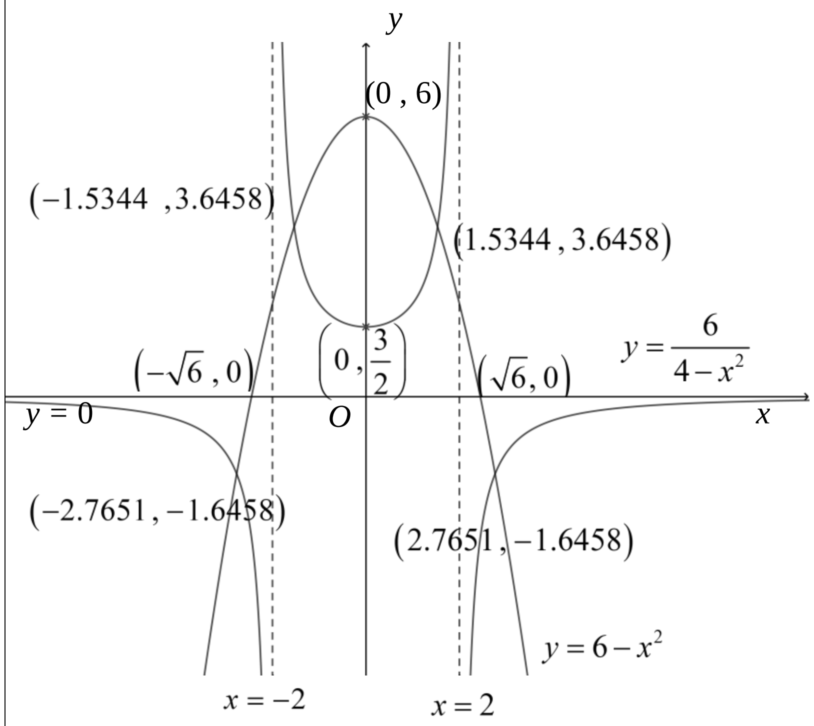
St Andrew's Junior College
2022 Preliminary Examination
H2 Mathematics Paper 1 (9758/01)

Q	Solution	Mark scheme
1(i)	$4(x+y)^2+(x-y)^2=20$ ---- (1) Differentiate (1) with respect to x $8(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)+2(x-y)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$ $8(x+y)+8(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+2(x-y)-2(x-y)\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $10x+6y+(8x+8y-2x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{10x+6y}{6x+10y}=-\frac{2(5x+3y)}{2(3x+5y)}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{5x+3y}{3x+5y}$ (Shown)	
(ii)	Since the tangents are perpendicular to the line $y=x$, hence the gradient of tangents = - 1 $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{5x+3y}{3x+5y}=-1$ $5x+3y=3x+5y$ $2x=2y$ $x=y---(*)$ Substituting into (1);	

Q	**Solution**	**Mark scheme**
	$4(x+x)^2+(x-x)^2=20$ $4(2x)^2=20$ $4x^2=5$ $x^2=\frac{5}{4}$ $x=\pm\frac{\sqrt{5}}{2}$ Given $y=x$ from (*) When $x=\frac{\sqrt{5}}{2},\ y=\frac{\sqrt{5}}{2}$ When $x=-\frac{\sqrt{5}}{2},\ y=-\frac{\sqrt{5}}{2}$ Hence, the points are $\left(\frac{\sqrt{5}}{2},\frac{\sqrt{5}}{2}\right)$ and $\left(-\frac{\sqrt{5}}{2},-\frac{\sqrt{5}}{2}\right)$ $y-\frac{\sqrt{5}}{2}=-\left(x-\frac{\sqrt{5}}{2}\right)$ $=-x+\frac{\sqrt{5}}{2}$ $y=-x+\sqrt{5}$	

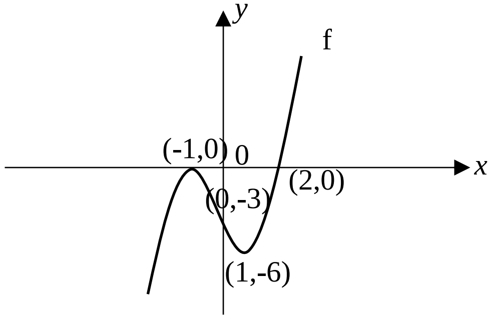
Q	Solution	Mark scheme
	$y-\left(-\frac{\sqrt{5}}{2}\right)=-\left(x-\left(-\frac{\sqrt{5}}{2}\right)\right)$ $=-x-\frac{\sqrt{5}}{2}$ $y=-x-\sqrt{5}$ The equation of the tangents are $y=-x+\sqrt{5}$ and $y=-x-\sqrt{5}$.	

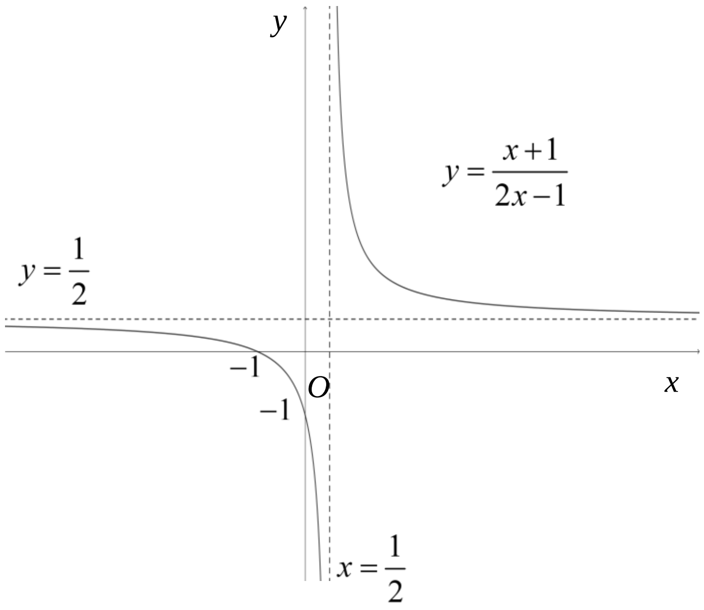
Q	**Solution**	**Mark scheme**
2(i)	$y=\frac{6}{4-x^2}=\frac{6}{(2-x)(2+x)}$ Asymptotes are $x=2, x=-2, y=0$ Intersections with axes: When x = 0, $y=\frac{6}{2(2)}=\frac{3}{2}$ (Also the stationary point) $y=6-x^2$ Intersections with axes: When x = 0, $y=6$ => (0 , 6) When y = 0, $6-x^2=0$ $x^2=6$ $x=\sqrt{6}$ or $-\sqrt{6}$ $(\sqrt{6}\,,0)$ or $(-\sqrt{6}\,,0)$	

<table>
<tr><th>Q</th><th>Solution</th><th>Mark scheme</th></tr>
<tr><td>2 (i)</td><td>
</td><td></td></tr>
<tr><td>(ii)</td><td>The x-coordinates of the intersection points between the graphs $y=\frac{6}{4-x^2}$ and $y=6-x^2$ are -2.77, -1.53, 1.53 and 2.77 (to 3 sig. fig.)
For $\frac{6}{4-x^2}<6-x^2$
From the graph above,
$-2.77<x<-2$ or $-1.53<x<1.53$ or $2<x<2.77$</td><td></td></tr>
<tr><td>(iii)</td><td>Replace x with $-x$,</td><td></td></tr>
</table>

Q	**Solution**	**Mark scheme**
	after the reflection about the *y*-axis, the solution is: $\Rightarrow 2 < x < 2.77$ or $-1.53 < x < 1.53$ or $-2.77 < x < -2$ Replace *x* with $x+4$, after the translation of 4 units in the negative *x* direction, $\Rightarrow 2 < x+4 < 2.77$ or $-1.53 < x+4 < 1.53$ or $-2.77 < x+4 < -2$ $\Rightarrow -2 < x < -1.23$ or $-5.53 < x < -2.47$ or $-6.77 < x < -6$ the solution set is therefore $\{x \in \mathbb{R} : -6.77 < x < -6 \text{ or } -5.53 < x < -2.47 \text{ or } -2 < x < -1.23\}$	

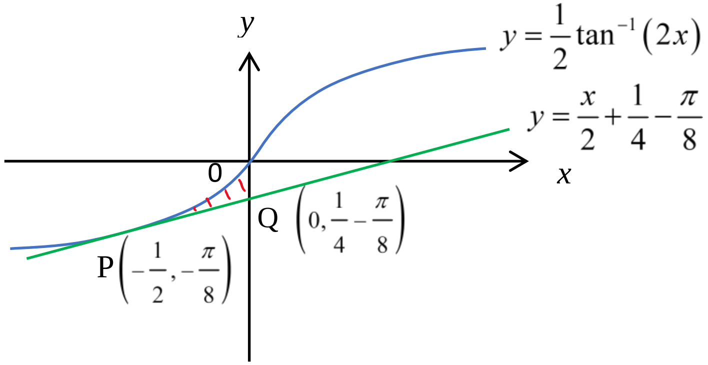
Q	Solution	Mark Scheme
3(i)	Given $\mathrm{f}\left(-1\right)=\mathrm{f}\left(2\right)=0$ but $-1\neq 2$ and $-1,2\in \mathrm{D}_{\mathrm{f}}$, f is not a one-to-one function. Hence, f does not have an inverse.	
(ii)	Let $y=\mathrm{f}(x)=ax^3+bx^2+cx+d$ Curve passes through $\left(0,-3\right)$ $d=-3$ $y=\mathrm{f}(x)=ax^3+bx^2+cx-3$ Curve passes through $\left(-1,0\right)$ $-a+b-c=3$ ----------- (1) Curve passes through $\left(2,0\right)$ $8a+4b+2c=3$ ----------- (2) $\frac{\mathrm{d}y}{\mathrm{d}x}=3ax^2+2bx+c$ Tangent to the curve at $x=1$ is a horizontal line, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $3a+2b+c=0$ ------------ (3) Solving (1), (2) and (3) using GC, $a=\frac{3}{2}$, $b=0$, $c=-\frac{9}{2}$, The equation of the curve is $y=\mathrm{f}(x)=\frac{3}{2}x^3-\frac{9}{2}x-3$	

Q	Solution	Mark Scheme
(iii)		
(iv)	Smallest $k = 1$	
(v)	The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections of each other about the line $y = x$.	
(vi)	Since the graphs $y = \mathrm{f}(x), y = \mathrm{f}^{-1}(x)$ and $y = x$ intersect at the same point, the solution of $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$ is the same as the solution of $\mathrm{f}(x) = x$. $\Rightarrow \frac{3}{2}x^3 - \frac{9}{2}x - 3 = x$ $\Rightarrow 3x^3 - 11x - 6 = 0$ (shown) Solving the equation using GC, $x = 2.14$(3 s.f.) since $x \geq 1$	
(vii)	$\mathrm{R_f} = [-6, \infty) \not\subseteq \mathrm{D_g} = (-5, \infty)$ Hence gf does not exist	

Q	Solution	Mark Scheme
4 (i)		
	From the graph above, $x \le -1$ or $x > \frac{1}{2}$	

Q	Solution	Mark Scheme
(ii)	$\int_{-2}^{0}\left\|\frac{x+1}{2x-1}\right\|\ \mathrm{d}x$ $=\int_{-2}^{0}\left\|\frac{1}{2}+\frac{3}{2(2x-1)}\right\|\ \mathrm{d}x$ $=\int_{-2}^{-1}\left(\frac{1}{2}+\frac{3}{2(2x-1)}\right)\ \mathrm{d}x+\int_{-1}^{0}-\left(\frac{1}{2}+\frac{3}{2(2x-1)}\right)\ \mathrm{d}x$ $=\left[\frac{1}{2}x+\frac{3}{4}\ln\|2x-1\|\right]_{-2}^{-1}-\left[\frac{1}{2}x+\frac{3}{4}\ln\|2x-1\|\right]_{-1}^{0}$ $=\left[-\frac{1}{2}+\frac{3}{4}\ln(3)-\left(-1+\frac{3}{4}\ln(5)\right)\right]-\left[0-\left(-\frac{1}{2}+\frac{3}{4}\ln 3\right)\right]$ $=-\frac{1}{2}+\frac{3}{4}\ln 3-\left(-1+\frac{3}{4}\ln(5)\right)-\frac{1}{2}+\frac{3}{4}\ln 3$ $=-1+\frac{3}{2}\ln 3+1-\frac{3}{4}\ln 5$ $=\frac{3}{2}\ln 3-\frac{3}{4}\ln 5$ (Shown)	

Q	Solution	Mark Scheme
5(i)	When $x=-\frac{1}{2}$, $y=\frac{1}{2}\tan^{-1}(-1)=\frac{1}{2}\left(-\frac{\pi}{4}\right)=-\frac{\pi}{8}$. When $x=0$, $y=\frac{0}{2}+\frac{1}{4}-\frac{\pi}{8}=\frac{1}{4}-\frac{\pi}{8}$ y-coordinate of P = $-\frac{\pi}{8}$; y-coordinate of Q = $\frac{1}{4}-\frac{\pi}{8}$	

(ii)	 $y=\frac{1}{2}\tan^{-1}(2x)$ $2y=\tan^{-1}(2x)$ $\tan(2y)=2x$ $x=\frac{1}{2}\tan(2y)$ Required volume	

	$= \pi \int_{-\frac{\pi}{8}}^{0} \left(\frac{1}{2}\tan(2y)\right)^2 \mathrm{d}y - \frac{\pi}{3}\left(\frac{1}{2}\right)^2 \left[\frac{1}{4} - \frac{\pi}{8} - \left(-\frac{\pi}{8}\right)\right]$ $= \frac{\pi}{4} \int_{-\frac{\pi}{8}}^{0} \tan^2(2y) \mathrm{d}y - \frac{\pi}{48}$ $= \frac{\pi}{4} \int_{-\frac{\pi}{8}}^{0} \left(\sec^2(2y) - 1\right) \mathrm{d}y - \frac{\pi}{48}$ $= \frac{\pi}{4} \left[\frac{1}{2}\tan(2y) - y\right]_{-\frac{\pi}{8}}^{0} - \frac{\pi}{48}$ $= \frac{\pi}{4} \left[0 - \left(-\frac{1}{2} + \frac{\pi}{8}\right)\right] - \frac{\pi}{48}$ $= \frac{\pi}{4} \left(\frac{1}{2} - \frac{\pi}{8}\right) - \frac{\pi}{48}$ $= \frac{\pi}{8} \left(\frac{5}{6} - \frac{\pi}{4}\right)$ units^3 Alternatively (more tedious mtd) : Required volume	

$$= \pi\int_{-\frac{\pi}{8}}^{0}\left(\frac{1}{2}\tan\left(2y\right)\right)^{2}\mathrm{d}y-\pi\int_{-\frac{\pi}{8}}^{\frac{1}{4}-\frac{\pi}{8}}4\left(y-\left(\frac{1}{4}-\frac{\pi}{8}\right)\right)^{2}\mathrm{d}y$$

$$= \frac{\pi}{4}\int_{-\frac{\pi}{8}}^{0}\tan^{2}\left(2y\right)\mathrm{d}y-4\pi\int_{-\frac{\pi}{8}}^{\frac{1}{4}-\frac{\pi}{8}}\left(y-\left(\frac{1}{4}-\frac{\pi}{8}\right)\right)^{2}\mathrm{d}y$$

$$= \frac{\pi}{4}\int_{-\frac{\pi}{8}}^{0}\left(\sec^{2}\left(2y\right)-1\right)\,\mathrm{d}y-4\pi\int_{-\frac{\pi}{8}}^{\frac{1}{4}-\frac{\pi}{8}}\left(y-\left(\frac{1}{4}-\frac{\pi}{8}\right)\right)^{2}\mathrm{d}y$$

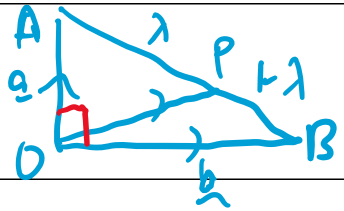
$$= \frac{\pi}{4}\left[\frac{1}{2}\tan\left(2y\right)-y\right]_{-\frac{\pi}{8}}^{0}-4\pi\left[\frac{\left(y-\left(\frac{1}{4}-\frac{\pi}{8}\right)\right)^{3}}{3}\right]_{-\frac{\pi}{8}}^{\frac{1}{4}-\frac{\pi}{8}}$$

$$= \frac{\pi}{4}\left[0-\left(-\frac{1}{2}+\frac{\pi}{8}\right)\right]-\frac{4}{3}\pi\left[\left(\frac{1}{4}-\frac{\pi}{8}-\left(\frac{1}{4}-\frac{\pi}{8}\right)\right)^{3}-\left(-\frac{\pi}{8}-\left(\frac{1}{4}-\frac{\pi}{8}\right)\right)^{3}\right]$$

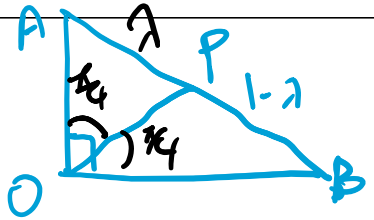
$$= \frac{\pi}{4}\left(\frac{1}{2}-\frac{\pi}{8}\right)-\frac{4}{3}\pi\left[\left(\frac{1}{4}\right)^{3}\right]$$

$$= \frac{\pi}{4}\left(\frac{1}{2}-\frac{\pi}{8}\right)-\frac{\pi}{48}$$

$$= \frac{\pi}{8}\left(\frac{5}{6}-\frac{\pi}{4}\right)\ \text{units}^{3}$$

6(i)	Using Ratio Theorem, $\overrightarrow{OP}=\frac{(1-\lambda)\mathbf{a}+\lambda\mathbf{b}}{1-\lambda+\lambda}$ $=(1-\lambda)\mathbf{a}+\lambda\mathbf{b}$ 	
	$\cos(\angle AOP)=\frac{\overrightarrow{OA}\bullet\overrightarrow{OP}}{\lvert\overrightarrow{OA}\rvert\lvert\overrightarrow{OP}\rvert}$ $=\frac{\mathbf{a}\bullet\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]}{\lvert\mathbf{a}\rvert\lvert(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\rvert}$ $=\frac{(1-\lambda)\mathbf{a}\bullet\mathbf{a}+\lambda\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\rvert}$ $=\frac{(1-\lambda)\lvert\mathbf{a}\rvert^2+0}{\lvert\mathbf{a}\rvert\lvert(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\rvert}$ since **a•b**=0 as **a** and **b** are perpendicular $=\frac{(1-\lambda)\lvert\mathbf{a}\rvert}{\lvert(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\rvert}$ (shown)	
(ii)	$\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]\bullet\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]$ $=(1-\lambda)\mathbf{a}\bullet(1-\lambda)\mathbf{a}+(1-\lambda)\mathbf{a}\bullet\lambda\mathbf{b}+\lambda\mathbf{b}\bullet(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\bullet\lambda\mathbf{b}$ $=(1-\lambda)^2\lvert\mathbf{a}\rvert^2+\lambda(1-\lambda)\mathbf{a}\bullet\mathbf{b}+\lambda(1-\lambda)\mathbf{b}\bullet\mathbf{a}+\lambda^2\lvert\mathbf{b}\rvert^2$ $=(1-\lambda)^2\lvert\mathbf{a}\rvert^2+\lambda^2\lvert\mathbf{b}\rvert^2$, since $\mathbf{a}\bullet\mathbf{b}=0$ given that **a** and **b** are perpendicular (Proven)	

Given also that OP bisects $\angle AOB$,



$$\angle AOP = \frac{\pi}{4},$$

$$\cos\frac{\pi}{4} = \frac{(1-\lambda)|\mathbf{a}|}{|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}|}$$

$$\frac{1}{\sqrt{2}} = \frac{(1-\lambda)|\mathbf{a}|}{|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}|}$$

$$\frac{1}{2} = \frac{(1-\lambda)^2|\mathbf{a}|^2}{|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}|^2}$$

$$= \frac{(1-\lambda)^2|\mathbf{a}|^2}{[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}]\bullet[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}]}$$

$$= \frac{(1-\lambda)^2|\mathbf{a}|^2}{(1-\lambda)^2|\mathbf{a}|^2+\lambda^2|\mathbf{b}|^2}$$

$$(1-\lambda)^2|\mathbf{a}|^2+\lambda^2|\mathbf{b}|^2 = 2(1-\lambda)^2|\mathbf{a}|^2$$

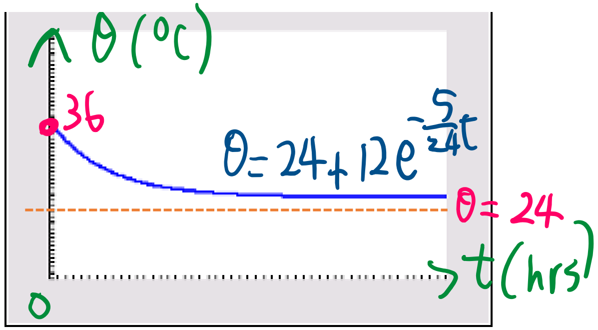
Hence $(1-\lambda)^2|\mathbf{a}|^2 = \lambda^2|\mathbf{b}|^2$

$$\frac{|\mathbf{a}|^2}{|\mathbf{b}|^2} = \frac{\lambda^2}{(1-\lambda)^2}$$

$$\frac{|\mathbf{a}|}{|\mathbf{b}|} = \left|\frac{\lambda}{(1-\lambda)}\right|$$

$\dfrac{|\mathbf{a}|}{|\mathbf{b}|} = \pm\dfrac{\lambda}{1-\lambda} = \dfrac{\lambda}{1-\lambda}$, reject $-\dfrac{\lambda}{1-\lambda}$ since $0<\lambda<1$ and ratio of length is positive.

7(i)	The rate of temperature change of a dead animal body is given by $\frac{d\theta}{dt} = -a(\theta - \theta_0)$, where $a > 0$. $\frac{1}{(\theta - \theta_0)} \frac{d\theta}{dt} = -a$ Integrating both sides $\int \frac{1}{\theta - \theta_0} \, d\theta = -a \, dt$ $\ln(\theta - \theta_0) = -at + C$, since $\theta - \theta_0 > 0$ where a and C are arbitrary constants. $\theta - \theta_0 = e^{-at+C} = Ae^{-kt}$, where $A = e^C$ and $k = a$ $\Rightarrow \theta = \theta_0 + Ae^{-kt}$ (Shown)	
(ii)	$\theta_0 = 24$. When t=0 , θ = 36 is $\frac{d\theta}{dt}$ = -2.5 ^{0}C, $\frac{d\theta}{dt} = -k(\theta - \theta_0)$, where $k > 0$. $-2.5 = -k$ (36-24) $\therefore k = \frac{5}{24}$ Using $\theta = \theta_0 + Ae^{-kt}$ 36 = 24 + A $\therefore A = 12$	

iii	 $\theta = 24 + 12e^{-\frac{5}{24}t}$	
iv	A constant rate of decrease is not possible as the temperature of the body of the dead animal will become $0°\text{C}$ or even negative at some point in time, which is lower than the surrounding temperature.	
8(a)	Since the equation has all real coefficients, complex roots occur in complex conjugate pairs. Since $z = \frac{5}{3} - \frac{\sqrt{11}}{3}i$ is a complex root $\Rightarrow$ its conjugate $z = \frac{5}{3} + \frac{\sqrt{11}}{3}i$ exists as a root of the equation.	

	$(z-(-2))(z-\left(\frac{5}{3}-\frac{\sqrt{11}}{3}i\right))(z-\left(\frac{5}{3}+\frac{\sqrt{11}}{3}i\right))=0$ $(z+2)(z^2-\frac{10}{3}z+4)=0---(*)$ $z^3-\frac{4}{3}z^2-\frac{8}{3}z+8=0$ $\Rightarrow 3z^3-4z^2-8z+24=0---(\#)$ $\therefore a=-4, b=-8, c=24$	
8b (i)	$w=\frac{e^{i\theta}+e^{i\phi}}{e^{i\theta}-e^{i\phi}}$ $=\frac{e^{i\left(\frac{\theta+\phi}{2}\right)}}{e^{i\left(\frac{\theta+\phi}{2}\right)}}\times\frac{e^{i\left(\frac{\theta-\phi}{2}\right)}+e^{-i\left(\frac{\theta-\phi}{2}\right)}}{e^{i\left(\frac{\theta-\phi}{2}\right)}-e^{-i\left(\frac{\theta-\phi}{2}\right)}}$ $=\frac{2\cos\left(\frac{\theta-\phi}{2}\right)}{2i\sin\left(\frac{\theta-\phi}{2}\right)}$ $=\frac{1}{i}\cot\left(\frac{\theta-\phi}{2}\right)$ $=-i\cot\left(\frac{\theta-\phi}{2}\right)$ $=e^{-i\frac{\pi}{2}}\cot\left(\frac{\theta-\phi}{2}\right)$	

Alternative method:

$$w=\frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\phi}}{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\phi}}$$

$$=\frac{(\cos\theta+i\sin\theta)+(\cos\phi+\sin\phi)}{(\cos\theta+i\sin\theta)-(\cos\phi+\sin\phi)}$$

$$=\frac{(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)}{(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)}$$

$$=\frac{2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)+i\left(2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)\right)}{-2\left(\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right)+i\left(2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right)}$$

$$=\frac{\cos\left(\frac{\theta-\phi}{2}\right)\left(\cos\left(\frac{\theta+\phi}{2}\right)+i\sin\left(\frac{\theta+\phi}{2}\right)\right)}{\sin\left(\frac{\theta-\phi}{2}\right)\left(-\sin\left(\frac{\theta+\phi}{2}\right)+i\cos\left(\frac{\theta+\phi}{2}\right)\right)}$$

$$=\cot\left(\frac{\theta-\phi}{2}\right)\frac{\left(\cos\left(\frac{\theta+\phi}{2}\right)+i\sin\left(\frac{\theta+\phi}{2}\right)\right)}{i\left(\cos\left(\frac{\theta+\phi}{2}\right)+i\sin\left(\frac{\theta+\phi}{2}\right)\right)}$$

$$=\frac{1}{i}\cot\left(\frac{\theta-\phi}{2}\right)=e^{i\left(-\frac{\pi}{2}\right)}\cot\left(\frac{\theta-\phi}{2}\right)\text{ (Shown)}$$

b (ii)	$\lvert w\rvert = \left\lvert e^{-i\frac{\pi}{2}}\cot\left(\frac{\theta-\phi}{2}\right)\right\rvert = \left\lvert e^{-i\frac{\pi}{2}}\right\rvert \left\lvert\cot\left(\frac{\theta-\phi}{2}\right)\right\rvert = \left\lvert\cot\left(\frac{\theta-\phi}{2}\right)\right\rvert$ $\arg(w) = \begin{cases} -\frac{\pi}{2}, \text{ if } \cot\left(\frac{\theta-\phi}{2}\right) > 0 \\ \frac{\pi}{2}, \text{ if } \cot\left(\frac{\theta-\phi}{2}\right) < 0 \end{cases}$	

<table>
<tr><td>9(i)</td><td>Let T_n denote the distance covered by a runner from Besto on the nth training session.

Since T_n follows an arithmetic progression with common difference 250,
$T_n = 400 + 250(n-1)$
Given that $T_n \geq 20000$,
$400 + 250(n-1) \geq 20000$

<table><tr><th>n</th><th>T_n</th></tr><tr><td>79</td><td>19900 < 20 000</td></tr><tr><td>80</td><td>20150 > 20 000</td></tr><tr><td>81</td><td>20400 > 20 000</td></tr></table>
The minimum value of n is 80.</td><td></td></tr>
<tr><td>(i)
Alt</td><td>Let T_n denote the distance covered by a runner from Besto on the nth training session.

Since T_n follows an arithmetic progression with common difference 250,
$T_n = 400 + 250(n-1)$
Given that $T_n \geq 20000$,
$400 + 250(n-1) \geq 20000$
$250(n-1) \geq 19600$
$n - 1 \geq 78.4$
$n \geq 79.4$</td><td></td></tr>
</table>

The minimum number of sessions for a runner to complete at least 20 km is 80.

(ii)

n	Total distance covered in the *nth* stage
1	$2(50)$
2	$2(50)+2(150)$ $=2(50)+2(3)(50)$ $=2(50)[1+3]$
3	$2(50)+2(150)+2(450)$ $=2(50)+2(3)(50)+2(3)^2(50)$ $=2(50)\left[1+3+3^2\right]$
…	…
n	$2(50)+2(3)(50)+2(3)^2(50)+\cdots+2(3)^{n-1}(50)$ $=2(50)\left[1+3+3^2+\cdots+3^{n-1}\right]$ $=100\left[\dfrac{1(3^n-1)}{3-1}\right]$ $=50(3^n-1)$

(iii)

To find the number of completed stages: $50(3^n-1)\le 42000$

n	$50(3^n-1)$
5	12100 < 42 000
6	36400 < 42 000
7	109300 > 42 000

After completing 6 stages, the runner completed 36 400 m.

<table>
<tr>
<td></td>
<td>Distance remaining = 42 000 – 36 400 = 5600
Given that $OP_7 = 50 \times 3^6 = 36450 > 5600$, the runner from Besto is running away from O at a distance of 5600 m and has not reached P_7.</td>
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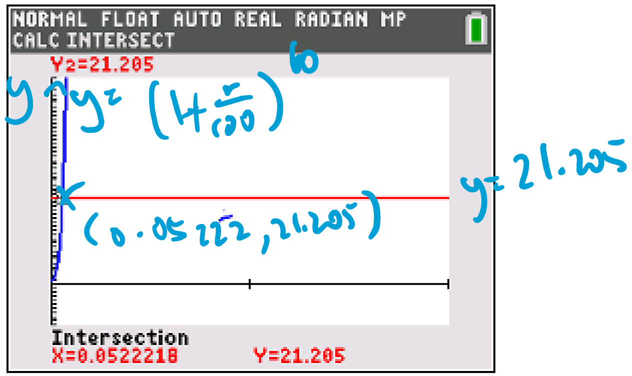
<tr>
<td>(iv)</td>
<td>On the 10th session, a runner from Choco would have completed $400(1.1)^9$ m
From 11th session onwards, using the new plan designed by Choco, a runner will complete $400(1.1)^9\left(1+\frac{r}{100}\right)^{60}$.
$$400(1.1)^9\left(1+\frac{r}{100}\right)^{60} \geq 20000$$
$$(1+\frac{r}{100})^{60} \geq 21.205$$
Let $x = \frac{r}{100}$


From GC, $x \geq 0.05222$</td>
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	$\frac{r}{100} \geq 0.05222$ $r \geq 5.222 = 5.22$ (to 3 sf)	

10i	$X = (13-11\sin\theta)(13-11\cos\theta)$ $\frac{\mathrm{d}X}{\mathrm{d}\theta} = \left(-11\cos\theta\right)(13-11\cos\theta) + (13-11\sin\theta)(11\sin\theta)$ $= -143\cos\theta + 121\cos^2\theta + 143\sin\theta - 121\sin^2\theta$ $= 143\sin\theta - 143\cos\theta + 121(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$ $= 143\sin\theta - 143\cos\theta - 121(\sin\theta - \cos\theta)(\cos\theta + \sin\theta)$ $= 11\left(13(\sin\theta\text{-}\cos\theta)\right) - 121(\sin\theta - \cos\theta)(\cos\theta + \sin\theta)$ $= 11(\sin\theta - \cos\theta)(13 - 11\sin\theta - 11\cos\theta)$ (Shown)	
(ii)	$\frac{\mathrm{d}X}{\mathrm{d}\theta} = 0$ $\Rightarrow 11(\sin\theta - \cos\theta)(13 - 11\sin\theta - 11\cos\theta) = 0$ $\Rightarrow \sin\theta - \cos\theta = 0$ or $13 - 11\sin\theta - 11\cos\theta = 0$ $\Rightarrow \tan\theta = 1$ *or* $11\sin\theta + 11\cos\theta = 13 - - - (\#)$ Using *R*-formula to equation (1), $\sqrt{2}\sin(\theta+\alpha) = \frac{13}{11}$ where $\tan\alpha = 1$ $\Rightarrow \alpha = \frac{\pi}{4}$	

$\therefore \sqrt{2}\sin(\theta+\frac{\pi}{4})=\frac{13}{11}$

$\sin(\theta+\frac{\pi}{4})=\frac{13}{11\sqrt{2}}---(*)$

where $k=\frac{13}{11\sqrt{2}}$ and $\alpha=\frac{\pi}{4}$

Solving $\tan\theta=1$,

$\Rightarrow \theta=\frac{\pi}{4}$

$\Rightarrow \sin(\theta+\frac{\pi}{4})=0.83567$

$\Rightarrow \theta+\frac{\pi}{4}=0.98935$

$\Rightarrow \theta=0.20396$ since $0\le\theta\le\frac{\pi}{4}$

iii

Using first derivative test

θ	0.78	$\frac{\pi}{4}$	0.79
$\frac{dX}{d\theta}$	0.215 >0	0	-0.183 <0

Hence X is a maximum when $\theta_1=\frac{\pi}{4}$

Using first derivative test

θ	0.203	0.20396	0.204
$\frac{dX}{d\theta}$	-0.06997 <0	0	0.00318 >0

Hence X is a minimum when $\theta_2 = 0.20396$

Alternatively, use second derivative test

$$\frac{dX}{d\theta} = -143\cos\theta \ +121\cos^2\theta + 143\sin\theta - 121\sin^2\theta$$

$$\frac{d^2X}{d\theta^2} = 143\sin\theta - 242\cos\theta\sin\theta + 143\cos\theta - 242\sin\theta\cos\theta$$

$$= 143\sin\theta - 484\cos\theta\sin\theta + 143\cos\theta$$

$$\left.\frac{d^2X}{d\theta^2}\right|_{\theta=\frac{\pi}{4}} = -39.8 < 0$$

$$\left.\frac{d^2X}{d\theta^2}\right|_{\theta=0.20396} = 72.999 > 0$$

Hence X is a maximum when $\theta = \frac{\pi}{4}$

and X is a minimum when $\theta = 0.20396$.

iv The greatest possible value of X is 27.3 m^2 when $\theta = \frac{\pi}{4}$.

	To find the minimum area covered by grass, we have to use the maximum area *X*. The area covered by grass $= 676 – 4(27.3) - \pi(11)^2$ $=187\ m^2$.	