ and $\mathrm{f}\left(\frac{1}{2}\right)\mathrm{f}(1) < 0$, by Intermediate Value Theorem, there exists a real root α in the interval $\left(\frac{1}{2}, 1\right)$.

(ii) (a)

$x + \ln x = 0$

$x = -\ln x$

$x_1 = 0.6931,\ x_2 = 0.3665$ (4 d.p.)

If the iteration continues, $x_3 = 1.0037,\ x_4 = -0.037 < 0$ and the iteration cannot continue since $\ln x$ is not defined for $x < 0$.

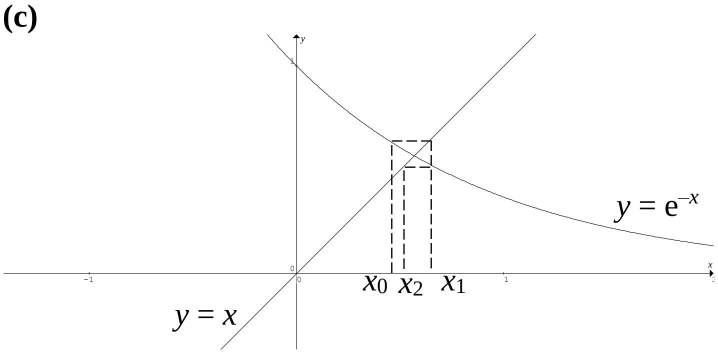
(b)

$-x = \ln x$

$x = \mathrm{e}^{-x} \Rightarrow \mathrm{F}(x) = \mathrm{e}^{-x}$

By G.C., $\alpha = 0.567$ (3 d.p.)

(c)



7 **(i)** Show algebraically that the equation $x^3 - x^2 - 6 = 0$ has exactly one real root in tl interval $(2,\ 3)$. [2]

(ii) Find an estimate of the root using 2 iterations of linear interpolation, correct to tw decimal places. [2]

(iii) Determine algebraically if the estimate in **(ii)** is an underestimate or overestimate the root. [3]

The equation $x^3 - x^2 - 6 = 0$, can be rearranged in the following ways (you are not required verify):

(A) $x = \left(x^2 + 6\right)^{\frac{1}{3}}$ (B) $x = x^2 - \dfrac{6}{x}$ (C) $x = \left(x + \dfrac{6}{x}\right)^{\frac{1}{2}}$

(iv) Determine with reasons which of the above expressions (A), (B) and (C) converg under the fixed point iteration method using the initial value $x_0 = 2$. If it converges, u a graph to demonstrate 3 iterations of its convergence to the root, labelling the poin x_0, x_1 and x_2 clearly. [5]

TJC/FM/2019/P1/7

[Solution]

(i) $\mathrm{g}(x) = x^3 - x^2 - 6$

$\mathrm{g}(2) = -2$ and $\mathrm{g}(3) = 12$

Since $y = \mathrm{g}(x)$ is continuous,

a root exists in the interval $(2,\ 3)$.

$\mathrm{g}'(x) = 3x^2 - 2x = 0 \Rightarrow x = 0,\ x = \dfrac{2}{3}$

Hence, $\mathrm{g}(x)$ is strictly increasing for $x > \dfrac{2}{3}$,

so there is exactly one root in the interval $(2,\ 3)$.

(ii) Since $x_1 = \dfrac{a\,|\,\mathrm{f}(b)\,| + b\,|\,\mathrm{f}(a)\,|}{|\,\mathrm{f}(b)\,| + |\,\mathrm{f}(a)\,|}$

$x_1 = 2.142857143$

$x_2 = 2.193415638$

Hence, an estimate of the root is $x = 2.19$.

(iii) $\mathrm{g}'(x) = 3x^2 - 2 > 0$ and $\mathrm{g}''(x) = 6x - 2 > 0$ for $x \in (2,3)$.

Since $\mathrm{g}(x)$ is strictly increasing and concave up in the interval $(2,\ 3)$,

the estimate is an underestimate of the root.

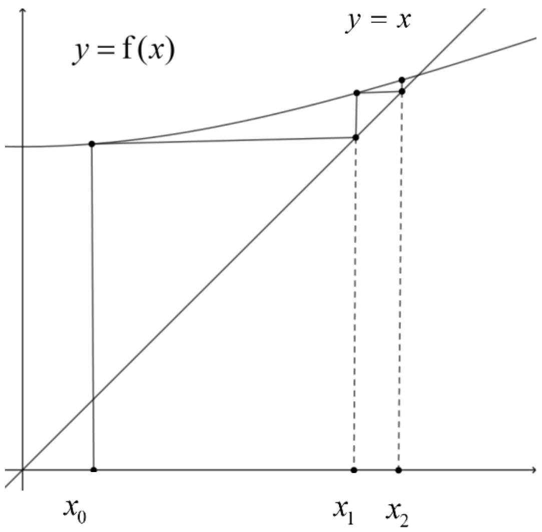
(iv) For (A), $\mathrm{f}'(x) = \dfrac{2}{3}x\left(x^2 + 6\right)^{-\frac{2}{3}} \Rightarrow \mathrm{f}'(2) = 0.287 < 1$.

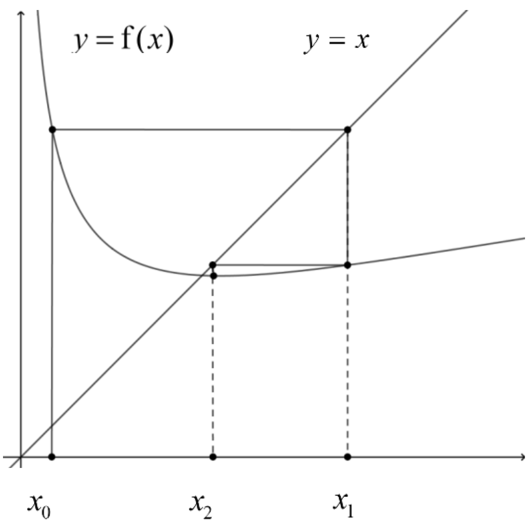
For (B), $\mathrm{f}'(x) = 2x + \dfrac{6}{x^2} \Rightarrow \mathrm{f}'(2) = 5.5 > 1$

For (C), $\mathrm{f}'(x)=\frac{1}{2}\left(x+\frac{6}{x}\right)^{-\frac{1}{2}}\left(1-\frac{6}{x^2}\right) \Rightarrow \mathrm{f}'(2)=-0.112<1$

Only (A) and (C) satisfies the criterion for convergence.

[M1 – Differentiation, **A1** – Correct Evaluation, **B1** – Correct Conclusion]





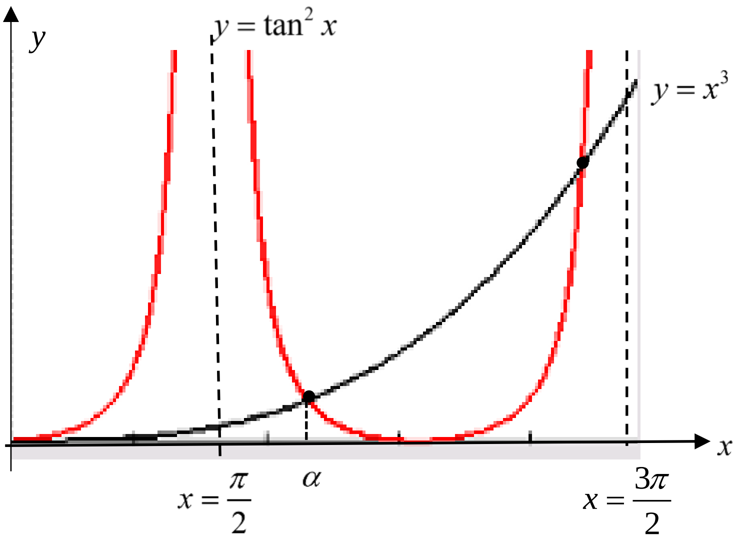
8 By considering the graphs of $y = \tan^2 x$ and $y = x^3$, show that the equation $\tan^2 x - x^3 = 0$ has exactly two roots in the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Denoting the smaller root by α, where $1 < \alpha < 2$, use linear interpolation once on the interval $[1, 2]$ to estimate the value of α, giving your answer correct to 3 decimal places. Comment on the suitability of the method used in this case. [3]

Taking $x_1 = 1.8$, where x_1 is the initial approximation of α, use the Newton-Raphson method to obtain a sequence of approximations for α, giving your answer correct to 3 decimal places. You should demonstrate that the root is found to the required degree of accuracy. [3]

With the aid of a sketch, explain why any initial approximation x_1 such that $\frac{\pi}{2} < x_1 < \alpha$ will produce a sequence converging to α whereas some approximations x_1 such that $\alpha < x_1 < \frac{3\pi}{2}$ will not converge to α. [3]

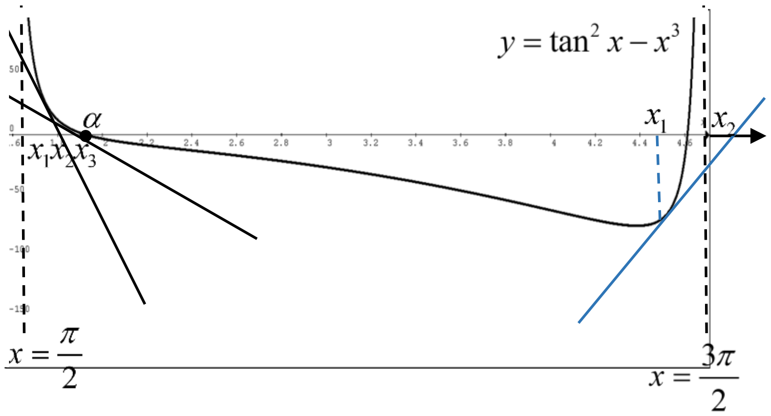
HCI et al/FM/2018/P1/3

[Solution]



From the sketch, the graphs of $y = \tan^2 x$ and $y = x^3$ intersect at exactly two points in $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Therefore, the equation $\tan^2 x - x^3 = 0$ has exactly two roots in the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

Let $f(x) = \tan^2 x - x^3$.

	$x_1 = \dfrac{f(2) - 2f(1)}{f(2) - f(1)} = 1.306$ This method is not suitable on [1, 2] because there is a vertical asymptote at $x = \dfrac{\pi}{2}$. Taking $x_1 = 1.8$, by Newton-Raphson method, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$, where $f'(x) = 2\tan x \sec^2 x - 3x^2$ $= 1.87134$ $x_3 = 1.91718$ $x_4 = 1.92783$ $x_5 = 1.92823$ f(1.9275) = 0.04 > 0 f(1.9285) = –0.015 < 0 $\Rightarrow 1.9275 < \alpha < 1.9285$ $\therefore \alpha = 1.928$ (to 3 d.p.)  From the sketch, the minimum point is at $x = 4.39$, thus it is clear from the graph that any initial approximation x_1 such that $\dfrac{\pi}{2} < x_1 < \alpha$ will produce a sequence converging to the positive root α. However an initial approximation x_1, where $4.39 < x < \dfrac{3\pi}{2}$ may result in an approximation that is greater than $\dfrac{3\pi}{2}$, a value that is much further away from α.

9 The function f is such that $\mathrm{f}(x) = 2\cos x - \mathrm{e}^{-x}$.

(i) Matthias concludes that the equation $\mathrm{f}(x) = 0$ has no roots in the interval $[1,5]$. Explain how Matthias may have arrived at this conclusion and draw a sketch to illustrate why he is wrong. [2]

The equation $2\cos x - \mathrm{e}^{-x} = 0$ has a root α in the interval $[1,2]$.

(ii) Alice uses linear interpolation twice on the interval $[1,2]$ to find an approximation to α. Find the approximation to α given by this method, giving your answer to 2 decimal places. Without any further interpolation, demonstrate how Alice can verify its correctness to 2 decimal places. [4]

(iii) Tabitha decides to use the Newton-Raphson method to find α with an initial approximation of $\alpha_1 = 1$, terminating the process when she has found two successive iterates that are equal when rounded to 5 decimal places. State the value of each of the iterates calculated correct to 5 decimal places. [3]

Buoyed by her success, Tabitha decided to try and find other roots of the equation by using other starting points. Knowing that there is another root β in the interval $[3,7]$, she decided to apply the Newton-Raphson method to find β with an initial approximation of $\beta_1 = 3$. Describe what she will discover and with an aid of a sketch, explain her discovery briefly. [2]

(iv) Henry decided to use the following recurrence relation to find the roots of $y = \mathrm{f}(x)$:

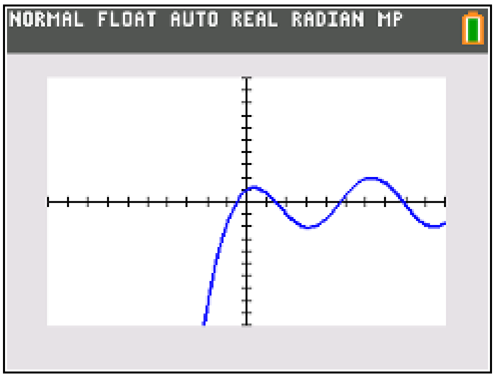
$$x_{n+1} = -\ln(2\cos x_n) \text{ for } n \geq 1.$$

Using $x_1 = 1.5$ as the starting point, calculate his obtained output for x_2 and explain why he will fail when attempting to calculate x_3. [2]

Using $x_1 = 1$ as the starting point, state the output correct to 2 decimal places. With an aid of a sketch, explain briefly the recursive process that led to this output. [2]

ACJC/FM/2019/P1/9

[Solution]

<table>
<tr><td>9(i)</td><td>


From the sketch above, it is evident that we can find a and b such that there are 2 roots in the interval $[a,b]$, with $\mathrm{f}(a)\mathrm{f}(\mathrm{b})>0$.
</td></tr>
<tr><td>9(ii)</td><td>
Note that $\mathrm{f}(1)=0.71273>0$, $\mathrm{f}(2)=-0.96763<0$

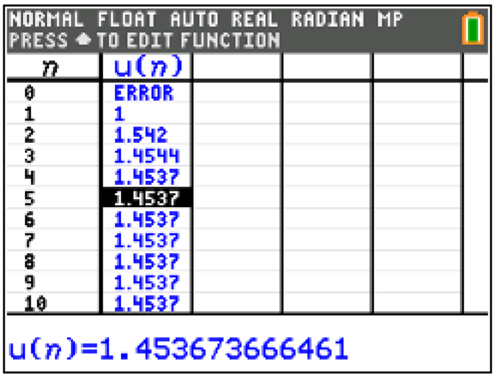
$$x_1=\frac{2\mathrm{f}(1)-\mathrm{f}(2)}{\mathrm{f}(1)-\mathrm{f}(2)}=1.42415$$

Since $\mathrm{f}(x_1)>0$, consider $[x_1,2]$.

$$x_2=\frac{2\mathrm{f}(x_1)-\mathrm{f}(2)}{\mathrm{f}(x_1)-\mathrm{f}(2)}=1.45327$$

Therefore, approximation to α is 1.45 (2 d.p.).
To verify: $\mathrm{f}(1.455)=-0.00232<0$
Hence, α to 2 d.p. should be 1.45.
</td></tr>
<tr><td>9(iii)</td><td>
$\mathrm{f}'(x)=-2\sin x+\mathrm{e}^{-x}$

$$\alpha_{n+1}=\alpha_n-\frac{\mathrm{f}(\alpha_n)}{\mathrm{f}'(\alpha_n)}=\alpha_n-\frac{2\cos\alpha_n-\mathrm{e}^{-\alpha_n}}{-2\sin\alpha_n+\mathrm{e}^{-\alpha_n}}$$



$\alpha_1=1, \alpha_2=1.54197, \alpha_3=1.45441, \alpha_4=1.45367, \alpha_5=1.45367$

Tabitha will discover that using $\beta_1=3$ will obtain the negative root of -0.53979 instead.
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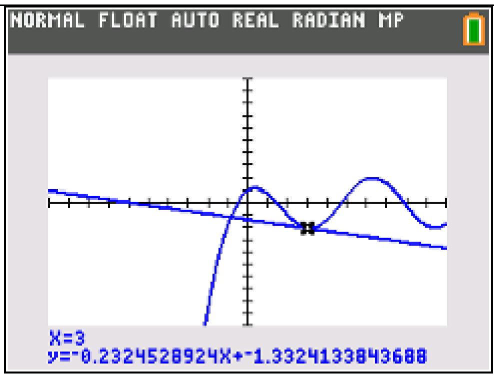
</table>



The tangent at $x=3$ to the graph of $y=\mathrm{f}\left(x\right)$ has an x-intercept that is negative. Subsequent iterations using the Newton-Raphson method will lead to the convergence to the negative root instead.

9(iv) $x_2=-\ln\left(2\cos 1.5\right)=1.95564$

Since $1.95564>\dfrac{\pi}{2}$, $\cos x_2<0$,

hence $\ln\left(2\cos x_2\right)$ will be undefined.

For $x_1=1$,

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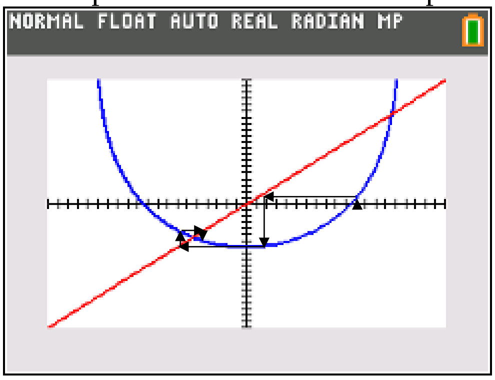
n	v(n)
8	⁻0.527
9	⁻0.547
10	⁻0.535
11	⁻0.542
12	⁻0.538
13	⁻0.541
14	⁻0.539
15	⁻0.54
16	⁻0.54
17	⁻0.54
18	⁻0.54

v(n)=⁻0.53991031538082

the output is -0.54 to 2 decimal places.



10 Show that the equation $2x^5-\sqrt{3}x^4+32x-16\sqrt{3}=0$ has a root α between 0 and 1. [1]

Use the Newton-Raphson method with $\mathrm{f}(x)=2x^5-\sqrt{3}x^4+32x-16\sqrt{3}$, to find α correct to 4 decimal places. [3]

Suggest an integer k such that α can be expressed in the form $\dfrac{\sqrt{k}}{2}$. Without the use of a calculator, find the other roots of the equation $\mathrm{f}(x)=0$. Leave your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi<\theta\le\pi$. [4]

On an Argand diagram, the point P_0 represents the root α and the points P_k $(k=1,2,3,4)$, taken in the anti-clockwise direction starting from P_0, represent the remaining roots.

Find the exact area of the pentagon $P_0P_1P_2P_3P_4$. [3]

VJC/FM/2018/P2/4

[Solution]

$\mathrm{f}(x)=2x^5-\sqrt{3}x^4+32x-16\sqrt{3}$

$\mathrm{f}(0)=-16\sqrt{3}<0$

$\mathrm{f}(1)=2-\sqrt{3}+32-16\sqrt{3}=4.555...>0$

Since f is a continuous function and there is a change in sign in the interval (0,1), there is a root between 0 and 1.(shown)

Using Newton-Raphson formula, $x_{n+1}=x_n-\dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$, where $\mathrm{f}'(x)=10x^4-4\sqrt{3}x^3+32$:

$x_0=1$

$x_1=0.870119$

$x_2=0.866028$

$x_3=0.866025$

$$\left.\begin{aligned}\mathrm{f}(0.86595)&=-0.00250<0\\ \mathrm{f}(0.86605)&=8.15\times10^{-4}>0\end{aligned}\right\}\therefore 0.86595<\alpha<86605$$

$\therefore\alpha$ is 0.8660 (correct to 4 dp.)

Given $\alpha=\dfrac{\sqrt{k}}{2}$, $k=3$.

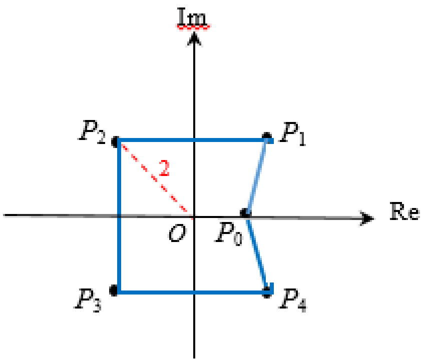
$\mathrm{f}(x)=2x^5-\sqrt{3}x^4+32x-16\sqrt{3}=(2x-\sqrt{3})(x^4+16)$

$x^4+16=0$

$x^4=-16=2^4\mathrm{e}^{\mathrm{i}\pi}$

$x=2\mathrm{e}^{\mathrm{i}\left(\frac{2k+1}{4}\right)\pi},\ k=0,\pm1,-2$

$=2\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}},\ 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},\ 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}},\ 2\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$



Area of pentagon $=\left(2\sqrt{2}\right)\left(2\sqrt{2}\right)-\dfrac{1}{2}\left(2\sqrt{2}\right)\left(\sqrt{2}-\dfrac{\sqrt{3}}{2}\right)=6+\dfrac{\sqrt{6}}{2}$ units^2

Answers

1 $\frac{16}{79}$; 2.13

2 (a) -0.2364 (b) (iii) $\alpha = 0.28917$

3 (a) $x_1 \approx -0.146,\ x_2 \approx -0.0418,\ x_3 \approx -0.00480,\ x_4 \approx -0.00008$ (b) $\alpha \approx 0.240$

4 (ii) $\alpha = 3.046$ (iii) 2.995

5 (ii) 3 (iv) 3.32

6 (ii) (a) $x_1 = 0.6931,\ x_2 = 0.3665$ (4 d.p.)

(b) $\mathrm{F}(x) = \mathrm{e}^{-x}$; $\alpha = 0.567$ (3 d.p.)

7 (ii) 2.19 (iii) underestimate

8 1.306; $\alpha = 1.928$ (to 3 d.p.)

9 **(ii)** 1.45 **(iii)** $\alpha_4 = 1.45367, \alpha_5 = 1.45367$ (iv) $x_2 = 1.95564$ (failed); -0.54 (2 d.p)

10 $\alpha = 0.8660$ (4 dp.); k (omitted), $x = 2\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}},\ 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},\ 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}},\ 2\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$; area = $6 + \frac{\sqrt{6}}{2}$ units^2