H2 Mathematics (9758)
Chapter 10 Integration Techniques
Discussion Questions (Solutions)

Level 1

Integration – Reverse of Differentiation

1 Find $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\theta\cos\theta)$. Hence, find $\displaystyle\int \theta\sin\theta\ \mathrm{d}\theta$.

	$\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\theta\cos\theta)=\cos\theta-\theta\sin\theta$ $\displaystyle\int \cos\theta-\theta\sin\theta\ \mathrm{d}\theta=\theta\cos\theta+C$ Integration is reverse of differentiation $\displaystyle\int \cos\theta\ \mathrm{d}\theta-\int\theta\sin\theta\ \mathrm{d}\theta=\theta\cos\theta+C$ $\displaystyle\int \theta\sin\theta\ \mathrm{d}\theta=\int\cos\theta\ \mathrm{d}\theta-\theta\cos\theta-C$ $=\sin\theta-\theta\cos\theta+C'$, where $C'=-C$

Integration of Standard Functions

2 Find the following integrals:

(a) $\displaystyle\int\left(2\sqrt{\mathrm{e}^x}+3\mathrm{e}^{5-3x}\right)\mathrm{d}x$

(b) $\displaystyle\int_k^1\left(1+\frac{2}{x}\right)^2\mathrm{d}x$, $k>0$

(c) $\displaystyle\int\frac{(2x-5)(x+2)}{\sqrt{x}}\mathrm{d}x$

Q2	Solutions
(a)	$\displaystyle\int(2\sqrt{\mathrm{e}^x}+3\mathrm{e}^{5-3x})\ \mathrm{d}x$ $\displaystyle=\int(2\mathrm{e}^{\frac{1}{2}x}+3\mathrm{e}^{5-3x})\ \mathrm{d}x$ $=\dfrac{2\mathrm{e}^{\frac{1}{2}x}}{\frac{1}{2}}+\dfrac{3\mathrm{e}^{5-3x}}{-3}+C$ $=4\mathrm{e}^{\frac{x}{2}}-\mathrm{e}^{5-3x}+C$

(b)	Need to expand first before integrating term by term. $\int_k^1 \left(1+\frac{2}{x}\right)^2 \mathrm{d}x = \int_k^1 \left(1+\frac{4}{x}+\frac{4}{x^2}\right)\mathrm{d}x$ $= \left[x + 4\ln\lvert x\rvert - \frac{4}{x}\right]_k^1$ $= (1-4) - \left(k + 4\ln\lvert k\rvert - \frac{4}{k}\right)$ $= \frac{4}{k} - k - 4\ln k - 3 \qquad (k > 0)$
(c)	$\int \frac{(2x-5)(x+2)}{\sqrt{x}}\mathrm{d}x$ $= \int \frac{2x^2 - x - 10}{\sqrt{x}}\mathrm{d}x$ $= \int 2x^{\frac{3}{2}} - x^{\frac{1}{2}} - 10x^{-\frac{1}{2}}\mathrm{d}x$ $= \frac{4}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 20\sqrt{x} + C$

Integration involving the function and its derivative

Formula to memorise (not in MF27) and apply

(1) For $n \in \mathbb{R},\ n \neq -1,\ \int \mathrm{f}'(x)[\mathrm{f}(x)]^n \, \mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + C$

(2) For $n \in \mathbb{R}, n = -1, \int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^{-1} \mathrm{d}x = \int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x = \ln\left|\mathrm{f}(x)\right| + C$

(3) $\int \mathrm{f}'(x) \mathrm{e}^{\mathrm{f}(x)} \, \mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + C$

3 Find the following integrals:

(a) $\int x\sqrt{3-7x^2} \, \mathrm{d}x$

(b) $\int \frac{x}{2x^2-6} \, \mathrm{d}x$

(c) $\int x^2 \mathrm{e}^{x^3+1} \mathrm{d}x$

Q3	**Solution**
(a)	$\int x\sqrt{3-7x^2} \, \mathrm{d}x$ $= -\frac{1}{14}\int -14x\left(3-7x^2\right)^{\frac{1}{2}} \, \mathrm{d}x$ $= \left(-\frac{1}{14}\right)\frac{\left(3-7x^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + C$ $= -\frac{1}{21}\left(3-7x^2\right)^{\frac{3}{2}} + C$
(b)	$\int \frac{x}{2x^2-6} \, \mathrm{d}x$ $= \frac{1}{4}\int \frac{4x}{2x^2-6} \, \mathrm{d}x$ $= \frac{1}{4}\ln\left|2x^2-6\right| + C$ **Note:** 1. $\frac{x}{2x^2-6}$ is a proper function 2. $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x = \ln\left|\mathrm{f}(x)\right| + C,\ C \in \mathbb{R}$ Remember to put modulus
(c)	$\int x^2 \mathrm{e}^{x^3+1} \mathrm{d}x$ $= \frac{1}{3}\int 3x^2 \mathrm{e}^{x^3+1} \mathrm{d}x$ $= \frac{1}{3}\mathrm{e}^{x^3+1} + C$

Integration of Rational Algebraic Functions (including MF27)

4 Find the following integrals:

(a) $\displaystyle\int \frac{1}{3-4t^2}\,\mathrm{d}t$ **(b)** $\displaystyle\int \frac{1}{(x+3)(x+4)}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{10}{x^2-2x+11}\,\mathrm{d}x$

Q4	Solutions
(a)	$\displaystyle\int \frac{1}{3-4t^2}\,\mathrm{d}t$ $\displaystyle=\int \frac{1}{\left(\sqrt{3}\right)^2-(2t)^2}\,\mathrm{d}t$ $\displaystyle=\frac{1}{2\sqrt{3}}\left(\frac{1}{2}\right)\left[\ln\left\|\frac{\sqrt{3}+2t}{\sqrt{3}-2t}\right\|\right]+C$ $\displaystyle=\frac{\sqrt{3}}{12}\ln\left\|\frac{\sqrt{3}+2t}{\sqrt{3}-2t}\right\|+C$
(b)	$\displaystyle\int \frac{1}{(x+3)(x+4)}\,\mathrm{d}x$ $\displaystyle=\int \frac{1}{(x+3)}-\frac{1}{(x+4)}\,\mathrm{d}x$ $=\ln\|x+3\|-\ln\|x+4\|+C$ $\displaystyle=\ln\left\|\frac{x+3}{x+4}\right\|+C$
(c)	$\displaystyle\int \frac{10}{x^2-2x+11}\,\mathrm{d}x$ $\displaystyle=10\int \frac{1}{(x-1)^2+10}\,\mathrm{d}x$ $\displaystyle=\frac{10}{\sqrt{10}}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right)+C$ $\displaystyle=\sqrt{10}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right)+C$

Integration of Trigonometric Functions

5 **(a)** $\int \sin^3 x \cos x \, \mathrm{d}x$ **(b)** $\int \sin^2 x \, \mathrm{d}x$

Q5	Solutions
(a)	$\int \sin^3 x \cos x \, \mathrm{d}x$ $= \frac{\sin^4 x}{4} + C$ $\mathrm{f}(x) = \sin x$ $\mathrm{f}'(x) = \cos x$
(b)	$\int \sin^2 x \, \mathrm{d}x$ $= \frac{1}{2}\int (1-\cos 2x) \, \mathrm{d}x$ $= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$ Use double angle formula $\cos 2A = 1 - 2\sin^2 A$

Integration by substitution

6 Using the substitution $u=\sqrt{x}$ to find $\int \frac{1}{(1-x)\sqrt{x}}\,\mathrm{d}x$.

Q6	**Solutions**
(a)	$u=\sqrt{x}$ $\Rightarrow u^2=x$ $\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=2u$ $\int \frac{1}{(1-x)\sqrt{x}}\,\mathrm{d}x$ $=\int \frac{1}{(1-u^2)u}\cdot 2u\,\mathrm{du}$ $=\int \frac{2}{(1-u^2)}\,\mathrm{du}$ $=\frac{2}{2}\ln\left\lvert\frac{1+u}{1-u}\right\rvert+C$ $=\ln\left\lvert\frac{1+\sqrt{x}}{1-\sqrt{x}}\right\rvert+C$

Integration by Parts

7 Find the following integrals:

(a) $\int (x+1)\mathrm{e}^{-x}\,\mathrm{d}x$

(b) $\int x\sin 2x\,\mathrm{d}x$

(c) $\int_1^{\mathrm{e}} x\ln x\,\mathrm{d}x$

Q7	Solutions
(a)	$\int (x+1)\mathrm{e}^{-x}\,\mathrm{d}x$ $= -\mathrm{e}^{-x}(x+1) + \int \mathrm{e}^{-x}\,\mathrm{d}x$ $= -\mathrm{e}^{-x}(x+1) - \mathrm{e}^{-x} + C$ $= -\dfrac{x+2}{\mathrm{e}^x} + C$
(b)	$\int x\sin 2x\,\mathrm{d}x$ $= x\left(-\dfrac{1}{2}\cos 2x\right) + \dfrac{1}{2}\int \cos 2x\,\mathrm{d}x$ $= -\dfrac{1}{2}x\cos 2x + \dfrac{1}{4}\sin 2x + C$ $= \dfrac{1}{4}\sin 2x - \dfrac{1}{2}x\cos 2x + C$
(c)	$\int x\ln x\,\mathrm{d}x = \dfrac{x^2}{2}\ln x - \int \dfrac{x^2}{2}\left(\dfrac{1}{x}\right)\mathrm{d}x$ **Strategy:** Consider solving the question without limits first. $= \dfrac{x^2}{2}\ln x - \int \dfrac{x}{2}\,\mathrm{d}x$ $= \dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} + C$ **Strategy:** Apply the limits only after solving the question. $\int_1^{\mathrm{e}} x\ln x\,\mathrm{d}x = \left[\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4}\right]_1^{\mathrm{e}}$ $= \left[\dfrac{\mathrm{e}^2}{2}\ln \mathrm{e} - \dfrac{\mathrm{e}^2}{4}\right] - \left[\dfrac{1}{2}\ln 1 - \dfrac{1}{4}\right]$ $= \left[\dfrac{\mathrm{e}^2}{2} - \dfrac{\mathrm{e}^2}{4}\right] - \left[0 - \dfrac{1}{4}\right]$ $= \dfrac{\mathrm{e}^2}{4} + \dfrac{1}{4}$ $= \dfrac{1}{4}\left(\mathrm{e}^2 + 1\right)$

Level 2

Integration – Reverse of Differentiation

8 Find $\frac{d}{dx}\left(x^2e^{x+1}\right)$. Hence, find $\int xe^x\left(x+2\right)dx$.

Q8	Solutions
	$\frac{d}{dx}\left(x^2e^{x+1}\right)=2xe^{x+1}+x^2e^{x+1}$ $=xe^{x+1}\left(2+x\right)$ Integration is reverse of differentiation $\int xe^{x+1}\left(x+2\right)dx=x^2e^{x+1}+C$ $e\int xe^x\left(x+2\right)dx=x^2e^{x+1}+C$ Rewrite e^{x+1} as e^1e^x and factorise the constant e. $\int xe^x\left(x+2\right)dx=\frac{1}{e}\left(x^2e^{x+1}\right)+\frac{C}{e}$ $=x^2e^x+B$ where $B=\frac{C}{e}$ **Alternative** $\int xe^x\left(x+2\right)dx=\frac{1}{e}\int xe^{x+1}\left(x+2\right)dx$ $=\frac{1}{e}\left[x^2e^{x+1}\right]+C$ $=x^2e^x+C$

Integration involving the function and its derivative

9 Find the following integrals:

(a) $\displaystyle\int \sin 2\theta \cos 2\theta \,\mathrm{d}\theta$

(b) $\displaystyle\int \mathrm{e}^{\cos\frac{x}{6}} \sin\frac{x}{6}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{1}{x(1+\ln 3x)}\,\mathrm{d}x$

(e) $\displaystyle\int \frac{\mathrm{e}^{-3x}}{\left(2-\mathrm{e}^{-3x}\right)^3}\,\mathrm{d}x$

(f) $\displaystyle\int \frac{7x+3}{7x^2+6x}\,\mathrm{d}x$

Formula to memorise (not in MF27) and apply

(1) For $n\in\mathbb{R},\ n\neq -1,\ \int \mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1}+C$

(2) For $n\in\mathbb{R},\ n=-1, \int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^{-1}\mathrm{d}x = \int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\left|\mathrm{f}(x)\right|+C$

(3) $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)}+C$

Q9	Solution
(a)	Method 1: $\int \sin 2\theta\cos 2\theta\,\mathrm{d}\theta$ $=\frac{1}{2}\int 2\cos 2\theta \sin 2\theta\,\mathrm{d}\theta$ $=\frac{1}{2}\frac{\sin^2 2\theta}{2}+C$ $=\frac{1}{4}\sin^2 2\theta + C$ $\mathrm{f}(x)=\sin 2\theta$ $\mathrm{f}'(x)=2\cos 2\theta$ Method 2: $\int \sin 2\theta\cos 2\theta\,\mathrm{d}\theta$ Use double angle formula $\sin 2A = 2\sin A\cos A$ $=\frac{1}{2}\int 2\sin 2\theta\cos 2\theta\,\mathrm{d}\theta$ $=\frac{1}{2}\int \sin 4\theta\,\mathrm{d}\theta$ $=-\frac{1}{8}\cos 4\theta + C$

(b)	$\int e^{\cos\frac{x}{6}}\sin\frac{x}{6}\,dx$ $=-6\int\left(-\frac{1}{6}\sin\frac{x}{6}\right)e^{\cos\frac{x}{6}}\,dx$ $=-6e^{\cos\frac{x}{6}}+C$ $f(x)=\cos\frac{x}{6}$ $f'(x)=-\frac{1}{6}\sin\frac{x}{6}$
(c)	$\int\frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ $=\int\left(\sin^{-1}x\right)\left(\frac{1}{\sqrt{1-x^2}}\right)dx$ $=\frac{\left(\sin^{-1}x\right)^2}{2}+C$ $f(x)=\sin^{-1}x$ $f'(x)=\frac{1}{\sqrt{1-x^2}}$
(d)	$\int\frac{1}{x(1+\ln 3x)}\,dx$ $=\int\frac{\frac{1}{x}}{1+\ln 3x}\,dx$ $=\ln\left\|1+\ln 3x\right\|+C$ $f(x)=1+\ln 3x$ $f'(x)=\frac{1}{3x}(3)=\frac{1}{x}$
(e)	$\int\frac{e^{-3x}}{\left(2-e^{-3x}\right)^3}\,dx=\int e^{-3x}\left(2-e^{-3x}\right)^{-3}dx$ $=\frac{1}{3}\int 3e^{-3x}\left(2-e^{-3x}\right)^{-3}dx$ $=\frac{1}{3}\frac{\left(2-e^{-3x}\right)^{-2}}{-2}+C$ $=-\frac{1}{6}\left(2-e^{-3x}\right)^{-2}+C$ $f(x)=2-e^{-3x}$ $f'(x)=3e^{-3x}$
(f)	$\int\frac{7x+3}{7x^2+6x}\,dx$ $=\frac{1}{2}\int\frac{14x+6}{7x^2+6x}\,dx$ $=\frac{1}{2}\ln\left\|7x^2+6x\right\|+C$ $f(x)=7x^2+6x$ $f'(x)=14x+6$

Integration of Rational Algebraic Functions (including MF27)

10 N2009/I/2

Find the exact value of p such that $\int_0^1 \frac{1}{4-x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1-p^2x^2}}\,\mathrm{d}x$. [5]

Q10	Solutions
	$\int_0^1 \frac{1}{4-x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1-p^2x^2}}\,\mathrm{d}x$

MF27

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$	$(x > a)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$

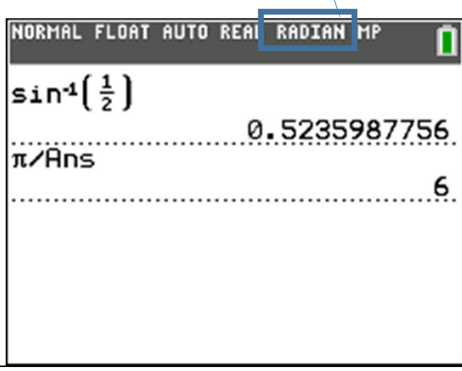
Remember to include **modulus** for ln function

$$\frac{1}{4}\left[\ln\left|\frac{2+x}{2-x}\right|\right]_0^1 = \frac{1}{p}\left[\sin^{-1}(px)\right]_0^{\frac{1}{2p}}$$

$$\frac{1}{4}\ln 3 = \frac{1}{p}\times\frac{\pi}{6}$$

$$p = \frac{2\pi}{3\ln 3}$$

Golden Rule:
Replace x by px
$\Rightarrow$ Divide by p



Let $\sin^{-1}\left(\frac{1}{2}\right) = \text{ANS}$

$\frac{\pi}{\text{ANS}} = 6$

$\Rightarrow \text{ANS} = \frac{\pi}{6}$

$\Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

11 Find the following integrals:

(a) $\displaystyle\int \frac{6x^3}{3x^2+1}\,\mathrm{d}x$ **(b)** $\displaystyle\int \frac{3x^2+2}{\sqrt{\left(x^3+2x-8\right)}}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{x+35}{x^2-25}\,\mathrm{d}x$ **(d)** $\displaystyle\int \frac{x-4}{x^2+6x+11}\,\mathrm{d}x$

Q11	Solutions
(a)	$\displaystyle\int \frac{6x^3}{3x^2+1}\,\mathrm{d}x$ Long division to make it to a proper fraction. $\displaystyle=\int 2x-\frac{2x}{3x^2+1}\,\mathrm{d}x$ $\displaystyle\int\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x=\ln\lvert\mathrm{f}(x)\rvert+C$ $\mathrm{f}(x)=3x^2+1\Rightarrow \mathrm{f}'(x)=6x$ $\displaystyle=\int 2x-\frac{1}{3}\left(\frac{6x}{3x^2+1}\right)\mathrm{d}x$ $\displaystyle=x^2-\frac{1}{3}\ln\lvert 3x^2+1\rvert+C$ You can remove modulus since $3x^2+1>0$ for all x $\displaystyle=x^2-\frac{1}{3}\ln\left(3x^2+1\right)+C$
(b)	$\displaystyle\int \frac{3x^2+2}{\sqrt{\left(x^3+2x-8\right)}}\,\mathrm{d}x$ $\displaystyle=\int\left(3x^2+2\right)\left(x^3+2x-8\right)^{-\frac{1}{2}}\,\mathrm{d}x$ $\displaystyle\int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n\,\mathrm{d}x=\frac{\left[\mathrm{f}(x)\right]^{n+1}}{n+1}+C$ $\displaystyle=\frac{\left(x^3+2x-8\right)^{\frac{1}{2}}}{\frac{1}{2}}+C$ $=2\left(x^3+2x-8\right)^{\frac{1}{2}}+C$
(c)	$\displaystyle\int \frac{x+35}{x^2-25}\,\mathrm{d}x$ $\displaystyle=\int\frac{x+35}{(x-5)(x+5)}\,\mathrm{d}x$ Partial fraction $\displaystyle=\int\frac{4}{x-5}-\frac{3}{x+5}\,\mathrm{d}x$ $=4\ln\lvert x-5\rvert-3\ln\lvert x+5\rvert+C$

	Alternative (not recommended since denominator can be factorized): $\int \frac{x+35}{x^2-25}\,dx$ $= \int \left(\frac{1}{2}\right)\left(\frac{2x}{x^2-25}\right) + \frac{35}{x^2-5^2}\,dx$ $= \frac{1}{2}\ln\left\|x^2-25\right\| + 35\left[\frac{1}{2(5)}\ln\left\|\frac{x-5}{x+5}\right\|\right] + C$ $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| + C$ $= \frac{1}{2}\ln\left\|(x-5)(x+5)\right\| + \frac{7}{2}\ln\left\|\frac{x-5}{x+5}\right\| + C$ $\int \frac{f'(x)}{f(x)}\,dx = \ln\left\|f(x)\right\| + C$
(d)	(1) For $\int \frac{px+q}{ax^2+bx+c}\,dx$, where ax^2+bx+c cannot be factorized into real linear factors: Method : Find constants A and B such that $px+q = A(2ax+b)+B$ and then use the "splitting the numerator method" $\int \frac{x-4}{x^2+6x+11}\,dx$ $= \int \frac{\frac{1}{2}(2x+6)-7}{x^2+6x+11}\,dx$ Consider $\int \frac{f'(x)}{f(x)}\,dx = \ln\left\|f(x)\right\| + C$ and MF26: $f(x) = x^2+6x+11 \Rightarrow f'(x) = 2x+6$ Rewrite: $x-4 = \frac{1}{2}(2x+6)-7$ $= \frac{1}{2}\int \frac{2x+6}{x^2+6x+11}\,dx - 7\int \frac{1}{(x+3)^2+\left(\sqrt{2}\right)^2}\,dx$ $= \frac{1}{2}\ln\left(x^2+6x+11\right) - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C$ Complete the square: $x^2+6x+11 = (x+3)^2+\left(\sqrt{2}\right)^2$ $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ Modulus not required since $x^2+6x+11>0$ for all x

12 N2014/II/2

Using partial fractions, find

$$\int_0^2 \frac{9x^2+x-13}{(2x-5)(x^2+9)}\,\mathrm{d}x$$

Give your answer in the form $a\ln b + c\tan^{-1} d$, where a, b, c and d are rational numbers to be determined. [9]

Q12	Solution

$$\frac{9x^2+x-13}{(2x-5)(x^2+9)} = \frac{3}{2x-5} + \frac{Ax+B}{x^2+9}$$

Partial fractions decomposition

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

$$9x^2+x-13 = 3(x^2+9)+(2x-5)(Ax+B)$$

when $x=0$,

$$-13 = 3(9)+(-5)(B)$$

$$B=8$$

when $x=1$,

$$9+1-13 = 3(10)+(-3)(A+8)$$

$$A=3$$

$$\frac{9x^2+x-13}{(2x-5)(x^2+9)} = \frac{3}{2x-5} + \frac{3x+8}{x^2+9}$$

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + C$$

$$\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_0^2 \frac{9x^2+x-13}{(2x-5)(x^2+9)}\,\mathrm{d}x = \int_0^2 \frac{3}{2x-5} + \frac{3x+8}{x^2+9}\,\mathrm{d}x$$

$$= \int_0^2 \frac{3}{2x-5} + \frac{3x}{x^2+9} + \frac{8}{x^2+3^2}\,\mathrm{d}x$$

$$= \left[\frac{3}{2}\ln|2x-5| + \frac{3}{2}\ln|x^2+9| + \frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^2$$

$$= \left[\frac{3}{2}\ln\left|(2x-5)(x^2+9)\right| + \frac{8}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^2$$

$$= \left[\frac{3}{2}\ln|(-1)(13)| + \frac{8}{3}\tan^{-1}\left(\frac{2}{3}\right)\right] - \left[\frac{3}{2}\ln|(-5)(9)| + \frac{8}{3}\tan^{-1}\left(\frac{0}{3}\right)\right]$$

$$= \frac{3}{2}\ln 13 - \frac{3}{2}\ln 45 + \frac{8}{3}\tan^{-1}\left(\frac{2}{3}\right) = \frac{3}{2}\ln\left(\frac{13}{45}\right) + \frac{8}{3}\tan^{-1}\left(\frac{2}{3}\right)$$

$$\therefore a = \frac{3}{2}, b = \frac{13}{45}, c = \frac{8}{3}, d = \frac{2}{3}$$

Integration of Trigonometric Functions

13 Find the following integrals:

(a) $\displaystyle\int \sec^2 x + 2\operatorname{cosec}^2\left(4x-\frac{\pi}{3}\right)\mathrm{d}x$ **(b)** $\displaystyle\int \cos^2 2x + \tan^2 2x \,\mathrm{d}x$

(c) $\displaystyle\int -\frac{1}{2}\sec\left(\frac{\pi}{6}-x\right)\tan\left(x-\frac{\pi}{6}\right)\mathrm{d}x$ **(d)** $\displaystyle\int \frac{1}{1+\cos 4x}\,\mathrm{d}x$

Q13 Solutions

(a)

$$\int \sec^2 x + 2\operatorname{cosec}^2\left(4x-\frac{\pi}{3}\right)\mathrm{d}x$$

$$= \tan x + 2\left(-\frac{1}{4}\right)\cot\left(4x-\frac{\pi}{3}\right)+C$$

$$= \tan x - \frac{1}{2}\cot\left(4x-\frac{\pi}{3}\right)+C$$

For $2\operatorname{cosec}^2\left(4x-\frac{\pi}{3}\right)$:

Replace x by $4x-\frac{\pi}{3}$

➔ Divide by 4

$\int \sec^2 x\,\mathrm{d}x = \tan x + C$

$\int \operatorname{cosec}^2 x\,\mathrm{d}x = -\cot x + C$

(b)

$$\int \cos^2 2x + \tan^2 2x \,\mathrm{d}x$$

$$= \int \frac{1+\cos 4x}{2}\,\mathrm{d}x + \int \sec^2 2x - 1\,\mathrm{d}x$$

$$= \frac{1}{2}\int 1+\cos 4x\,\mathrm{d}x + \int \sec^2 2x - 1\,\mathrm{d}x$$

$$= \frac{1}{2}\left[x+\frac{1}{4}\sin 4x\right]+\frac{1}{2}\tan 2x - x + c$$

$$= \frac{1}{8}\sin 4x + \frac{1}{2}\tan 2x - \frac{x}{2} + c$$

Use of formula in MF27:

- $\cos 2A = 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$
- $1+\tan^2 A = \sec^2 A$

For $\cos 4x$:
Replace x by $4x$
➔ Divide by 4

$\int \sec^2 x\,\mathrm{d}x = \tan x + C$

(c)

$$\int -\frac{1}{2}\sec\left(\frac{\pi}{6}-x\right)\tan\left(x-\frac{\pi}{6}\right)\mathrm{d}x$$

$$= \int -\frac{1}{2}\sec\left(\frac{\pi}{6}-x\right)\left[-\tan\left(\frac{\pi}{6}-x\right)\right]\mathrm{d}x$$

$$= \int \frac{1}{2}\sec\left(\frac{\pi}{6}-x\right)\tan\left(\frac{\pi}{6}-x\right)\mathrm{d}x$$

$$= -\frac{1}{2}\sec\left(\frac{\pi}{6}-x\right)+C$$

Angle differs by a negative sign. Make them the same: $\tan\left(x-\frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}-x\right)$

For $\sec\left(\frac{\pi}{6}-x\right)\tan\left(\frac{\pi}{6}-x\right)$:

Replace x by $\frac{\pi}{6}-x$

➔ Divide by -1

$\int \sec x \tan x\,\mathrm{d}x = \sec x + C$ (MF26)

(d)

$\int \frac{1}{1+\cos 4x} \, dx$

$= \int \frac{1}{1+\left(2\cos^2 2x - 1\right)} \, dx$

$= \int \frac{1}{2}\sec^2 2x \, dx$

$= \frac{1}{4}\tan 2x + C$

No known formula to integrate $\frac{1}{1+\cos 4x}$. Note that this is not of the form $\frac{f'(x)}{f(x)}$. Use trigo identity:

$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Replace x by $2x$
➔ Divide by 2

$\int \sec^2 x \, dx = \tan x + C$

Integration by substitution

14 Using the suggested substitution, find:

(a) $\int \tan^3 x \, \mathrm{d}x$, let $u = \tan x$

(b) $\int \frac{1}{x^2\sqrt{25-x^2}} \, \mathrm{d}x$, let $x = 5\cos\theta$

(c) $\int \frac{1}{\mathrm{e}^x + 2\mathrm{e}^{-x}} \, \mathrm{d}x$, let $u = \mathrm{e}^x$

(d) $\int_{\pi/2}^{\pi} \frac{\sin\theta}{1+\cos^2\theta} \, \mathrm{d}\theta$, let $x = \cos\theta$

Q14 Solution

(a)

$$\int \tan^3 x \, \mathrm{d}x \qquad \text{Let } u = \tan x, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x = 1 + \tan^2 x = 1 + u^2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{1+u^2}$$

$$= \int \frac{u^3}{1+u^2} \, \mathrm{d}u$$

$$= \int \left(u - \frac{u}{1+u^2} \right) \mathrm{d}u$$

> Long division to change expression into a proper fraction.

$$= \int u \, \mathrm{d}u - \frac{1}{2} \int \frac{2u}{1+u^2} \, \mathrm{d}u$$

> Apply the formula $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x = \ln|\mathrm{f}(x)| + C$

$$= \frac{u^2}{2} - \frac{1}{2} \ln|1+u^2| + C$$

$$= \frac{\tan^2 x}{2} - \frac{1}{2} \ln\left|1 + \tan^2 x\right| + C$$

> Remember to substitute back and express final answer in terms of *x*.

$$= \frac{\tan^2 x}{2} - \frac{1}{2} \ln\left|\sec^2 x\right| + C$$

$$= \frac{\tan^2 x}{2} - \frac{1}{2} \ln\left|\cos x\right|^{-2} + C$$

> $\ln\left|\sec^2 x\right| = \ln\left|\frac{1}{\cos^2 x}\right| = \ln\left|\cos x\right|^{-2} = -2\ln\left|\cos x\right|$

$$= \frac{\tan^2 x}{2} + \ln\left|\cos x\right| + C$$

(b)

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,\mathrm{d}x$$

$$= \int \frac{-5\sin\theta}{25\cos^2\theta\sqrt{25-25\cos^2\theta}}\,\mathrm{d}\theta$$

$$= \int \frac{-5\sin\theta}{(25\cos^2\theta)(5\sin\theta)}\,\mathrm{d}\theta$$

$$= -\frac{1}{25}\int \sec^2\theta\,\mathrm{d}\theta$$

$$= -\frac{1}{25}\tan\theta + c$$

$$= -\frac{1}{25}\left(\frac{\sqrt{25-x^2}}{x}\right) + c$$

$$= -\frac{\sqrt{25-x^2}}{25x} + c$$

Let $x = 5\cos\theta$

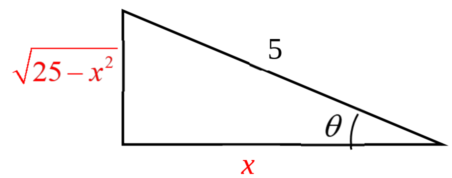
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -5\sin\theta$$

$$\sqrt{25-25\cos^2\theta} = \sqrt{25\left(1-\cos^2\theta\right)} = \sqrt{25\sin^2\theta} = 5\sin\theta$$

When the given substitution is in trigo form, use the triangle to find out the other trigo ratio.

$$x = 5\cos\theta \Rightarrow \cos\theta = \frac{x}{5}$$

$$\therefore \tan\theta = \frac{\sqrt{25-x^2}}{x}$$



Remember to substitute back and express final answer in terms of *x*.

(c)

$$\int \frac{1}{e^x + 2e^{-x}}\,\mathrm{d}x$$

$$= \int \frac{1}{u + \frac{2}{u}}\left(\frac{1}{u}\right)\mathrm{d}u$$

$$= \int \frac{1}{u^2 + 2}\,\mathrm{d}u$$

$$= \int \frac{1}{\left(\sqrt{2}\right)^2 + u^2}\,\mathrm{d}u$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{e^x}{\sqrt{2}}\right) + C$$

$$\frac{1}{u+\frac{2}{u}}\left(\frac{1}{u}\right) = \frac{1}{u\left(u+\frac{2}{u}\right)} = \frac{1}{u^2+2}$$

Steps:

(1) using $u = e^x$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = e^x = u$$

(2) Replace expression

$$\frac{1}{e^x + 2e^{-x}} = \frac{1}{u + \frac{2}{u}}$$

(3) Replace variable *u* back to original variable *x*

Remember to substitute back and express final answer in terms of *x*.

(d)	Let $x = \cos\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta$ When $\theta = \frac{\pi}{2}$, $x = \cos\frac{\pi}{2} = 0$ When $\theta = \pi$, $x = \cos\pi = -1$. $\int_{\pi/2}^{\pi} \frac{\sin\theta}{1+\cos^2\theta}\,\mathrm{d}\theta = \int_0^{-1} \frac{-1}{1+x^2}\mathrm{d}x$ $= -\left[\tan^{-1}x\right]_0^{-1}$ $= \frac{\pi}{4}$

Integration by parts

15 Find the following integrals:

(a) $\int x^2 \cos x \, \mathrm{d}x$

(b) $\int_0^{\frac{1}{\sqrt{2}}} x \sin^{-1}\left(x^2\right) \mathrm{d}x$

(c) $\int \mathrm{e}^{2x} \sin x \, \mathrm{d}x$

Q15	Solution
(a)	$\int x^2 \cos x \, \mathrm{d}x = x^2 \sin x - \int 2x \sin x \, \mathrm{d}x$ $= x^2 \sin x - 2\int x \sin x \, \mathrm{d}x$ $= x^2 \sin x - 2\left[-x\cos x - \int -\cos x \, \mathrm{d}x\right]$ $= x^2 \sin x + 2x\cos x - 2\int \cos x \, \mathrm{d}x$ $= x^2 \sin x + 2x\cos x - 2\sin x + C$ **L I A T E**: x^2 → A, $\cos x$ → T **(Keep)** x^2 **Integration by parts:** $\mathrm{K}_{\text{eep } u}\mathrm{I}_{\text{ntegrate } v} - \int \mathrm{D}_{\text{ifferentiate } u}\mathrm{I}_{\text{ntegrate } v}$
(b)	$\int x \sin^{-1}\left(x^2\right) \mathrm{d}x$ $= \frac{x^2}{2}\sin^{-1}\left(x^2\right) - \int \frac{x^2}{2} \frac{2x}{\sqrt{1-x^4}} \mathrm{d}x$ $= \frac{x^2}{2}\sin^{-1}\left(x^2\right) - \int \frac{x^3}{\sqrt{1-x^4}} \mathrm{d}x$ $= \frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{4}\int \frac{-4x^3}{\sqrt{1-x^4}} \mathrm{d}x$ $= \frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{4}\int \left(-4x^3\right)\left(1-x^4\right)^{-\frac{1}{2}} \mathrm{d}x$ $= \frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{4}\left[\frac{\left(1-x^4\right)^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$ $= \frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{2}\sqrt{1-x^4} + C$ $\int_0^{\frac{1}{\sqrt{2}}} x \sin^{-1}\left(x^2\right) \mathrm{d}x = \left[\frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{2}\sqrt{1-x^4}\right]_0^{\frac{1}{\sqrt{2}}}$ $= \frac{\pi}{24} + \frac{\sqrt{3}}{4} - \frac{1}{2}$ **L I A T E**: $\sin^{-1}\left(x^2\right)$ → I, x → A **(Keep)** $\sin^{-1}\left(x^2\right)$ **Strategy:** Consider solving the question without limits first. **Strategy:** Apply the limits only after solving the question.

(c)	$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}\cos x\,dx$	Apply integration by parts once
	$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x + \int \frac{1}{2}e^{2x}\sin x\,dx\right]$	Apply integration by parts again
	$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$	STOP when we see the required integral appearing again
	$\Rightarrow \left(1+\frac{1}{4}\right)\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) + C'$ $\Rightarrow \int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) + C$	Combine the required integral on the LHS and make the required integral the subject

Level 3

16 2009/MJC/II/1

(i) Differentiate $e^{\cos x}$ with respect to x. [1]

(ii) Find $\int e^{\cos x} \sin 2x \, dx$. [3]

Q16	Solutions
(i)	$\frac{d}{dx} e^{\cos x} = -e^{\cos x} \sin x$ (Hence $\int -e^{\cos x} \sin x \, dx = e^{\cos x} + c$ (Integration is the reverse of differentiation))
(ii)	$\int e^{\cos x} \sin 2x \, dx$ $= \int e^{\cos x} (2\sin x \cos x) \, dx$ $= \int \left(-e^{\cos x} \sin x\right)\left(-2\cos x\right) dx$ $= \left(-2\cos x\right)\left(e^{\cos x}\right) - \int \left(2\sin x\right)\left(e^{\cos x}\right) dx$ $= -2e^{\cos x} \cos x - 2\int e^{\cos x} \sin x \, dx$ $= -2e^{\cos x} \cos x - 2\left(-e^{\cos x}\right) + C$ $= -2e^{\cos x} \cos x + 2e^{\cos x} + C$ From (i), since $\frac{d}{dx} e^{\cos x} = -e^{\cos x} \sin x$, $\int -e^{\cos x} \sin x \, dx = e^{\cos x} + c$ Hence, we let $u = -2\cos x$ and $v = -e^{\cos x} \sin x$ $\frac{du}{dx} = 2\sin x$ and $\int v \, dx = e^{\cos x}$

17 N2019/II/1

You are given that $I=\int x(1-x)^{\frac{1}{2}}\,\mathrm{d}x$.

(i) Use integration by parts to find an expression for *I*. [2]

(ii) Use the substitution $u^2=1-x$ to find another expression for *I*. [2]

(iii) Show algebraically that your answers to parts **(i)** and **(ii)** differ by a constant. [2]

Q17	Solutions
(i)	$I=\int x(1-x)^{\frac{1}{2}}\,\mathrm{d}x$ $=\frac{-2x(1-x)^{\frac{3}{2}}}{3}+\int\frac{2}{3}(1-x)^{\frac{3}{2}}\,\mathrm{d}x$ $=\frac{-2x}{3}(1-x)^{\frac{3}{2}}-\frac{4(1-x)^{\frac{5}{2}}}{15}+C$ **Integration by parts:** $\mathrm{K}_{\text{eep }u}\mathrm{I}_{\text{ntegrate }v}-\int\mathrm{D}_{\text{ifferentiate }u}\mathrm{I}_{\text{ntegrate }v}$ **Special Remark:** Notice that both x and $(1-x)^{\frac{1}{2}}$ are **algebraic terms**. Next, we observe that by letting x to be the term that is to be **differentiated**, **it allows for easier integration to take places in the "Integration by parts formula"**. Hence we let x to be the term that is to be **differentiated** and $(1-x)^{\frac{1}{2}}$ to be the term that is to be **integrated.**
(ii)	$I=\int x(1-x)^{\frac{1}{2}}\,\mathrm{d}x$ $=\int(1-u^2)u(-2u)\,\mathrm{d}u$ $=2\int(u^4-u^2)\,\mathrm{d}u$ $=2\left[\frac{u^5}{5}-\frac{u^3}{3}\right]+D$ $=2\left[\frac{(1-x)^{\frac{5}{2}}}{5}-\frac{(1-x)^{\frac{3}{2}}}{3}\right]+D$ Note you **should not use the same letter** as part **(i)** to represent your arbitrary constant as in general the arbitrary constant is not the same. <u>Steps:</u> (1) using $u=(1-x)^{\frac{1}{2}}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$ $=-\frac{1}{2}\frac{1}{(1-x)^{\frac{1}{2}}}=-\frac{1}{2u}$ (2) Replace expression $u=(1-x)^{\frac{1}{2}}$ $u^2=(1-x)$ $x=1-u^2$ (3) Replace variable *u* back to original variable *x*

(iii)

$$\frac{-2x}{3}(1-x)^{\frac{3}{2}}-\frac{4(1-x)^{\frac{5}{2}}}{15}+C-\left[2\left[\frac{(1-x)^{\frac{5}{2}}}{5}-\frac{(1-x)^{\frac{3}{2}}}{3}\right]+D\right]$$

$$=\frac{-2x}{3}(1-x)^{\frac{3}{2}}-\frac{4(1-x)^{\frac{5}{2}}}{15}+C-\frac{2(1-x)^{\frac{5}{2}}}{5}+\frac{2(1-x)^{\frac{3}{2}}}{3}-D$$

$$=(1-x)^{\frac{5}{2}}\left[-\frac{4}{15}-\frac{2}{5}\right]+(1-x)^{\frac{3}{2}}\left[\frac{-2x}{3}+\frac{2}{3}\right]+C-D$$

$$=\frac{-2}{3}(1-x)^{\frac{5}{2}}+\frac{2}{3}[1-x](1-x)^{\frac{3}{2}}+C-D$$

$$=\frac{-2}{3}(1-x)^{\frac{5}{2}}+\frac{2}{3}(1-x)^{\frac{5}{2}}+C-D$$

$$=C-D=\text{constant}\quad(\text{shown})$$

Question asks to prove that answers in part **(i)** and part **(ii)** differ by a constant so you can just subtract the 2 expressions from part **(i)** and part **(ii)** **and check if it results in a constant.**

Factorise and combine the liked terms, i.e. terms with $(1-x)^{\frac{5}{2}}$ and $(1-x)^{\frac{3}{2}}$.

Combine

$[1-x](1-x)^{\frac{3}{2}}=(1-x)^{\frac{5}{2}}$

<u>Alternative Method</u>

Let $u^2=1-x$ for the answer in **(i):**

$$\frac{-2x}{3}(1-x)^{\frac{3}{2}}-\frac{4(1-x)^{\frac{5}{2}}}{15}+C-\left[2\left[\frac{(1-x)^{\frac{5}{2}}}{5}-\frac{(1-x)^{\frac{3}{2}}}{3}\right]+D\right]$$

$$=\frac{-2}{3}(1-u^2)u^3-\frac{4}{15}u^5+C-\left[2\left[\frac{u^5}{5}-\frac{u^3}{3}\right]+D\right]$$

$$=\frac{-2}{3}(u^3-u^5)-\frac{4}{15}u^5+C-\frac{2}{5}u^5+\frac{2}{3}u^3-D$$

$$=\frac{-2}{3}u^3+\frac{2}{3}u^5-\frac{4}{15}u^5+C-\frac{2}{5}u^5+\frac{2}{3}u^3-D$$

$$=C-D=\text{constant}\quad(\text{shown})$$

Observe that we can replace $u^2=1-x$ in both the expressions obtained in part **(i)** and part **(ii). This will help to simplify the expressions greatly.**

For e.g.

$(1-x)^{\frac{3}{2}}=u^3$ and $(1-x)^{\frac{5}{2}}=u^5$