EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2018

General Certificate of Education Advanced Level

Higher 2

MATHEMATICS **9758/01**

Paper 1 [100 marks]

12 September 2018

3 hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

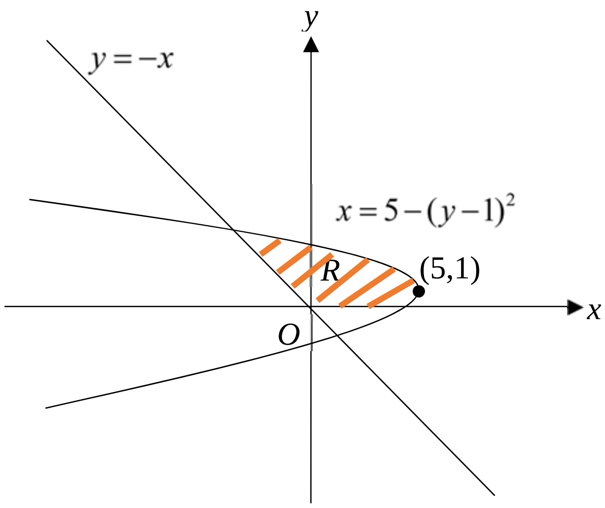
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1



The shaded region R bounded by the curve $y = -x$, the line $y=-x$ and the x-axis is rotated about the x-axis through 360^{O}. Find the volume of the solid formed, leaving your answer to 2 decimal places. [4]

Suggested Solution
$y = -x$ $y=1\pm\sqrt{5-x}$ $y=1+\sqrt{5-x}$ intersect $y=-x$ at $(-4,4)$ $y=1-\sqrt{5-x}$ cuts the x-axis at $(4,0)$ Volume required $=\pi\int_{-4}^{5}\left(1+\sqrt{5-x}\right)^2 \mathrm{d}x-\pi\int_{4}^{5}\left(1-\sqrt{5-x}\right)^2 \mathrm{d}x-\frac{1}{3}\pi(4)^2(4)$ $=201.06$ (to 2 d.p.)

2 **(i)** Solve the inequality $\dfrac{x^2-ax-a}{x-a}\geq a$, where a is a positive real constant, leaving your answer in terms of a. [4]

(ii) Hence, by using a suitable value for a, solve the inequality

$$\frac{4\mathrm{e}^{2x}-\mathrm{e}^{x}-1}{4\mathrm{e}^{x}-1}\geq\frac{1}{4}$$

leaving your answer in exact form. [3]

Suggested Solution

(i) $\dfrac{x^2 - ax - a}{x-a} \geq a \quad (x \neq a)$

$$\frac{\left(x^2 - ax - a\right) - a(x-a)}{x-a} \geq 0$$

$$\frac{x^2 - 2ax + a^2 - a}{x-a} \geq 0$$

Consider $x^2 - 2ax + \left(a^2 - a\right) = 0$

$$x = \frac{-(-2a) \pm \sqrt{(-2a)^2 - 4(1)\left(a^2 - a\right)}}{2}$$

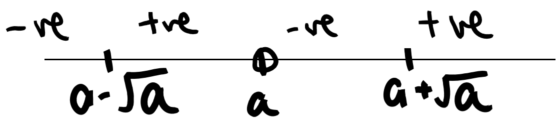
$$= \frac{2a \pm \sqrt{4a}}{2} = a \pm \sqrt{a}$$

Method 1 (test critical points)

$\dfrac{x^2 - 2ax + a^2 - a}{x-a} \geq 0$ can be rewritten as

$$\frac{\left(x - \left(a + \sqrt{a}\right)\right)\left(x - \left(a - \sqrt{a}\right)\right)}{x-a} \geq 0$$

Using sign test, we can check whether each factor is positive or negative for the different range of values of x

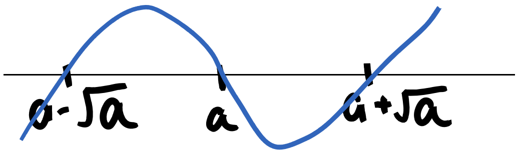


Thus, we have $a - \sqrt{a} \leq x < a$ or $x \geq a + \sqrt{a}$

Method 2

$$\frac{\left(x^2 - ax - a\right) - a(x-a)}{x-a} \geq 0$$

$$(x-a)\left[x^2 - 2ax + \left(a^2 - a\right)\right] \geq 0 \quad (x \neq a)$$



$a-\sqrt{a} \le x < a$ or $x \ge a+\sqrt{a}$

(ii)

Given $\dfrac{4e^{2x}-e^{x}-1}{4e^{x}-1} \ge \dfrac{1}{4}$.

$$\frac{\left(e^{x}\right)^2-\frac{1}{4}\left(e^{x}\right)-\frac{1}{4}}{\left(e^{x}\right)-\frac{1}{4}} \ge \frac{1}{4}$$.

Replace x with e^{x} and let $a=\dfrac{1}{4}$,

$-\dfrac{1}{4} \le e^{x} < \dfrac{1}{4}$ or $e^{x} \ge \dfrac{3}{4}$

$0 < e^{x} < \dfrac{1}{4}$ or $e^{x} \ge \dfrac{3}{4}$

$x < \ln\left(\dfrac{1}{4}\right)$ or $x \ge \ln\left(\dfrac{3}{4}\right)$

3 The parametric equations of a curve C are $x=at$, $y=at^3$, where a is a positive constant.

(i) The point P on the curve has parameter p and the tangent to the curve at point P cuts the y-axis at S and the x-axis at T. The point M is the midpoint of ST. Find a Cartesian equation of the curve traced by M as p varies. [5]

(ii) Find the exact area bounded by the curve C, the line $x=0$, $x=3$ and the x-axis, giving your answer in terms of a. [3]

Suggested Solution

Solutions:

(i) $x=at,\quad y=at^3$

$\dfrac{dx}{dt}=a,\quad \dfrac{dy}{dt}=3at^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3at^2}{a}$$
$$= 3t^2$$

Therefore, gradient of the tangent at $P = 3p^2$

Equation of tangent at P:

$y - ap^3 = 3p^2\left(x - ap\right)$

$y - ap^3 = 3p^2x - 3ap^3$

$y = 3p^2x - 2ap^3$

For point S, $x = 0, y = -2ap^3$.

$\therefore S$ is $\left(0, -2ap^3\right)$

For point T, $y = 0, x = \dfrac{2ap^3}{3p^2} = \dfrac{2}{3}ap$

$\therefore T$ is $\left(\dfrac{2}{3}ap, 0\right)$

Midpoint M $= \left(\dfrac{1}{3}ap, -ap^3\right)$.

$x = \dfrac{1}{3}ap, \quad y = -ap^3$

$\dfrac{3x}{a} = p,$

$y = -a\left(\dfrac{3x}{a}\right)^3 = -\dfrac{27x^3}{a^2}$

(ii) Required area $= \int_0^3 y \,\mathrm{d}x$

$= \int_0^{\frac{3}{a}} at^3\left(a\right) \mathrm{d}t$

$= \int_0^{\frac{3}{a}} at^3\left(a\right) \mathrm{d}t$

$= \mathrm{a}^2\left[\dfrac{t^4}{4}\right]_0^{\frac{3}{a}} = \dfrac{81}{4a^2}$

Alternatively, find the cartesian of the given curve and use it to find the required area.

$x = at, \quad y = at^3$

$\therefore y = a\left(\frac{x}{a}\right)^3 = \frac{x^3}{a^2}$

Required area $= \int_0^3 \frac{x^3}{a^2}\,\mathrm{d}x = \frac{1}{a^2}\left[\frac{x^4}{4}\right]_0^3 = \frac{1}{a^2}\left[\frac{3^4}{4} - 0\right] = \frac{81}{4a^2}$

4 It is given that $y=\sin^{-1}x\cos^{-1}x$, where $-1\le x\le 1$.

(i) Show that $\sqrt{1-x^2}\dfrac{dy}{dx}=\cos^{-1}x-\sin^{-1}x$. [1]

(ii) Show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=-2$ [2]

(iii) Hence find the exact value of A, B and C if y can be expressed as $Ax+Bx^2+Cx^3$, up to (and including) the term in x^3. [4]

(iv) A student used **(iii)** to estimate that $\sin^{-1}(0.8)\cos^{-1}(0.8)\approx 0.8A+0.8^2B+0.8^3C$. Explain, with working, if his estimate is a good one. [1]

Suggested Solution

(i)

$$y=\sin^{-1}x\cos^{-1}x\Rightarrow\frac{dy}{dx}=\frac{\cos^{-1}x}{\sqrt{1-x^2}}+\left(\frac{-\sin^{-1}x}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow\sqrt{1-x^2}\frac{dy}{dx}=\cos^{-1}x-\sin^{-1}x\text{---(1) [shown]}$$

(ii)

Diff (1) wrt x,

$$\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\frac{dy}{dx}+\sqrt{1-x^2}\frac{d^2y}{dx^2}=\frac{-1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow(-x)\frac{dy}{dx}+\left(1-x^2\right)\frac{d^2y}{dx^2}=-1-1=-2\Rightarrow\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-2\text{ --(2)[shown]}$$

(iii) Diff (2) wrt x,

$$-2x\frac{d^2y}{dx^2}+\left(1-x^2\right)\frac{d^3y}{dx^3}-\left(\frac{dy}{dx}+x\frac{d^2y}{dx^2}\right)=0$$

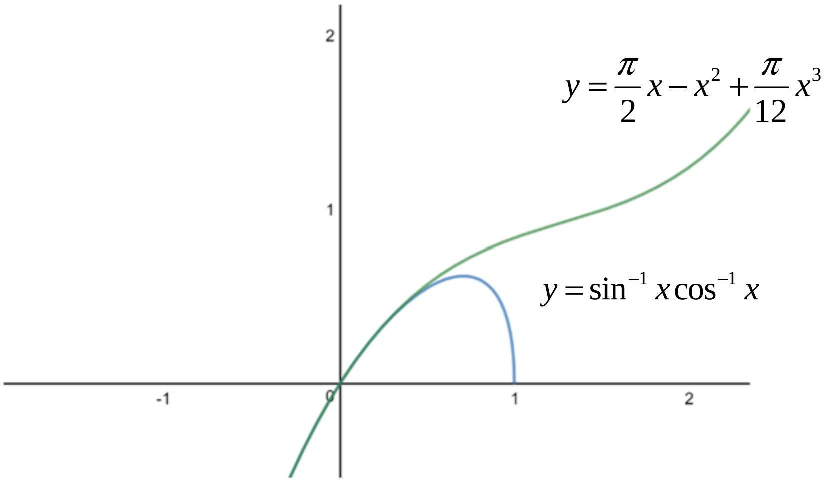
When $x=0, y=0, \dfrac{dy}{dx}=\dfrac{\pi}{2}, \dfrac{d^2y}{dx^2}=-2, \dfrac{d^3y}{dx^3}=\dfrac{\pi}{2}$.

Therefore, $\sin^{-1}x\cos^{-1}x\approx\left(\dfrac{\pi}{2}\right)x+\dfrac{(-2)}{2!}x^2+\dfrac{\left(\frac{\pi}{2}\right)}{3!}x^3=\dfrac{\pi}{2}x-x^2+\dfrac{\pi}{12}x^3$.

(iv) The estimate is not good as $\sin^{-1}(0.8)\cos^{-1}(0.8)=0.597$ (to 3 sf)

But $\frac{\pi}{2}(0.8)-(0.8)^2+\frac{\pi}{12}(0.8)^3=0.751$ (to 3 sf)

Alternative Explanation



The graphs illustrated that at $x=0.8$, the two graphs are quite different from each other.

5 **(a)** Referred to the origin *O*, the points *A* and *B* have position vectors $\mathbf{a}$ and $\mathbf{b}$. Point *C* is on the line which contains *A* and is parallel to **b.** It is given that the vectors $\mathbf{a}$ and $\mathbf{b}$ are both of magnitude 2 units and are at an angle of $\sin^{-1}(1/6)$ to each other. If the area of triangle *OAC* is 3 units^2, use vector product to find the possible position vectors of *C* in terms of $\mathbf{a}$ and $\mathbf{b}$. [5]

(b) Referred to the origin *O*, the points *P* and *Q* have position vectors $\mathbf{p}$ and $\mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel, non-zero vectors. Point *R* is on *PQ* produced such that $PQ:QR = 1:\lambda$. Point *M* is the mid-point of *OR*.

(i) Find the position vector of *R* in terms of λ, $\mathbf{p}$ and $\mathbf{q}$. [1]

F is a point on *OQ* such that *F*, *P* and *M* are collinear.

(ii) Find the ratio *OF*:*FQ*, in terms of λ. [4]

<table>
<tr><td><u>Suggested Solution</u></td></tr>
<tr><td>(a)
$\overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$.

Area of triangle OAC
$= \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OC}\right|$
$= \frac{1}{2}\left|\mathbf{a} \times (\mathbf{a} + \lambda\mathbf{b})\right|$
$= \frac{1}{2}\left|\mathbf{a} \times \mathbf{a} + \lambda(\mathbf{a} \times \mathbf{b})\right|$
$= \frac{1}{2}|\lambda|\left|(\mathbf{a} \times \mathbf{b})\right|$
$= \frac{1}{2}|\lambda||\mathbf{a}||\mathbf{b}|\sin\theta$
$= \frac{1}{2}|\lambda|(2)(2)\frac{1}{6} = \frac{1}{3}|\lambda|$

Since area of triangle OAC= 3,
$\frac{1}{3}|\lambda| = 3$
$\lambda = 9 \text{ or } -9$
$\overrightarrow{OC} = \mathbf{a} \pm 9\mathbf{b}$</td></tr>
<tr><td>(b)</td></tr>
</table>

By the ratio theorem, $\overrightarrow{OQ}=\dfrac{\overrightarrow{OR}+\lambda\overrightarrow{OP}}{1+\lambda}$ $\overrightarrow{OR}=(1+\lambda)\mathbf{q}-\lambda\mathbf{p}$
Since the point F lies on line OQ, $\overrightarrow{OF}=t\mathbf{q}$, for some $t\in\mathbb{R}$. $\overrightarrow{PM}=\overrightarrow{OM}-\overrightarrow{OP}$ $=\frac{1}{2}\overrightarrow{OR}-\mathbf{p}$ $=\dfrac{(1+\lambda)}{2}\mathbf{q}-\dfrac{\lambda}{2}\mathbf{p}-\mathbf{p}$ $=\dfrac{(1+\lambda)}{2}\mathbf{q}-\left(\dfrac{\lambda}{2}+1\right)\mathbf{p}$ Since the point F also lies on line PM, $\overrightarrow{OF}=\mathbf{p}+s\overrightarrow{PM}$, for some $s\in\mathbb{R}$. $=\mathbf{p}+s\left[\dfrac{(1+\lambda)}{2}\mathbf{q}-\left(\dfrac{\lambda}{2}+1\right)\mathbf{p}\right]$ $=\left(1-\dfrac{s\lambda}{2}-s\right)\mathbf{p}+\dfrac{s(1+\lambda)}{2}\mathbf{q}$ Since $\mathbf{p}$ and $\mathbf{q}$ are non-parallel & non-zero vectors, comparing coefficients of $\mathbf{p}$ and $\mathbf{q}$ against $\overrightarrow{OF}=t\mathbf{q}$, we have $1-\dfrac{s\lambda}{2}-s=0$ $s\left(\dfrac{\lambda}{2}+1\right)=1$ $s=\dfrac{1}{\frac{\lambda}{2}+1}=\dfrac{2}{\lambda+2}$ $\overrightarrow{OF}=\dfrac{s(1+\lambda)}{2}\mathbf{q}=\dfrac{2}{\lambda+2}\left(\dfrac{1+\lambda}{2}\right)\mathbf{q}$ $=\dfrac{1+\lambda}{2+\lambda}\mathbf{q}=\dfrac{1+\lambda}{2+\lambda}\overrightarrow{OQ}$ Thus, $OF:FQ=1+\lambda:1$

6 **Do not use a calculator in answering this question.**

(a) It is given that two complex numbers z and w satisfy the following equations

$$13z = (4-7\mathrm{i})w,$$
$$z - 2w = 5 - 4\mathrm{i}.$$

Find z and w. [4]

(b) It is given that $q = -\sqrt{3} - \mathrm{i}$.

(i) Find an exact expression for q^6, giving your answer in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [3]

(ii) Find the three smallest positive whole number values of n for which $\dfrac{q^n}{q^*}$ is purely imaginary. [4]

Suggested Solution
(a) From (2): $z = 5 - 4\mathrm{i} + 2w$ Sub into equation (1): $13(5-4\mathrm{i}+2w) = (4-7\mathrm{i})w$ $(22+7\mathrm{i})w = -13(5-4\mathrm{i})$ $w = \dfrac{-13(5-4\mathrm{i})}{22+7\mathrm{i}} \times \dfrac{22-7\mathrm{i}}{22-7\mathrm{i}}$ $= \dfrac{-13(110-35\mathrm{i}-88\mathrm{i}-28)}{22^2+7^2}$ $= \dfrac{-13}{533}(82-123\mathrm{i}) = -2+3\mathrm{i}$ Sub into $z = 5 - 4\mathrm{i} + 2w$ $z = 5 - 4\mathrm{i} + 2(-2+3\mathrm{i})$ $=1+2\mathrm{i}$
(b)(i) $q = -\sqrt{3} - \mathrm{i}$ $\arg(q) = -\dfrac{5\pi}{6}$ $\lvert q \rvert = 2$

Thus, $q = 2e^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$

$$q^6 = \left(2e^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}\right)^6 = 2^6 e^{\mathrm{i}(-5\pi)} = 64e^{\mathrm{i}(\pi)}$$

(ii)

$$\begin{aligned}\arg\left(\frac{q^n}{q*}\right) &= \arg(q^n) - \arg(q*) \\ &= n\arg(q) + \arg(q) \\ &= (n+1)\arg(q) \\ &= (n+1)\left(-\frac{5\pi}{6}\right) \\ &= -(n+1)\frac{5\pi}{6}\end{aligned}$$

For $\frac{q^n}{q*}$ to be imaginary, $\arg\left(\frac{q^n}{q*}\right) = \pm\frac{\pi}{2}$

Thus

$$-(n+1)\frac{5\pi}{6} = \frac{(2k+1)\pi}{2}, k \in ¢$$

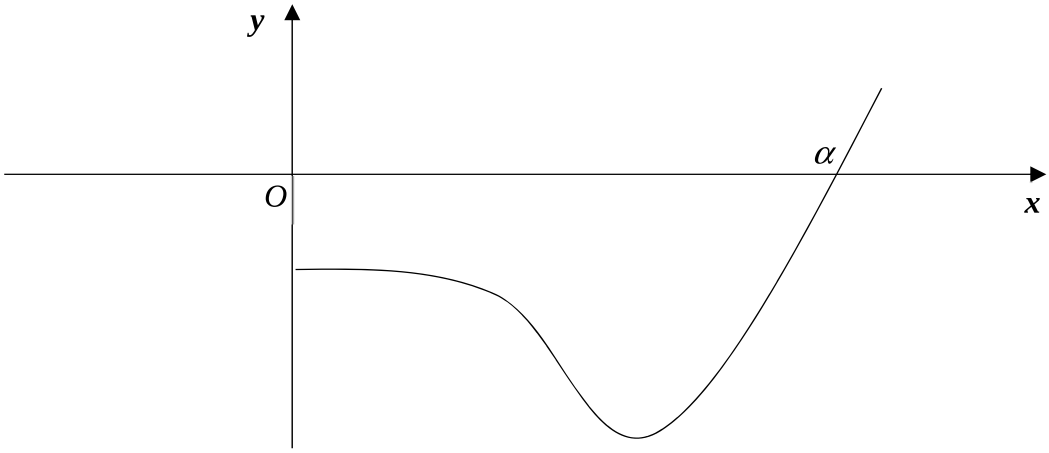
$$(n+1) = -\frac{3(2k+1)}{5}, k \in ¢$$

$$n+1 = 3, 9, 15...$$

$$n = 2, 8, 14, ..$$

The smallest positive whole number values of *n* are 2, 8, 14.

7



It is given that $\mathrm{f}(x)=2x^6-4x^4-6x^2-7$. The diagram shows the curve with equation $y=\mathrm{f}(x)$ for $x\geq 0$. The curve crosses the positive x-axis at $x=\alpha$.

(i) Find the value of α, giving your answer correct to 3 decimal places. [1]

(ii) Show that $\mathrm{f}(x)=\mathrm{f}(-x)$ for all real values of x. What can be said about the six roots of the equation $\mathrm{f}(x)=0$? [4]

It is given that $\mathrm{g}'(x)=\mathrm{f}(x)$, for all real values of x.

(iii) Determine the x-coordinates of all the stationary points of graph of $y=\mathrm{g}(x)$ and determine their nature. [3]

(iv) For which values of x is the graph of $y=\mathrm{g}(x)$ concave upwards? [3]

Suggested Solution
(i) Using GC, $\alpha=1.804$ (3 d.p)
(ii) $\mathrm{f}(x)=2x^6-4x^4-6x^2-7$ $\mathrm{f}(-x)=2(-x)^6-4(-x)^4-6(-x)^2-7$ $=2x^6-4x^4-6x^2-7$ $=\mathrm{f}(x)$ $\Rightarrow \mathrm{f}(x)=\mathrm{f}(-x)$ (shown) Since, $\mathrm{f}(x)=\mathrm{f}(-x)$, two real roots of $\mathrm{f}(x)=0$ are 1.804 and -1.804. (From (i))

The remaining 4 complex roots are in a form of 2 complex conjugate pairs and also are negatives of each other. The four complex roots can be written as x_1, x_2, x_3, x_4, where $x_1 = -x_3$ while $x_2 = -x_4$. Also $x_1{}^* = x_2$ while $x_3{}^* = x_4$ In other words, the complex roots are of the forms: $x_1 = a + b\text{i}$ $x_2 = a - b\text{i}$ $x_3 = -a - b\text{i}$ $x_4 = -a + b\text{i}$

(iii)

$y = \text{g}(x)$ has stationary points when $\text{g}'(x) = \text{f}(x) = 0$, i.e. at $x = 1.804$ or -1.804.

From the graph, we can see

	$x = 1.804^-$	x=1.804	$x = 1.804^+$
$\text{g}'(x) = \text{f}(x)$	-ve	0	+ve

$y = \text{g}(x)$ has a minimum point at $x = 1.804$.

From the graph, we can see

	$x = -1.804^-$	$x = -1.804$	$x = -1.804^+$
$\text{g}'(x) = \text{f}(x)$	+ve	0	-ve

$y = \text{g}(x)$ has a maximum point at $x = -1.804$.

(iv)

$y = \text{g}(x)$ is concave upwards when $\text{g}''(x) = \text{f}'(x)$ is positive.

$\text{f}'(x) = 12x^5 - 16x^3 - 12x$

$\text{f}'(x) = 0$

Using GC the real roots are $x = 0, -1.37, 1.37$ (these are also the x-coordinates of the stationary points of $\text{f}(x)$.)

From graph, we can tell that $f'(x)$ is positive for $x \in [-1.37, 0] \cup [1.37, \infty)$

8 **(a)** **(i)** Show that, for $r \in$ ¢, $r \geq 2$,

$$\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!} = \frac{r^2-r-1}{(r+1)!}. \qquad [1]$$

Let $S_n = \sum_{r=2}^{n} \frac{r^2-r-1}{(r+1)!}$.

(ii) Hence find S_n in terms of n. [3]

(iii) Show that S_n converges to a limit L, where L is to be determined. [2]

(iv) Find the least integer value of n such that S_n differs from L by less than 10^{-10}. [2]

(b) **(i)** Suppose that f is a continuous, strictly decreasing function defined on $[1, \infty)$, with $\mathrm{f}(x) > 0$, $x \geq 1$. According to the Maclaurin-Cauchy test, then the infinite series $\sum_{n=1}^{\infty} \mathrm{f}(n)$ is convergent if and only if the integral $\int_1^{\infty} \mathrm{f}(x)\,\mathrm{d}x$ is finite. By applying the Maclaurin-Cauchy test on the function f defined by $\mathrm{f}(x) = \frac{1}{x}$, $x \geq 1$, determine if the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent. [2]

(ii) Let p be a positive number. By considering the Maclaurin-Cauchy test, show that if $p > 1$, the infinite series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \frac{1}{n^p} + \ldots$ is convergent. [2]

Suggested Solution

(i)

For $r \in$ ¢, $r \geq 2$,

$$\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!}$$

$$= \frac{r(r+1) - 2(r+1) + 1}{(r+1)!}$$

$$= \frac{r^2 + r - 2r - 2 + 1}{(r+1)!}$$

$$= \frac{r^2 - r - 1}{(r+1)!}$$

(ii)

$$Sn = \sum_{r=2}^{n} \frac{r^2 - r - 1}{(r+1)!}$$

$$= \sum_{r=2}^{n} \left[\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!} \right]$$

$$= \left\{ \left[\frac{1}{1!} - \frac{2}{2!} + \frac{1}{3!} \right] \right.$$

$$+ \left[\frac{1}{2!} - \frac{2}{3!} + \frac{1}{4!} \right]$$

$$+ \left[\frac{1}{3!} - \frac{2}{4!} + \frac{1}{5!} \right]$$

$$+ \left[\frac{1}{4!} - \frac{2}{5!} + \frac{1}{6!} \right]$$

M

$$+ \left[\frac{1}{(n-3)!} - \frac{2}{(n-2)!} + \frac{1}{(n-1)!} \right]$$

$$+ \left[\frac{1}{(n-2)!} - \frac{2}{(n-1)!} + \frac{1}{n!} \right]$$

$$\left. + \left[\frac{1}{(n-1)!} - \frac{2}{n!} + \frac{1}{(n+1)!} \right] \right\}$$

$$= \frac{1}{1!} - \frac{2}{2!} + \frac{1}{2!} + \frac{1}{n!} - \frac{2}{n!} + \frac{1}{(n+1)!}$$

$$= \frac{1}{2} - \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$= \frac{1}{2} - \frac{n}{(n+1)!}$$

(iii)

$$Sn = \frac{1}{2} - \frac{n}{(n+1)!}$$

As $n \to \infty$, $\dfrac{n}{(n+1)!} \to 0$.

$\therefore\ S_n \to \dfrac{1}{2}$, thus S_n converges and $L = \dfrac{1}{2}$

(iv)

For $\left\lvert S_n - \frac{1}{2} \right\rvert < 10^{-10}$ $\frac{n}{(n+1)!} < 10^{-10}$ From GC, $n \geq 14$ $\therefore$ the least value of $n = 14$.
(b)(i) $\int_1^{\infty} \frac{1}{x} \mathrm{d}x = [\ln x]_1^{\infty}$ $[\ln x]_1^{\infty}$ is not finite and thus, by the Maclaurin-Cauchy test, $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.
(b)(ii) $\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x = \int_1^{\infty} x^{-p} \mathrm{d}x$ $= \frac{1}{(-p+1)} \left[x^{-p+1} \right]_1^{\infty}$ $= 0 - \frac{1}{(-p+1)}$ $= \frac{1}{p-1}$ is finite, whenever $p > 1$. Thus, by the Maclaurin-Cauchy test, whenever $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

9 A drilling company plans to install a straight pipeline *AB* through a mountain. Points (x, y, z) are defined relative to a main control site at the foot of the mountain at $(0,0,0)$, where units are metres. The *x*-axis points East, the *y*-axis points North and the *z*-axis points vertically upwards. Point *A* has coordinates $(-200,150,10)$ while point *B* has coordinates $(100,10,a)$, where *a* is an integer. Point *B* is at a higher altitude than Point *A*.

(i) Given that the pipeline *AB* is of length 337 metres, find the coordinates of *B*. [3]

A thin flat layer of rock runs through the mountain and is contained in the plane with equation $20x+y+2z=-837$.

(ii) Find the coordinates of the point where the pipeline meets the layer of rock. [4]

To stabilise the pipeline, the drilling company decides to build 2 cables to join points *A* and *B* to the layer of rock. Point *A* is joined to Point *P* while point *B* is joined to Point *Q*.

(iii) Assuming that the minimum length of cable is to be used, find the length *PQ*. [2]

(iv) Show that the pipeline is at an angle of 10.8^0 to the horizontal plane. [2]

(v) After the pipeline is completed, a ball bearing is released from point *B* to roll down the pipeline to check for obstacles. The ball bearing loses altitude at a rate of 0.3*t* metres per second, where *t* is the time (in seconds) after its release. Find the speed at which the ball bearing is moving along the pipeline 10 seconds after its release. [3]

Suggested Solution
(i) Given that length of BA is 337 $\overrightarrow{AB}=\begin{pmatrix}300\\-140\\a-10\end{pmatrix}$ $\sqrt{300^2+(-140)^2+(a-10)^2}=337$ $(a-10)^2=3969$ $a-10=\pm 63$ $a=73 \text{ or } -53$ Since B is of higher altitude, $a>10$ and thus a= 73. Coordinates of B are $(100,10,73)$.

(ii)

$$\overrightarrow{AB} = \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix}$$

Equation of line AB:

$$\mathbf{r} = \begin{pmatrix} -200 \\ 150 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix}, \lambda \in \mathbb{R}$$

Let C be the point of intersection between the line and plane. Since C is on the plane,

$$\left[\begin{pmatrix} -200 \\ 150 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix} \right] \cdot \begin{pmatrix} 20 \\ 1 \\ 2 \end{pmatrix} = -837$$

$$-4000 + 150 + 20 + \lambda(6000 - 140 + 126) = -837$$

$$5986\lambda = 2993 \Rightarrow \lambda = \frac{1}{2}$$

$$\overrightarrow{OC} = \begin{pmatrix} -200 \\ 150 \\ 10 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix} = \begin{pmatrix} -50 \\ 80 \\ 41.5 \end{pmatrix}$$

The coordinates of C are (-50,80,41.5).

(iii)

Length of projection of AB onto the plane

$$= \frac{\left| \overrightarrow{AB} \times \begin{pmatrix} 20 \\ 1 \\ 2 \end{pmatrix} \right|}{\sqrt{405}}$$

$$= \frac{\left| \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix} \times \begin{pmatrix} 20 \\ 1 \\ 2 \end{pmatrix} \right|}{\sqrt{405}} = \frac{\left| \begin{pmatrix} -343 \\ 660 \\ 3100 \end{pmatrix} \right|}{\sqrt{405}}$$

$$= \frac{\sqrt{10163249}}{\sqrt{405}} = 158 \text{ m (to 3 sf)}$$

(iv)

Let acute angle between horizontal plane and AB be α .

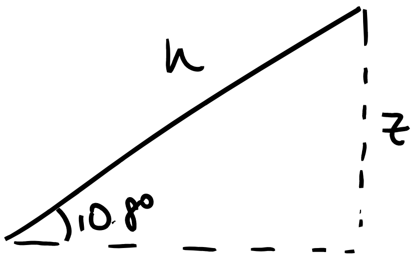
Normal to horizontal plane is **k.**

$$\sin\alpha = \frac{\left|\overrightarrow{AB} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\left|\overrightarrow{AB}\right|}$$

$$\sin\alpha = \frac{\begin{pmatrix}300\\-140\\63\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}}{337} = \frac{63}{337}$$

$\alpha = 10.8^0$

(v) Let h refer to the sloped distance the ball bearing has moved and z the altitude the ball bearing has moved.



Given that $\frac{\mathrm{d}z}{\mathrm{d}t} = -0.3t$, we need to find $\frac{\mathrm{d}h}{\mathrm{d}t}$.

$$\sin\left(10.8^0\right) = \frac{z}{h}$$

$$\frac{63h}{337} = z$$

Differentiate with respect to t:

$$\frac{63}{337}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{337}{63}\left(-0.3t\right)$$

When t=10

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{337}{63}\left(-0.3(10)\right)$$

$$= -16.0 ms^{-1}$$

Speed of the bearing= 16.0 m/s

10 An epidemiologist is studying the spread of a disease, dengue fever, which is spread by mosquitoes, in town A. P is defined as the number of infected people (in thousands) t years after the study begins. The epidemiologist predicts that the rate of increase of P is proportional to the product of the number of infected people and the number of uninfected people. It is known that town A has 10 thousand people of which a thousand were infected initially.

(i) Write down a differential equation that is satisfied by P. [1]

(ii) Given that the epidemiologist projects that it will take 2 years for half the town's population to be infected, solve the differential equation in **(i)** and express P in terms of t. [6]

(iii) Hence, sketch a graph of P against t. [2]

A second epidemiologist proposes an alternative model for the spread of the disease with the following differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{2\cos t}{(2-\sin t)^2} \qquad (*).$$

(iv) Using the same initial condition, solve the differential equation (*) to find an expression of P in terms of t. [3]

(v) Find the greatest and least values of P predicted by the alternative model. [2]

(vi) The government of town A deems the alternative model as a more realistic model for the spread of the disease as it more closely follows the observed pattern of the spread of the disease. What could be a possible factor contributing to this? [1]

Suggested Solution
(i) $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(10-P)$
(ii) $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(10-P)$ **<u>Method 1 to integrate:</u>** $\int \frac{1}{P(10-P)}\,\mathrm{d}P = k\int \mathrm{d}t$ $\frac{1}{10}\int \frac{1}{P} + \frac{1}{10-P}\,\mathrm{d}P = k\int \mathrm{d}t$ $\frac{1}{10}\left[\ln\lvert P\rvert - \ln\lvert(10-P)\rvert\right] = kt + C$ $\frac{1}{10}\ln\left\lvert\frac{P}{10-P}\right\rvert = kt + c$

$$\frac{1}{10}\ln\left(\frac{P}{10-P}\right)=kt+C$$
$$\ln\left(\frac{P}{10-P}\right)=10kt+C_1$$
$$\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$$

<u>Method 2 to integrate</u>

$$\int\frac{1}{P(10-P)}\,\mathrm{d}P=k\int\mathrm{d}t$$
$$\int\frac{1}{25-(P-5)^2}\,dP=k\int\mathrm{d}t$$
$$\frac{1}{10}\ln\left|\frac{5+(P-5)}{5-(P-5)}\right|=kt+c$$
$$\frac{1}{10}\ln\left|\frac{P}{10-P}\right|=kt+c$$

From either Method 1 or 2,

since $P>0, 10-P\geq 0$

$$\frac{1}{10}\ln\left(\frac{P}{10-P}\right)=kt+C$$
$$\ln\left(\frac{P}{10-P}\right)=10kt+C_1$$
$$\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$$

<u>Substitute in values into solution</u>

Sub $t=0, P=1$

$$\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$$
$$\frac{1}{9}=Ae^0\Rightarrow A=\frac{1}{9}$$
$$\frac{P}{10-P}=\frac{1}{9}e^{10kt}$$

Sub $t=2, P=5$

$$\frac{5}{10-5}=\frac{1}{9}e^{10(2)k}$$

$$1=\frac{1}{9}e^{20k}$$

$$e^{20k}=9 \Rightarrow k=\frac{1}{20}\ln(9)\approx 0.10986$$

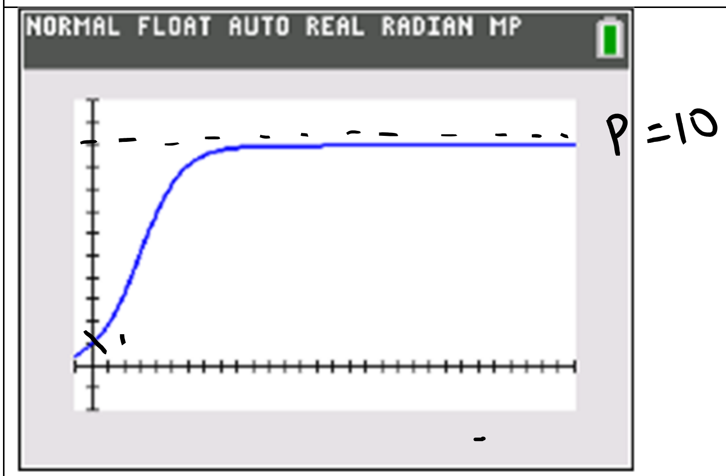
So we have

$$\frac{P}{10-P}=\frac{1}{9}e^{\frac{t}{2}\ln(9)}$$

$$9P=(10-P)e^{\frac{t}{2}\ln(9)}$$

$$P\left(9+e^{\frac{t}{2}\ln(9)}\right)=10e^{\frac{t}{2}\ln(9)}$$

$$P=\frac{10e^{\frac{t}{2}\ln(9)}}{9+e^{\frac{t}{2}\ln(9)}}$$



(iv)

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{2\cos t}{(2-\sin t)^2}=(-2)(-\cos t)(2-\sin t)^{-2}$$

$$P=\frac{-2(2-\sin t)^{-1}}{-1}=\frac{2}{2-\sin t}+c$$

Sub $t=0, P=1$

$$1=\frac{2}{2-\sin 0}+c$$

$$c=1-1=0$$

Thus $P=\dfrac{2}{2-\sin t}$

(v)

Since $-1\le \sin t\le 1$,

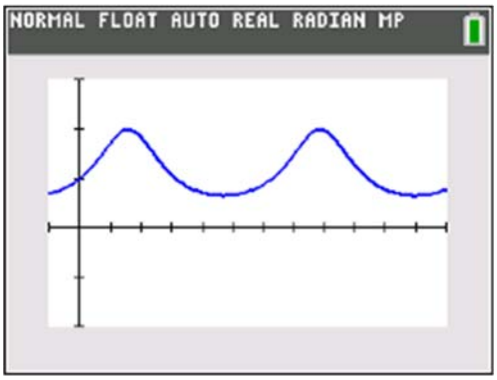
Largest value of P is when $\sin t=1$

Largest value of P= 2

Smallest value of P is when $\sin t=-1$

Smallest value of P= 2/3

(vi)



We can use the GC to plot $P=\dfrac{2}{2-\sin t}$.

The second model could be deemed more suitable, as it shows oscillating values of *P*, which could correspond to the population of the mosquitoes which could vary seasonally. (For example, when the season is hot and rainy, the environment is more conducive for the breeding of mosquitoes.)