Anglo-Chinese School

(Independent)

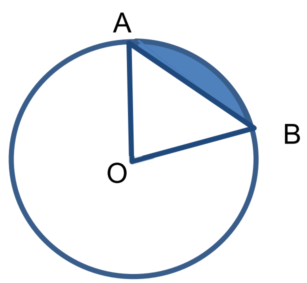
PRELIM 2022 PAPER 2

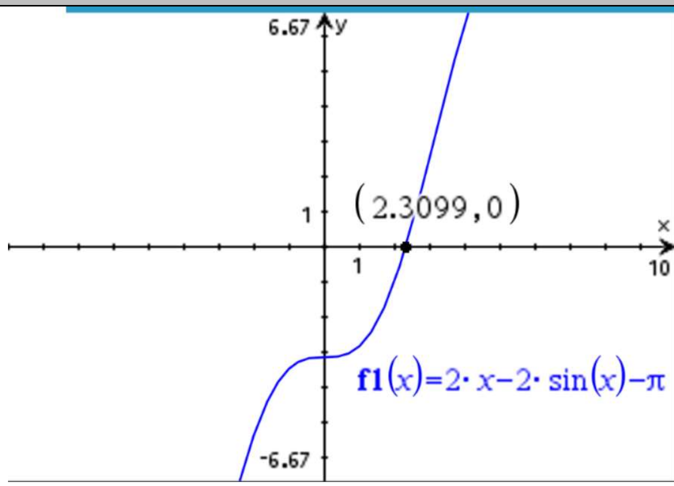
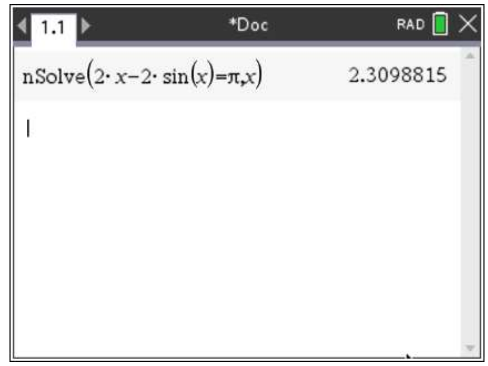
YEAR 6 IB DIPLOMA PROGRAMME

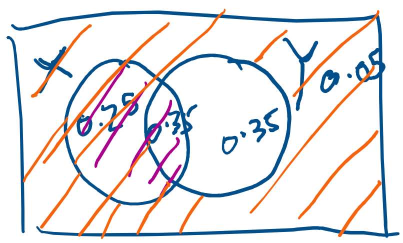
MATHEMATICS : ANALYSIS AND APPROACHES

HIGHER LEVEL **STUDENT SOLUTIONS**

<u>SECTION A</u>

Qn	Solution	COMMENTS
1.		**[4 marks]**
(a)	 Area of the minor segment = Area of the sector – Area of $\triangle OAB$ $=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta$	Some mistook sector for segment.
(b)	$\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta=\frac{1}{4}\pi r^2$ $\Rightarrow 2r^2\left(\theta-\sin\theta\right)=\pi r^2$ $\therefore 2\theta-2\sin\theta=\pi$ (shown)	This is correct if (a) is correct

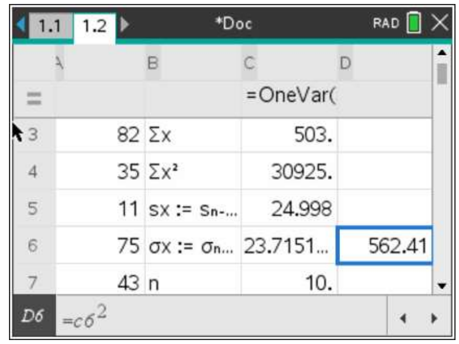
Qn	Solution	COMMENTS
(c)	 $\theta = 2.30988 = 2.31$ (3sf)	Many could not do. Must use Graph or nsolve 
2.		**[4 marks]**
	$f'(x) = \sin x\sqrt{\cos x}$ Observe that $\frac{d}{dx}(\cos x) = -\sin x$. $f(x) = \int \sin x\sqrt{\cos x}\ dx$ $= -\int(-\sin x)(\cos x)^{\frac{1}{2}}\ dx$ $= -\frac{(\cos x)^{\frac{3}{2}}}{\frac{3}{2}} + c$ $= -\frac{2}{3}(\cos x)^{\frac{3}{2}} + c$ Substituting $x = \frac{\pi}{3}, y = 0$: $0 = -\frac{2}{3}\left(\cos\frac{\pi}{3}\right)^{\frac{3}{2}} + c$ $\Rightarrow c = \frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} = \frac{1}{3\sqrt{2}}$ Thus in exact form we have $f(x) = -\frac{2}{3}(\cos x)^{\frac{3}{2}} + \frac{1}{3\sqrt{2}}$	Many did not see this as in the form of $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$ Not acceptable for decimal value of $\frac{1}{3\sqrt{2}}$. All other surd forms of $\frac{1}{3\sqrt{2}}$ are accepted.
3.		**[6 marks]**

Qn	Solution	COMMENTS
(a)	$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$ $\Rightarrow 0.95 = 0.6 + 0.7 - P(X \cap Y)$ $\therefore P(X \cap Y) = 0.35$	Most got it correct
(b)	$P(Y \mid X) = \dfrac{P(X \cap Y)}{P(X)}$ $= \dfrac{0.35}{0.6}$ $= \dfrac{7}{12}$	$\frac{7}{12}$ is an exact value so no need to convert to 0.583. Some students only gave as 0.58.
(c)	$P(X \cup Y') = 0.25 + 0.35 + 0.05 = 0.65$  Alternatively, accept $P(X \cup Y') = 1 - P(Y) + P(X \cap Y)$ $= 1 - 0.7 + 0.35$ $= 0.65$	Badly done.
4.		**[7 marks]**

<table>
<tr><th>Qn</th><th>Solution</th><th>COMMENTS</th></tr>
<tr><td>(a)</td><td>

A time	B	C	D
=		=OneVar(a[	
2	21	x̄	50.3
3	82	Σx	503.
4	35	Σx²	30925.
5	11	sx := sn-...	24.998
6	75	σx := σn...	23.7152
7	43	n	10.
8	73	MinX	11.

From GDC we have
Mean $\mu = 50.3$
Variance $\sigma^2 = (23.7152)^2 = 562.4107 = 562\text{ (3sf)}$
</td><td>
Key in data and use one-var stats. Scroll down and the standard deviation is 23.7152 so need to square it to obtain the variance. Please correct to 3 significant figures



	A	B	C	D
=			=OneVar(	
3	82	Σx	503.	
4	35	Σx²	30925.	
5	11	sx := sn-...	24.998	
6	75	σx := σn...	23.7151...	562.41
7	43	n	10.	

D6 =c6^2
</td></tr>
<tr><td>(b)</td><td>

9	59	Q₁X	33.
10	33	MedianX...	51.
11		Q₃X	73.
12		MaxX	82.
13		SSX := Σ...	5624.1

From GDC we have
$a = 33,\ b = 51,\ c = 73$
</td><td></td></tr>
<tr><td>(c)</td><td>
New mean $\mu_{new} = 50.3 - q$

New standard deviation $\sigma_{new} = 23.7\text{ (3sf)}$
</td><td>Standard deviation measures the spread of the data so it is still the same if every datum is decreased by the same value.</td></tr>
<tr><td>5.</td><td></td><td>[7 marks]</td></tr>
<tr><td>(a)</td><td>

$$v = A - \ln(t + B)$$

$$\Rightarrow \frac{dv}{dt} = -\frac{1}{t+B}$$

Substituting $t = 10$, $\frac{dv}{dt} = -\frac{1}{20}$:

$$-\frac{1}{20} = -\frac{1}{B+10}$$

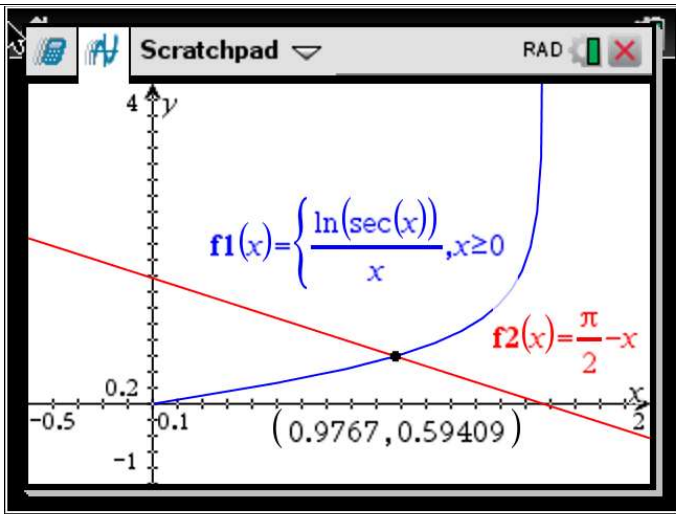
$$\Rightarrow B = 10$$

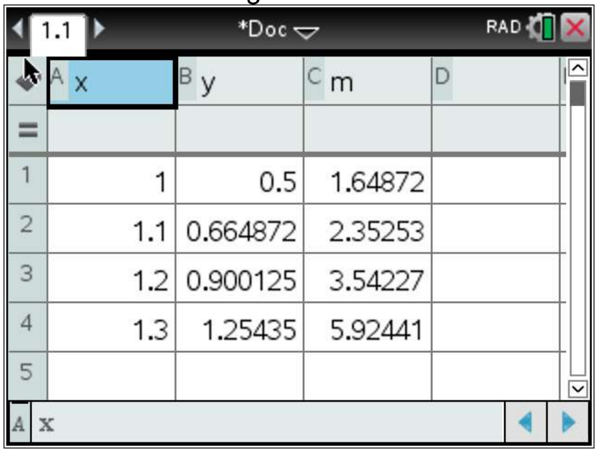
Substituting $t = 100$, $v = 0$:

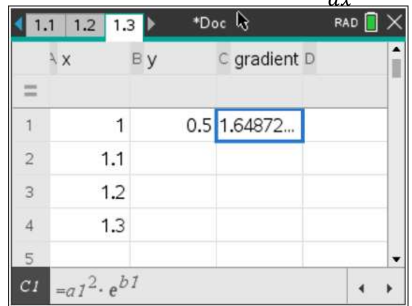
$$0 = A - \ln(100 + 10)$$

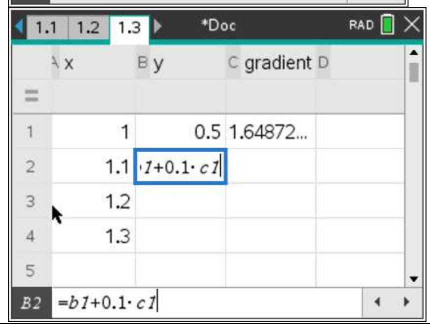
$$\Rightarrow A = \ln(110)$$
</td><td>Most got it correct but some could not differentiate v.</td></tr>
</table>

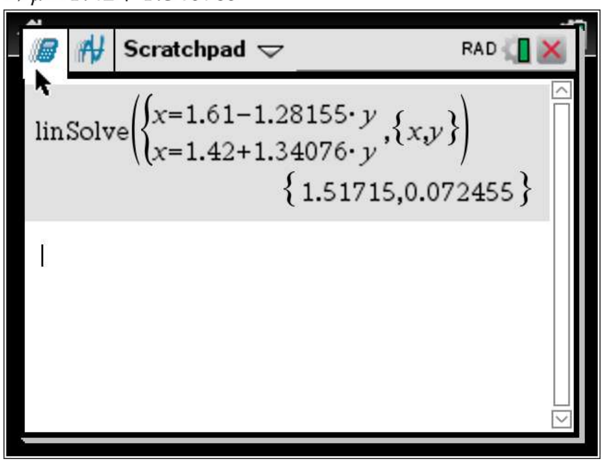
Qn	Solution	COMMENTS
(b)	$\int_3^4 (\ln(110)-\ln(x+10))\,dx$ 2.09802 Distance = 2.10 m (3sf)	4^{th} second means t = 3 to t = 4 because first second is t = 0 to t = 1
6.		**[5 marks]**
(a)	Number of ways is $13! = 6227020800$	This is an exact answer. Do not correct to 6.23×10^9
(b)	Number of ways is $6 \times 2 \times 3! \times 8! = 2903040$ (6 possible configurations of 5 seats in a row, 2 position swaps for Mr & Mrs Lee, 3! - for the Lee children, 8! - for remaining members of tour group)	Lees: $3! \times 2$ Others in first row: 8C5 Permutation of first row: 6! Because LEES is one block. 2^{nd} row: 3P3 Ans: $3! \times 2 \times 8C5 \times 6! \times 3!$
7.		**[9 marks]**
(a)	$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\ln(\sec x)}{x}$ $= \lim_{x\to 0} \frac{\frac{\sec x \tan x}{\sec x}}{1}$ $= \lim_{x\to 0} \tan x$ $= 0$	Recall: $\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$ Here f(x) is sec x whose differentiation is sec x tan x. Qn asks you to use l'Hopital's rule so no need to show indeterminate.
(b)	Let $g(x) = \ln(\sec x)$ $g'(x) = \frac{\sec x \tan x}{\sec x} = \tan x$ $g''(x) = \sec^2 x$ $g^{(3)}(x) = 2(\sec x \tan x)\sec x = 2\sec^2 x \tan x$ $g^{(4)}(x) = 4\sec x(\sec x \tan x)\tan x + 2\sec^2 x(\sec^2 x)$ $= 2\sec^2 x(2\tan^2 x + \sec^2 x)$	Some used differentiation at a point on GDC to get all the specific derivatives at x = 0

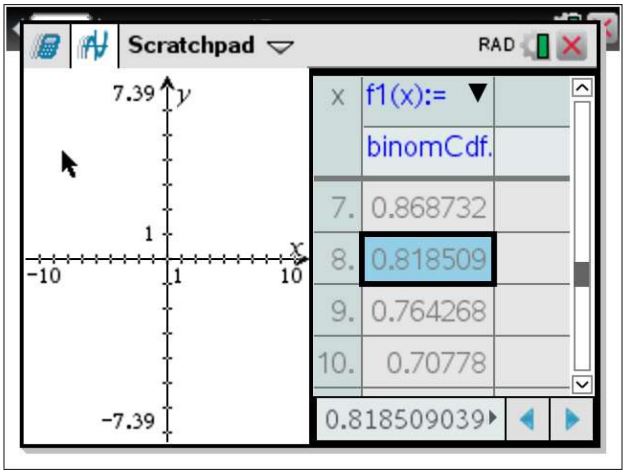
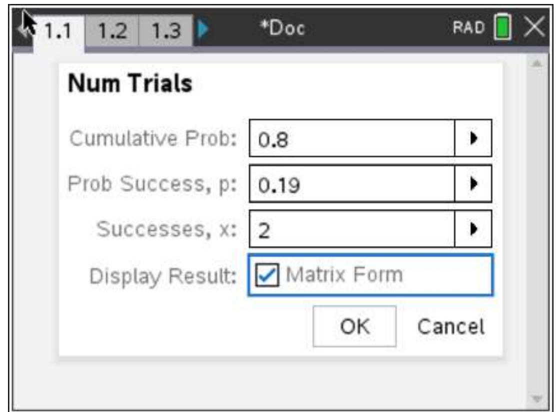
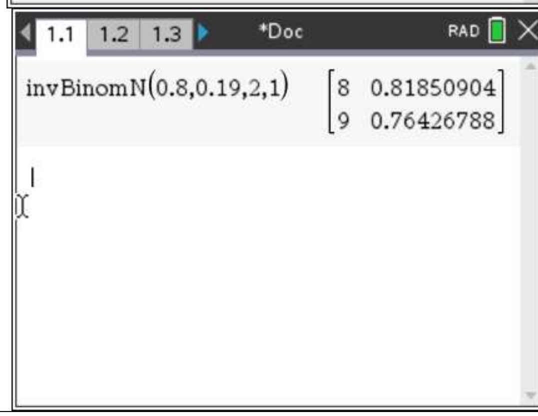
Qn	Solution	COMMENTS
(b)	Substituting $x=0$ into the above expressions: $g(0)=\ln(\sec 0)=\ln 1=0$ $g'(0)=\tan 0=0$ $g''(0)=\sec^2(0)=1$ $g^{(3)}(0)=2\sec^2(0)\tan 0=0$ $g^{(4)}(0)=2\sec^2(0)\left[2\tan^2 0+\sec^2 0\right]=2$ By Maclaurin's series, the expansion is: $g(x)=\ln(\sec x)=\frac{1}{2!}x^2+\frac{2}{4!}x^4+\ldots$ $=\frac{1}{2}x^2+\frac{1}{12}x^4+\ldots\text{ (shown)}$	
(c)	 So the volume generated when the region is revolved around the x-axis is $\pi\int_0^{0.9767}\left(\frac{\ln(\sec x)}{x}\right)^2 dx+\pi\int_{0.9767}^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)^2 dx$ $=0.525293$ $=0.525\text{ units}^3\text{ (3sf)}$	Big conceptual error for some. How can you find area and multiply by 2π? Please do not use u^3 for units^3.
8.		**[6 marks]**

<table>
<tr><th>Qn</th><th>Solution</th><th>COMMENTS</th></tr>
<tr><td>(a)</td><td>Euler's method using GDC:


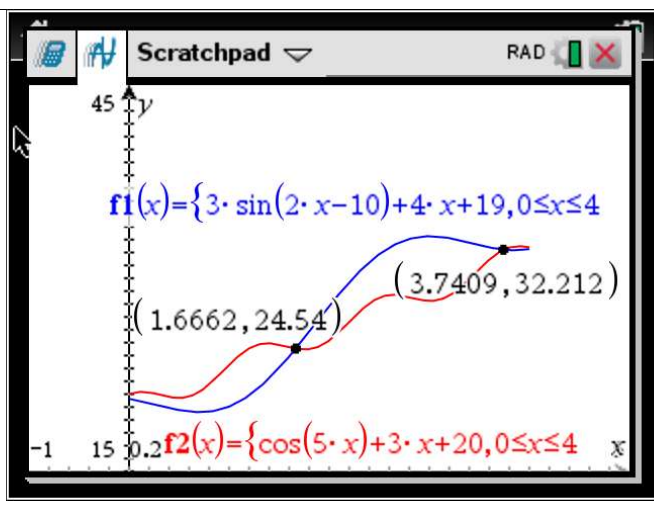
When $x=1.3,\ y=1.25\ (3\text{sf})$</td><td>New y = Old y + $\frac{dy}{dx}\times step$


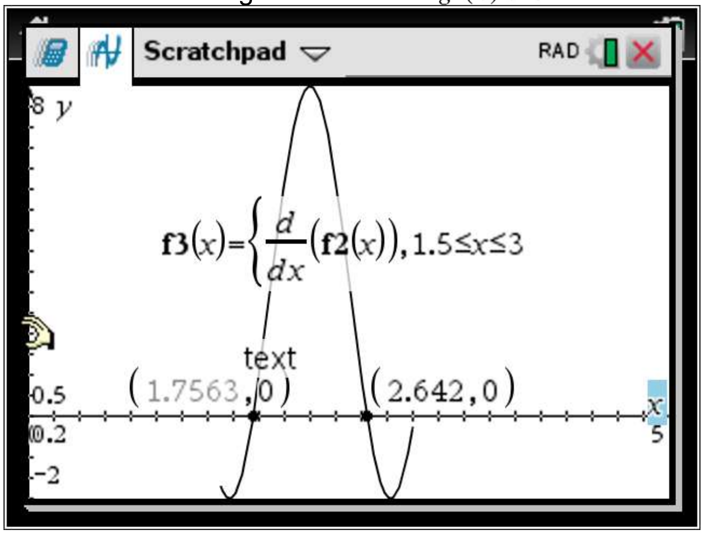
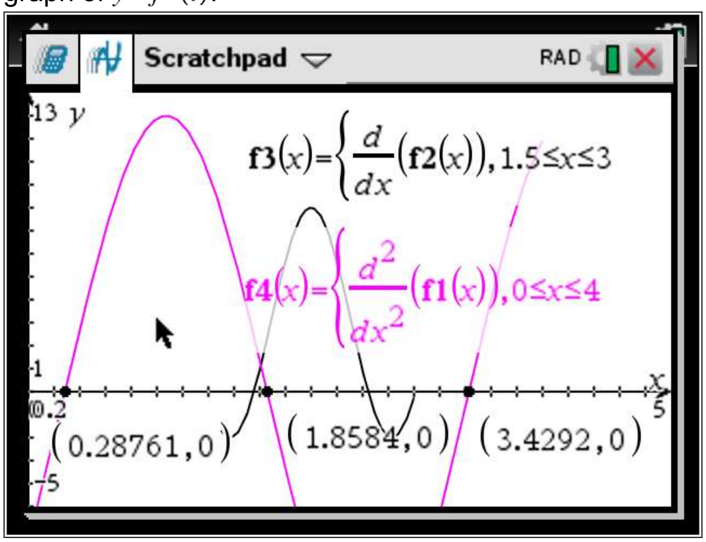

</td></tr>
<tr><td>(b)</td><td>$\frac{dy}{dx}=x^2e^y$
By variable separable we have
$\int e^{-y}dy=\int x^2dx$
$\Rightarrow -e^{-y}=\frac{x^3}{3}+c$
Sub in boundary condition $x=1,\ y=0.5$:
$-e^{-0.5}=\frac{1}{3}+c$
$\Rightarrow c=-e^{-0.5}-\frac{1}{3}=-0.939864=-0.940\ (3\text{sf})$
Thus the particular solution for the differential equation is
$-e^{-y}=\frac{x^3}{3}-0.940$
$\Rightarrow -y=\ln\left(0.940-\frac{x^3}{3}\right)$
$\therefore y=-\ln\left(0.940-\frac{x^3}{3}\right)$
Also accept $y=-\ln\left(e^{-0.5}+\frac{1}{3}-\frac{x^3}{3}\right)$</td><td>Find particular solution means find the equation for y with the value of c. Use the given condition x = 1, y = 0.5, not the approximated x = 1.3, y = 1.25</td></tr>
<tr><td>9.</td><td></td><td>[7 marks]</td></tr>
</table>

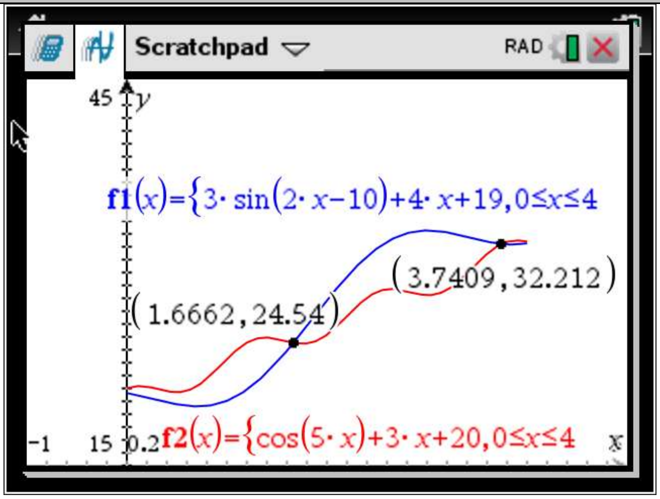
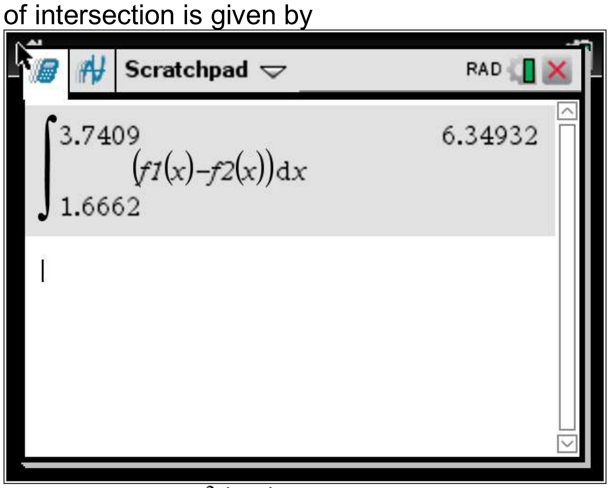
Qn	Solution	COMMENTS
(a)	$P(X \leq 1.61) = 0.9$ $P\left(Z \leq \dfrac{1.61-\mu}{\sigma}\right) = 0.9$ $\dfrac{1.61-\mu}{\sigma} = 1.28155$ ($invnorm(0.9, 0, 1)$) $\Rightarrow \mu = 1.61 - 1.28155\sigma$ $P(X \leq 1.42) = 0.09$ $P\left(Z \leq \dfrac{1.42-\mu}{\sigma}\right) = 0.09$ $\dfrac{1.42-\mu}{\sigma} = -1.34076$ ($invnorm(0.09, 0, 1)$) $\Rightarrow \mu = 1.42 + 1.34076\sigma$  $\mu = 1.51715 = 1.52$ ($3sf$) $\sigma = 0.072455 = 0.0725$ ($3sf$)	Well done except for not correcting to 3 significant figures Use linsolve in algebra menu.

Qn	Solution	COMMENTS
(b)	Let Y be the number of pineapples rejected out of n pineapples in the crate. Then $Y \sim B(n, 0.19)$. We have $P(Y \leq 2) > 0.80$. GDC type in $y = binompdf\,(x, 0.19, 0, 2)$  Largest $n = 8$.	Can use invbinomN. Check the matrix form.  

SECTION B

Qn	Solution	COMMENTS
10.		[15 marks]
(a)	$f(0) = 20.6321 = 20.6(3sf)$ $g(0) = 21$	Note final answers should be rounded to 3 significant figures
(b)	 Points of intersection of the two graphs are $t = 1.6662 \; or \; 3.7409$ $t = 1.67\,(3sf) \; or \; t = 3.74\,(3sf)$	<u>Common error</u>: Students were penalized for not sketching graph or showing any working or attempt at solving.

Qn	Solution	COMMENTS
(c)	$g(t)$ is an increasing function when $g'(t) > 0$.  Thus the range of values is $1.76 < t < 2.64\ (3sf)$	Students should state explicitly that $g'(t) > 0$ if they did not sketch the graph. Common error: Students were penalized if the inequality was not strict.
(d)	For points of inflexion, there is a change in concavity which can be graphically represented by a change of sign in the graph of $y = f''(t)$:  Thus the points of inflexion occur at $t = 0.288, 1.86, 3.43\ (3sf)$	Common error: Justification for point of inflexion was not explicitly stated even when the correct graph was plotted. In future, please state change in concavity / change in sign of 2[nd] derivative as seen from the graph or table.

Qn	Solution	COMMENTS
(e)	 Required area bounded by the two graphs between the points of intersection is given by  Area $=6.35\ units^2\ (3sf)$	Common error: Even when GDC is used to evaluate the area, the integral with limits should be stated to get full credit for the question.
11.		**[21 marks]**
(a)	To find direction vector, $\overrightarrow{PQ}=\begin{pmatrix}10\\8\\1\end{pmatrix}-\begin{pmatrix}4\\5\\7\end{pmatrix}=\begin{pmatrix}6\\3\\-6\end{pmatrix}=3\begin{pmatrix}2\\1\\-2\end{pmatrix}$ $l: \underset{\sim}{r}=\begin{pmatrix}4\\5\\7\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}, \lambda\in\mathbb{R}$	Well done.

Qn	Solution	COMMENTS
(b)	Normal of plane, $\underset{\sim}{n}=\begin{pmatrix}1\\-2\\2\end{pmatrix}\times\begin{pmatrix}2\\-3\\2\end{pmatrix}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ Acute angle between the line and the plane, $\theta=arc\sin\left(\frac{\left\|\begin{pmatrix}2\\2\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-2\end{pmatrix}\right\|}{\left\|\begin{pmatrix}2\\2\\1\end{pmatrix}\right\|\times\left\|\begin{pmatrix}2\\1\\-2\end{pmatrix}\right\|}\right)$ $=arc\sin\left(\frac{4}{\sqrt{9}\times\sqrt{9}}\right)$ $=26.3878$ $=26.4°\ (3sf)$	Common error: Wrong formula for angle between line and plane used or error in calculating vector product for the normal of plane resulting in loss of many more marks in subsequent parts.
(c)	Equation of plane is given by: $\underset{\sim}{r}\bullet\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}1\\7\\-3\end{pmatrix}\bullet\begin{pmatrix}2\\2\\1\end{pmatrix}=2+14-3=13$ Intersection of line and plane: $\begin{pmatrix}4+2t\\5+t\\7-2t\end{pmatrix}\bullet\begin{pmatrix}2\\2\\1\end{pmatrix}=13$ $\Rightarrow 8+4t+10+2t+7-2t=13$ $\Rightarrow 4t=-12$ $\Rightarrow t=-3$ Coordinates of $R=(4+2(-3),\ 5+(-3),\ 7-2(-3))=(-2,\ 2,\ 13)$	Generally well done. Final answers should be in coordinates form as specified by the question although there was no penalization this time round.

Qn	Solution	COMMENTS
(d)(i)	Foot of perpendicular F has position vector $\overrightarrow{OF}=\begin{pmatrix}4\\5\\7\end{pmatrix}+s\begin{pmatrix}2\\2\\1\end{pmatrix}, s\in\mathbb{R}.$ $\begin{pmatrix}4+2s\\5+2s\\7+s\end{pmatrix}\bullet\begin{pmatrix}2\\2\\1\end{pmatrix}=13$ $\Rightarrow 8+4s+10+4s+7+s=13$ $\Rightarrow 9s=-12$ $\Rightarrow s=\frac{-4}{3}$ Thus, position vector is $\overrightarrow{OF}=\begin{pmatrix}4+2\left(\frac{-4}{3}\right)\\5+2\left(\frac{-4}{3}\right)\\7+\left(\frac{-4}{3}\right)\end{pmatrix}=\begin{pmatrix}\frac{4}{3}\\\frac{7}{3}\\\frac{17}{3}\end{pmatrix}$	Not as well done as imagined even though this is a bookwork question. Students really need to revise basic formulae for vectors.
(d)(ii)	$\overrightarrow{PF}=\begin{pmatrix}\frac{-8}{3}\\\frac{-8}{3}\\\frac{-4}{3}\end{pmatrix}$ Shortest distance required $\lvert\overrightarrow{PF}\rvert=\sqrt{\left(\frac{-8}{3}\right)^2+\left(\frac{-8}{3}\right)^2+\left(\frac{-4}{3}\right)^2}$ $=\sqrt{\frac{144}{9}}$ $=4\ units$	<u>Common error</u>: Some students are confused about which vector represents the shortest distance.

Qn	Solution	COMMENTS
(d)(ii)	Alternatively, using the answer in (b), $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}$ $=\begin{pmatrix}-2\\2\\13\end{pmatrix}-\begin{pmatrix}4\\5\\7\end{pmatrix}$ $=\begin{pmatrix}-6\\-3\\6\end{pmatrix}$ Shortest distance required $\left\lvert\overrightarrow{PR}\right\rvert\sin\theta=\sqrt{(-6)^2+(-3)^2+(6)^2}\times\frac{4}{9}$ $=\sqrt{81}\times\frac{4}{9}$ $=4\ units$	Not a common approach for this part.
(e)	First, find the position vector of the reflected point, P'. $\overrightarrow{OP'}=\overrightarrow{OP}+2\overrightarrow{PF}$ $=\begin{pmatrix}4\\5\\7\end{pmatrix}+2\begin{pmatrix}\frac{-8}{3}\\\frac{-8}{3}\\\frac{-4}{3}\end{pmatrix}=\begin{pmatrix}\frac{-4}{3}\\\frac{-1}{3}\\\frac{13}{3}\end{pmatrix}$ Direction vector of the reflected line, $\overrightarrow{RP'}=\overrightarrow{OP'}-\overrightarrow{OR}$ $=\begin{pmatrix}\frac{-4}{3}\\\frac{-1}{3}\\\frac{13}{3}\end{pmatrix}-\begin{pmatrix}-2\\2\\13\end{pmatrix}=\begin{pmatrix}\frac{2}{3}\\\frac{-7}{3}\\\frac{-26}{3}\end{pmatrix}=-\frac{1}{3}\begin{pmatrix}-2\\7\\26\end{pmatrix}$ Thus the equation of the reflected line is $l':\ \underset{\sim}{r}=\begin{pmatrix}-2\\2\\13\end{pmatrix}+\lambda\begin{pmatrix}-2\\7\\26\end{pmatrix},\lambda\in\mathbb{R}$ The integers are $x=-2,\ y=7,\ z=26$.	Poorly done. Many students did not attempt this part of the question. For this part, try to look at earlier parts of the question to see if there are links. Students may also use Midpoint Theorem to find the reflected point before connecting to common point to get the direction vector.
12.		**[19 marks]**

Qn	Solution	COMMENTS
(a)	$LHS = 1 + e^{in\theta}$ $= e^{i0} + e^{in\theta}$ $= e^{\frac{in\theta}{2}}\left(e^{\frac{in\theta}{2}} + e^{-\frac{in\theta}{2}}\right)$ $= e^{\frac{in\theta}{2}}\left(2\cos\frac{n\theta}{2}\right) = RHS$ (shown)	Not as well done as expected. Not many students wrote 1 as e^{i0}. More students manipulated using half-angle formulae.
(b)(i)	Recognize that the LHS is a geometric progression with $a = z,\ r = -z$ and no. of terms $n = 7$. $LHS = z - z^2 + z^3 - ... + z^7$ $= \frac{a\left(1 - r^n\right)}{1 - r}$ $= \frac{z\left(1 - (-z)^7\right)}{1 - (-z)}$ $= \frac{z\left(1 + z^7\right)}{1 + z}$ $= \frac{z + z^8}{1 + z} = RHS$ (shown)	Some students approached the part using 2 geometric progressions when using 1 would have been faster.
(b)(ii)	$LHS = \sum_{k=1}^{7}(-1)^{k-1}\cos(k\theta)$ $= \text{Re}\left[\sum_{k=1}^{7}(-1)^{k-1} z^k\right]$ $= \text{Re}\left[\frac{z + z^8}{1 + z}\right]$ $= \text{Re}\left[\frac{z\left(1 + z^7\right)}{1 + z}\right]$ $= \text{Re}\left[\frac{e^{i\theta} e^{\frac{i7\theta}{2}}\left(2\cos\frac{7\theta}{2}\right)}{e^{\frac{i\theta}{2}}\left(2\cos\frac{\theta}{2}\right)}\right]$ $= \text{Re}\left[\frac{e^{i4\theta}\left(\cos\frac{7\theta}{2}\right)}{\left(\cos\frac{\theta}{2}\right)}\right]$ $= \frac{\cos 4\theta \cos\frac{7\theta}{2}}{\cos\frac{\theta}{2}} = RHS$ (shown)	Poorly attempted. Most students failed to recognize the link to (b) and started creating their own formulae. Students need to be sensitive to linking cosines to the Real part of a complex number and sines to the Imaginary part of a complex number.

Qn	Solution	COMMENTS
(c)	Using the result from (a), we have $LHS=\dfrac{1+z^n}{1-z^n}$ $=\dfrac{2\cos\dfrac{n\theta}{2}e^{i\frac{n\theta}{2}}}{-2i\sin\dfrac{n\theta}{2}e^{i\frac{n\theta}{2}}}$ $=\dfrac{-1}{i}\cot\dfrac{n\theta}{2}$ $=i\cot\dfrac{n\theta}{2}=RHS$	Generally well done.
(d)	$z^3=-\dfrac{1}{\sqrt{2}}(1+i)$ $\Rightarrow z^3=e^{i\frac{-3\pi}{4}}$ $\Rightarrow z^3=e^{i\left(2k\pi-\frac{3\pi}{4}\right)},\ k=-1,0,1$ $\Rightarrow z=e^{i\frac{2k\pi-\frac{3\pi}{4}}{3}},\ k=-1,0,1$ Thus the three roots are: $k=-1:\ z=e^{i\left(\frac{-11\pi}{12}\right)}$ $k=0:\ z=e^{i\left(\frac{-\pi}{4}\right)}$ $k=1:\ z=e^{i\left(\frac{5\pi}{12}\right)}$	<u>Common error</u>: Incorrect approach to finding modulus and argument. Modulus of a complex number cannot be negative. Students need to revise this concept thoroughly. Very few students used the cPolyroots GDC approach.
(e)	Let $z=\dfrac{w-1}{w+1}$. Then we make w the subject of the formula. $zw+z=w-1$ $\Rightarrow z+1=w(1-z)$ $\Rightarrow w=\dfrac{1+z}{1-z}$ Using (c) and letting $n=1$, we have $w=i\cot\dfrac{\theta}{2}$. Thus, using the values for θ from (d), the exact values of the 3 roots are: $w=i\cot\left(\dfrac{-11\pi}{24}\right),i\cot\left(\dfrac{-\pi}{8}\right),i\cot\left(\dfrac{5\pi}{24}\right)$.	Poorly attempted. Students need to be able to connect the dots with earlier parts of the question and find links between w and z before making use of the solutions from (d).

THE END

GOOD LUCK FOR YOUR IB EXAMS! 😊