2024 Victoria Junior College Preliminary Examination H2 Mathematics Paper 2 (modified)

Section A: Pure Mathematics (40 marks)

1 **(a)** Show that $\dfrac{1}{\sqrt{1+x^2}-\sqrt{1-x^2}}=\dfrac{1}{2x^2}\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)$. [1]

(b) Hence use appropriate expansions from the List of Formulae (MF26) to find the first two non-zero terms in the series expansion of $\dfrac{x^2}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, $x \neq 0$ in ascending powers of *x*. [3]

(c) State the set of values of *x* for which the series expansion is valid. [1]

(d) It is given that the two terms found in part **(b)** are equal to the first two terms in the series expansion of $\cos\left(ax^b\right)$. Find the possible value(s) of the constants *a* and *b*. [2]

2 **Do not use a calculator in answering this question.**

The complex numbers z_1, z_2 and z_3 are such that $z_1=-\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$, $z_2=-\sqrt{3}+\mathrm{i}$ and $z_3=\dfrac{z_1}{z_2}$.

(a) Express each of z_1, z_2 and z_3 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [3]

(b) Sketch an Argand diagram showing the points P_1, P_2 and P_3 where P_1, P_2 and P_3 represent the complex numbers z_1, z_2 and z_3 respectively. [2]

(c) Find the area of triangle OP_1P_2. [2]

(d) Find the smallest positive integer n for which $\left(z_2^*\right)^n$ is purely imaginary. [2]

3 The line l_1 has equation $\mathbf{r}=3\mathbf{i}-4\mathbf{j}-5\mathbf{k}+\lambda\left(\mathbf{i}-2\mathbf{j}-\mathbf{k}\right)$, where λ is a real parameter. The point *A* has position vector $\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

(a) The plane *p* contains the line l_1 and the point *A*. Find a cartesian equation of the plane *p*. [3]

(b) Find the position vector of the point A', the reflection of the point *A* in the line l_1. [4]

(c) The plane *q* is such that *q* is parallel to *p* and passes through the point with position vector $-3\mathbf{j}+\mathbf{k}$. Find a cartesian equation of *q* and the exact shortest distance between *p* and *q*. [3]

(d) The line l_2 has the equation $\dfrac{y-3}{2}=\dfrac{z-7}{3}$, $x=2$. Given that l_2 intersects *p* at point *S*, find the area of the triangle *OAS*. [4]

4 The curve C is defined by the parametric equations

$$x = a\left(1 + \frac{1}{t}\right) \quad \text{and} \quad y = a\left(t - \frac{1}{t^2}\right)$$

where a is a positive constant and $t \neq 0$.

(a) Show that $\dfrac{dy}{dx} = -\left(\dfrac{2+t^3}{t}\right)$. [3]

(b) Find, in terms of a, the coordinates of the turning point on C, and explain why it is a maximum. [4]

(c) Sketch C. [3]

Section B: Statistics (60 marks)

5 Two married couples, two single adults and two children formed a team of 8 to take part in a series of games.

(a) In the first game, the team sits in a circle. Find the number of arrangements that can be formed if each married couple must be seated together. [2]

(b) A group of three people are to be selected from the team for the second game. Find the number of different groups that can be formed if there must **not** be a married couple in the group. [2]

(c) In the third game, each team member selects a unique number from the set $\{1, 2, \ldots, 8\}$. Find the number of different ways this can be done if the numbers selected by the children are both greater than the numbers selected by the two single adults. [2]

6 A random variable X has the probability distribution given in the following table.

x	1	4	6	8
$\mathrm{P}(X = x)$	a	b	c	d

Given that $\mathrm{E}(X) = 4$, $\mathrm{Var}(X) = \dfrac{19}{4}$ and $\mathrm{P}(X < 4) = \mathrm{P}(X > 4)$, find the values of a, b, c and d. [5]

7 For events A, B and C, it is given that $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B) = 0.5$, $\mathrm{P}(C \mid A') = 0.6$ and $\mathrm{P}(A \mid C') = 0.76$.

(a) Find the greatest and least possible values of $\mathrm{P}(A \cap B)$. [2]

(b) Find $\mathrm{P}(C \cap A')$. [1]

(c) Find $\mathrm{P}(C' \cap A')$. [2]

(d) Find $\mathrm{P}(C)$. [3]

8 A small company makes wine glasses. Each day, n randomly chosen wine glasses are checked and the number of wine glasses found to be cracked is denoted by X.

(a) State, in context of the question, two assumptions needed for X to be well modelled by a binomial distribution. [2]

Assume now that X has the distribution $\mathrm{B}(n, p)$, where $n \geq 3$.

(b) Given that the mean of X and the variance of X are 1.8 and 1.773 respectively, find the value of n and the value of p. [2]

(c) Given instead that the probability of finding 2 cracked wine glasses is thrice the probability of finding 3 cracked wine glasses, find p in terms of n. [2]

9 **(a)** S and T are independent random variables with the distributions $\mathrm{N}(18, 3^2)$ and $\mathrm{N}(\mu, \sigma^2)$ respectively. It is given that $\mathrm{P}(T < 4) = \mathrm{P}(T > 9)$ and $\mathrm{P}(S < 3T) = 0.65$. Calculate the values of μ and σ. [4]

(b) A fruit stall sells grapes that are packed in packets with masses in grams that follow the distribution $\mathrm{N}(850, 30^2)$. The grapes are sold at \$18 per kilogram.

(i) Find the probability that a customer pays more than \$30 for two packets of grapes. [2]

(ii) The fruit stall accepts payment by cash or PayNow. The number of customers who pay by PayNow in a day is a random variable with mean 12 and variance 4.8. In a month of 30 days, find the probability that the average number of customers per day who pay by PayNow is more than 12.3. [3]

10 The yield per hectare, y kg, of a crop is believed to depend on the average rainfall, x mm, in the month of June. For 10 regions, records are kept of the values of x and y, and these are shown in the table below. The yield from the tenth region was accidentally deleted from the records after the data was analysed, and this is indicated by the value p.

Average rainfall (x mm)	149	110	188	135	156	140	168	118	122	174
Yield of crop (y kg)	13.8	6.5	15.2	12.2	14.4	12.2	14.7	9.5	9.9	p

Given that the equation of the regression line of y on x is $y = -2.5652 + 0.10168x$, show that $p = 14.4$. [2]

(a) Draw a scatter diagram for these values, labelling the axes clearly. Calculate the product moment correlation coefficient between x and y. [2]

(b) It is thought that a model of the form $y = a + b\ln x$ may also be a suitable fit to the data. Calculate least square estimates of a and b, and find the value of the product moment correlation coefficient between y and $\ln x$. [3]

(c) Use your answers to parts **(a)** and **(b)** to explain which of

$$y = -2.5652 + 0.10168x \text{ or } y = a + b\ln x$$

is the better model. [2]

(d) Using an appropriate regression line, estimate the yield for a region that experienced 200 mm of rainfall in June. Comment on the reliability of your estimate. [2]

(e) In some regions, rainfall is measured in inches instead of in mm. Given that there are 25.4 mm in an inch, show how the regression line found in part **(b)** can be re-written so that it can be used when x, the average rainfall in June, is given in inches. [1]

11 In the swimming training school AquaV, the time taken to swim a lap of the pool by the trainees is found to have a mean of 35 seconds. The school adopted a new international training programme Breakthru for 3 months and wanted to analyse if Breakthru is effective in improving the timings of the trainees.

A sample of 30 trainees is taken and the times taken, x seconds, to swim a lap of the pool by the trainees are summarised by

$$\sum(x-30)=94, \quad \sum(x-30)^2=758.$$

(a) Test, at the 5% significance level, whether there is any evidence that the mean time taken to swim a lap of the pool has improved after the trainees underwent 3 months of Breakthru, defining any parameters you use. [7]

(b) State an assumption used in carrying out the test. [1]

In another swimming training school AquaZ, the mean time taken to swim a lap of the pool by the trainees is 38 seconds. AquaZ similarly adopted Breakthru for 3 months and then also carried out a test at the 5% significance level to determine whether there is an improvement in the swimming times. A sample of 30 trainees was taken and their timings were measured. The sample standard deviation was found to be 4 seconds and the mean time was denoted by $\bar{y}$. Assume that the times taken to swim a lap by trainees in this school follow a normal distribution.

(c) Find the set of values of $\bar{y}$ for which the result of the test would be to reject the null hypothesis. [4]

(d) If the times taken by the 30 trainees is summarised by $\sum(y-30)=234$, determine the conclusion of the test. [2]