X JUNIOR COLLEGE
YEAR 6 PRELIMINARY EXAMINATION
Mock Arrangement
in preparation for Candidates' Examination
Higher (than) 2

CANDIDATE NAME

MATHEMATICS 9758/01

Paper 1 **Set I**

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Write your answers in the space provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need of clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

ABOUT THIS PAPER

X Junior College (XJC) is an unofficial initiative aimed at preparing pre-university and/or junior college students in Singapore for school-level and/or national-level H2 Mathematics examinations through self-prepared mock papers. It has no affiliation with any existing institution in Singapore or worldwide.

This mock paper follows closely the 9758 H2 Mathematics GCE Advanced Level syllabus, most suitable for preparation towards preliminary examinations and A–Levels. The paper intends to explore the unconventional ways and/or applications in which topics within the syllabus can be tested, which may affect the difficulty of this paper to varying degrees. **While it is ideal to attempt this paper under examination constraints, prospective candidates are reminded not to use this potentially non-conforming paper as a definitive gauge for actual performance.**

(For enquiries, mail to: xjuniorcollege@gmail.com)

This document consists of **25** printed pages and **3** blank pages.

1 The lines L_1 and L_2 on the xy–plane has vector equations

$$L_1 : \mathbf{r} = \begin{pmatrix} k \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ k \end{pmatrix}, \quad \lambda \in \mathbb{R}, k > 1,$$

$$L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ k \end{pmatrix} + \mu \begin{pmatrix} k \\ 1 \end{pmatrix}, \quad \mu \in \mathbb{R}, k > 1.$$

By considering cartesian equations, describe a pair of transformations which transforms L_1 onto L_2. [4]

2 It is given that x satisfies the inequality $\dfrac{a^2 - x - 2}{x^2 - x - 2} \geq 1$, for some real constant $a > 0$.

Write down, in terms of a where appropriate, the solution intervals of the above inequality for all possible a. [5]

3 An arithmetic sequence with first term a and common difference d is such that the sum of its first n terms is m, and the sum its first m terms is n.

(i) Find a and d in terms of m and n. [4]

(ii) Hence, show that the sum of the first $(m + n)$ terms of the sequence is $-(m + n)$. [2]

4 A curve C has equation $x^2 + 3xy - y^2 + 4x = 1$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [1]

The tangents to C at two distinct points meet at $(6, -4)$.

(ii) Show that these points satisfy the equation $2x + 13y = 11$. [3]

(iii) Hence, find the equation of these tangents, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers to be determined. [3]

5 The complex numbers u and v have the same modulus r and arguments α and β respectively, with $0 < \beta < \alpha < \frac{1}{2}\pi$.

(i) Express $\dfrac{u-v}{u+v}$ in the form $k\tan\left(\dfrac{\alpha-\beta}{2}\right)$, where k is a complex number to be found. [4]

On an Argand diagram, the points Z and W represent the complex numbers $(u-v)$ and $(u+v)$ respectively and angle $OWZ = \theta$.

(ii) Use the result in **(i)** to show that triangle OWZ is a right triangle, stating the right angle. [1]

(iii) By considering the ratio of the lengths $OZ:OW$, deduce an expression for the angle θ in terms of α and β. Hence, express $|u-v|$ and $|u+v|$ in terms of r and θ. [3]

6 Relative to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively such that O, A and B are not collinear. The point C lies on the line segment AB such that $AC\colon CB = (1-\lambda)\colon \lambda$ and that OC bisects angle AOB.

(i) Find λ in terms of $|\mathbf{a}|$ and $|\mathbf{b}|$. [4]

(ii) Another point D lies on the line segment AB such that $AC = BD$. By considering suitable dot products, show that $OD^2 - OC^2 = (|\mathbf{b}| - |\mathbf{a}|)^2$. [4]

7 **(a)** The functions f and g are defined such that gf exists and

$$\mathrm{f}(x) = \frac{x+1}{2x-1}, \qquad x \in \mathbb{R}, x \neq \frac{1}{2},$$

$$\mathrm{gf}(x) = \frac{2x-7}{x-2}, \qquad x \in \mathbb{R}, x \neq 2.$$

(i) Explain why f has an inverse. Find its inverse in a similar form, stating its domain. [3]

(ii) Find a simplified expression for $\mathrm{g}(x)$. [2]

(b) A function h is given such that

$$\mathrm{h}(x) - 2\mathrm{h}\left(\frac{x-1}{x}\right) + \mathrm{h}\left(\frac{1}{1-x}\right) = \frac{x^3+1}{x^2-x}, \qquad x \in \mathbb{R}, x \neq 0, 1.$$

(i) Find the value of h(2), h(0.5) and h(−1). [3]

(ii) Deduce an expression for hhh(x). Hence, given that $\mathrm{hh}(x) = \dfrac{1}{1-x}$, find h($x$). [2]

8 The integral I_n, where $n = 0, 1, 2, 3, \ldots$, is defined by

$$I_n = \int_0^a x^n \sqrt{ax - x^2}\,\mathrm{d}x,$$

for some constant $a > 0$.

(i) By substituting $x = a\sin^2\theta$, find the exact value of I_0 in terms of a. [4]

(ii) Show that $I_{n+1} = \left(\dfrac{2n+3}{2n+6}\right) aI_n$ for $n \geq 0$. [4]

8 [Continued]

(iii) Find the exact area of the region enclosed by the curve $y = \sqrt{(2x - x^2)^5}$ and the x–axis. [4]

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[Question 9 starts on the next page.]

9 A free-hanging rope is held by two people standing at a distance of $2a$ units from one another. Due to its weight, the rope curves downwards in the shape of a *catenary* defined by parametric equations

$$x = a\ln t, \qquad y = \frac{a}{2}\left(t + \frac{1}{t}\right), \qquad \text{for } t_1 \le t \le t_2,$$

where parameters t_1 and t_2 correspond to the two ends of the rope at which it is being held.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [1]

(ii) Given that the lowest point on the rope is horizontally in the middle of both people, find t_1 and t_2. [3]

(iii) Find, in degrees, the acute angle that the rope ends make with the horizontal as it curves downwards. [2]

The rope is rotated about the line $y = k$ at which height it is being held. The rope remains taut throughout its motion and the shape of the curve is maintained.

(iv) Show that $y = \frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right)$. Hence find k exactly in terms of a. [2]

9 **[Continued]**

(v) Determine the exact volume of the space enclosed by the rotating rope, expressing your answer in the form $Aa^3\left(B + C\mathrm{e}^2 + D\mathrm{e}^{-2}\right)$, where A, B, C and D are exact constants to be determined. [5]

10 In Ecology, the *Lotka-Volterra* equations are often used to model the population dynamics of a predator-prey ecosystem. Under this model, the population of prey and predators are each defined by a differential equation: the prey population grows at a rate proportional to its own population and declines at a rate proportional to the product of its own population and the predator population, while the predator population declines at a rate proportional to its own population and grows at a rate proportional to the product of its own population and the prey population.

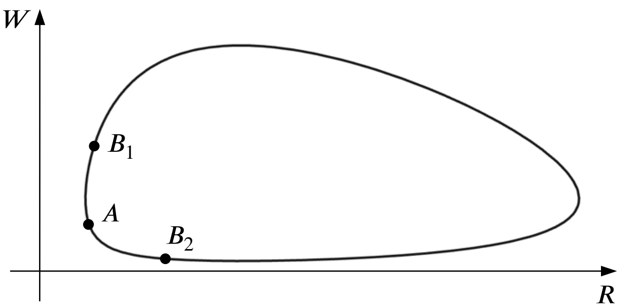
(i) An ecosystem initially has 1 000 prey rabbits and 1 000 predator wolves. After t years, the population of rabbits R and wolves W, each measured in thousands, are such that the rabbit population grows and declines with a coefficient of 0.6 and 0.4 respectively, while the wolf population grows and declines with a coefficient of 0.2 and 0.8 respectively.

(a) Write down, for this ecosystem, the two Lotka–Volterra equations relating R, W and t. [2]

(b) Hence, show that $2W + R - 3\ln W - 4\ln R = 3$. [5]

10 [Continued]

(ii) The graph below shows the population dynamics of the ecosystem described in **(i)**. Point A corresponds to the initial population, while points B_1 and B_2 correspond to the population in other years.



(a) Find, to the nearest hundred, the smallest and largest population size possible for each species. [4]

(b) Which point, B_1 or B_2, corresponds to the population in the year after the initial? Explain. [2]

11 Mr Wong is considering investing money in a savings plan to purchase a car. At the first day of January 2023, he puts \$100,000 into a bank account which pays compound interest at a rate of 2% per month on the last day of each month. He then puts \$1,000 into the account on the first day of each subsequent month.

The price of the car he wishes to buy is \$300,000.

(i) Find the month and year in which the total in the account will first exceed the price of the car. Explain whether this occurs on the first day or the last day of the month. [4]

Instead of buying the car immediately after his account balance first exceeds the price of the car, Mr Wong decides to buy much later. Starting from the first month he pays for the car, he no longer puts additional \$1,000 into the account on the first day of that month and subsequent months. The bank interest rate still applies.

The car will be purchased under a hire purchase scheme. In this scheme, Mr Wong pays a deposit of \$75,000 at the beginning of the first payment month. On subsequent months, the remaining amount will be paid by equal monthly instalments of \$15,000 at the beginning of each month. Starting from the first payment month, interest rate is charged at the end of each month at 0.5% of the outstanding amount. All payments are deducted from Mr Wong's account.

(ii) Show that, under the hire purchase scheme, Mr Wong would have completely paid for the car at the beginning of the seventeenth payment month, stating the amount to be paid on this month to the nearest cent. [5]

11 [Continued]

(iii) After paying the amount due on the last payment month, Mr Wong has \$527,328.50 in his bank account. Find the month and year in which Mr Wong made the first payment. [5]

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