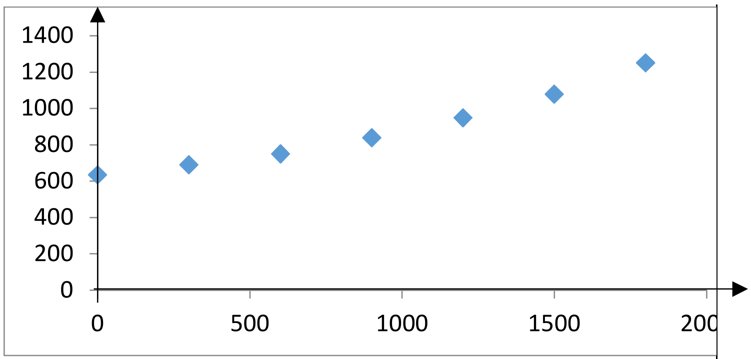
ANNEX B

TPJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Complex numbers	(i)Since the **coefficients** of $az^3 - 31z^2 + 212z + b = 0$ are **all real**, **complex roots occur in conjugate pair**. Since a **cubic equation has three roots**, the third root must be a real root. (ii) $a = 25$, $b = 190$, $-\frac{19}{25}$
2	Vectors	(ii) 7 (iii) length of perpendicular from *O* to *AN*.
3	Maclaurin series	(a) $3x + \frac{3}{2}x^2 + 6x^3 + ...$ (b)(ii) $a = 2$, $b = 6$
4	Application of Integration	(a) $\frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + D$ (b)(iii) $\sqrt{3} - 1 - \frac{\pi}{6}$
5	DRV	(i) $\frac{1}{36}$ (ii) $\frac{161}{36}$ (iii) 0.112
6	Binomial Expansion	(i)0.161 (ii)60 (iii) $\therefore \frac{3}{11} < p < \frac{4}{11}$
7	Correlation & Linear Regression	S

		 (ii)(a)0.9809 (b)0.9960 (iii) The scatter diagram shows that S increases at an increasing rate as h increases, and for $S = ch^2 + d$, $r \approx 0.9960$ which is closer to 1, so the model $S = ch^2 + d$ is a better model. (iv) $S = 0.000182h^2 + 672$ (v)1550 Estimate for when $h = 2200$ metres is not reliable since $h = 2200$ metres is outside the range of the given data and extrapolation is not a good practice.
8	Normal Distribution	(i)0.309 (ii)0.214 (iii)0.303 (iv) 314
9	Hypothesis Testing	(i) Every dustbin has **an equal probability of being selected** and the selections of each dustbin are **made independently**. (ii) Since $n = 50$ is large, by Central Limit Theorem, the **mean** mass of rubbish in dustbins will be approximately normally distributed. (iii) 18.49, 23.6 Since p-value = 0.013937 > 0.01, we do **not** reject H_0 and conclude that there is **insufficient** evidence at 1% level of significance to claim that there has been a reduction in the mass of rubbish in dustbins. (iv) $n \geq 56,\ n \in \mathbb{Z}^+$

10	P&C, Probability	(i) $\frac{63}{800}$ (ii) $\frac{28}{61}$ (iii) 504 (iv) 3360

H2 Mathematics 2017 Preliminary Exam Paper 2 Solutions

1(i)	Since the **<u>coefficients</u>** of $az^3 - 31z^2 + 212z + b = 0$ are **<u>all real</u>**, **<u>complex roots occur in conjugate pair</u>**. Since a **<u>cubic equation has three roots</u>**, the third root must be a real root.
1(ii)	Since $1-3\text{i}$ is a root of $az^3 - 31z^2 + 212z + b = 0$, $a(1-3\text{i})^3 - 31(1-3\text{i})^2 + 212(1-3\text{i}) + b = 0$ $a(-26+18\text{i}) - 31(-8-6\text{i}) + 212(1-3\text{i}) + b = 0$ $(-26a+460+b) + (18a-450)\text{i} = 0$ Comparing real and imaginary parts: $-26a + 460 + b = 0$ ----------- (1) $18a - 450 = 0$ -----------------(2) From (2), $a = 25$, $b = 190$ $(z-(1-3\text{i}))(z-(1+3\text{i}))$ $= z^2 - 2z + 10$ $25z^3 - 31z^2 + 212z + 190 = (z^2 - 2z + 10)(cz + d)$ Comparing coefficient of z^3: $c = 25$ Comparing constant: $190 = 10d$ $d = 19$ The real root is $-\frac{19}{25}$.
2(i)	$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{a}+\mathbf{c}$, $\overrightarrow{OC} = \mathbf{c}$ $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ $= \overrightarrow{OA} + \frac{5}{2}\overrightarrow{AC}$ $= \mathbf{a} + \frac{5}{2}(\mathbf{c}-\mathbf{a})$ $= \frac{1}{2}(5\mathbf{c}-3\mathbf{a})$ **<u>Alternatively:</u>** **By Ratio Theorem:** $\overrightarrow{OC} = \frac{2\overrightarrow{OX} + 3\overrightarrow{OA}}{5}$ $\overrightarrow{OX} = \frac{5\overrightarrow{OC} - 3\overrightarrow{OA}}{2}$ $\overrightarrow{OX} = \frac{1}{2}(5\mathbf{c}-3\mathbf{a})$ By midpoint theorem: $\overrightarrow{ON} = \frac{\overrightarrow{OB}+\overrightarrow{OX}}{2}$ $\overrightarrow{ON} = \frac{1}{2}\left[\mathbf{a}+\mathbf{c}+\frac{1}{2}(5\mathbf{c}-3\mathbf{a})\right]$ $= \frac{1}{4}(7\mathbf{c}-\mathbf{a})$
2(ii)	Area of triangle OAB = $\frac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$

	$4=\frac{1}{2}\lvert\mathbf{a}\times(\mathbf{a}+\mathbf{c})\rvert$ $=\frac{1}{2}\lvert\mathbf{a}\times\mathbf{c}\rvert \qquad (\because \mathbf{a}\times\mathbf{a}=\underset{\sim}{0})$ $\Rightarrow \lvert\mathbf{a}\times\mathbf{c}\rvert=8$ Area of triangle $OAN=\frac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{ON}\right\rvert$ $=\frac{1}{2}\left\lvert\mathbf{a}\times\frac{1}{4}(7\mathbf{c}-\mathbf{a})\right\rvert$ $=\frac{7}{8}\lvert\mathbf{a}\times\mathbf{c}\rvert \qquad (\because \mathbf{a}\times\mathbf{a}=\underset{\sim}{0})$ $=\frac{7}{8}(8)$ $=7$ square units
2(iii)	$\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the length of perpendicular from O to AN. **<u>Alternative answer:</u>** $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the shortest distance from O to AN. $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert$ is the area of a parallelogram formed with vector $\overrightarrow{OA}$ and unit vector $\overrightarrow{AN}$ as its adjacent sides. (**Not recommended here**) Area of triangle $OAN=7$ $\frac{1}{2}\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert\left\lvert\overrightarrow{AN}\right\rvert=7$ $\left\lvert\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{\lvert\overrightarrow{AN}\rvert}\right\rvert=\frac{14}{\lvert\overrightarrow{AN}\rvert}$ $=\frac{14}{\lvert\overrightarrow{ON}-\overrightarrow{OA}\rvert}$ $=\frac{14}{\left\lvert\frac{1}{4}(7\mathbf{c}-\mathbf{a})-\mathbf{a}\right\rvert}$ $=\frac{56}{\lvert 7\mathbf{c}-5\mathbf{a}\rvert}$ **(shown)**
3(a)	$e^{2x}\ln(1+3x)$ $=\left(1+2x+\frac{(2x)^2}{2!}+...\right)\left(3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}-\ldots\right)$ where $-1<3x\le 1$ $=\left(1+2x+2x^2+...\right)\left(3x-\frac{9}{2}x^2+9x^3-...\right)$

	$= 3x - \frac{9}{2}x^2 + 9x^3 + 6x^2 - 9x^3 + 6x^3 + \ldots$ $= 3x + \frac{3}{2}x^2 + 6x^3 + \ldots$ where $-\frac{1}{3} < x \le \frac{1}{3}$
3(b)(i)	$\frac{QR}{\sin\frac{3\pi}{4}} = \frac{PR}{\sin\left(\pi - \frac{3\pi}{4} - 2\theta\right)}$ $\frac{QR}{\sin\frac{3\pi}{4}} = \frac{PR}{\sin\left(\frac{\pi}{4} - 2\theta\right)}$ $QR = \frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}\cos 2\theta - \cos\frac{\pi}{4}\sin 2\theta}$ $QR = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\cos 2\theta - \frac{1}{\sqrt{2}}\sin 2\theta}$ $QR = \frac{1}{\cos 2\theta - \sin 2\theta}$ (shown)
3(b)(ii)	When θ is small, $QR \approx \frac{1}{\left(1 - \frac{(2\theta)^2}{2!}\right) - 2\theta}$ $= \frac{1}{1 - 2\theta - 2\theta^2}$ $= \left(1 - \left(2\theta + 2\theta^2\right)\right)^{-1}$ $= 1 + \left(2\theta + 2\theta^2\right) + \left(2\theta + 2\theta^2\right)^2 + \ldots$ $= 1 + 2\theta + 2\theta^2 + 4\theta^2 + \ldots$ $= 1 + 2\theta + 6\theta^2 + \ldots$ $a = 2$, $b = 6$
4(a)	$\int e^x \sin x \, dx$ $= e^x \sin x - \int e^x \cos x \, dx$ $= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx\right]$ $= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$ Hence, $\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + C$ $2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$ $\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + D$

4(b)(i)	Area of first rectangle, $x = \frac{k}{n}$: $A_1 = \frac{k/n}{\sqrt{3-2\left(k/n\right)-\left(k/n\right)^2}} \cdot \frac{k}{n} = \frac{k^2/n^2}{\sqrt{\frac{3n^2-2nk-k^2}{n^2}}} = \frac{k^2}{n\sqrt{3n^2-2nk-k^2}}$
4(b)(ii)	Area of second rectangle, $x = \frac{2k}{n}$: $A_2 = \frac{2k/n}{\sqrt{3-2\left(2k/n\right)-\left(2k/n\right)^2}} \cdot \frac{k}{n} = \frac{2k^2}{n\sqrt{3n^2-2n(2k)-(2k)^2}}$ Area of third rectangle, $x = \frac{3k}{n}$: $A_3 = \frac{3k/n}{\sqrt{3-2\left(3k/n\right)-\left(3k/n\right)^2}} \cdot \frac{k}{n} = \frac{3k^2}{n\sqrt{3n^2-2n(3k)-(3k)^2}}$ By observation, combined area of *n* rectangles: $A = \sum_{r=1}^{n} \frac{rk^2}{n\sqrt{3n^2-2nrk-r^2k^2}}$, where $a = 2$ *and* $b = 1$
4(b)(iii)	$\sum_{r=1}^{\infty} \frac{rk^2}{n\sqrt{\left(3n^2-anrk-br^2k^2\right)}}$ $=$ Area under curve from $x=0$ to $x=\sqrt{3}-1$ $= \int_0^{\sqrt{3}-1} \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$ $= \int_0^{\sqrt{3}-1} \frac{-1/2(-2-2x)-1}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$ $= -\frac{1}{2}\int_0^{\sqrt{3}-1} \frac{-2-2x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x - \int_0^{\sqrt{3}-1} \frac{1}{\sqrt{4-(x+1)^2}}\,\mathrm{d}x$ $= -\frac{1}{2}\left[\frac{\sqrt{3-2x-x^2}}{1/2}\right]_0^{\sqrt{3}-1} - \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$ $= -\left[\sqrt{3-2x-x^2}\right]_0^{\sqrt{3}-1} - \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$ $= -\left[1-\sqrt{3}\right] - \left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\frac{1}{2}\right]$ $= \sqrt{3}-1-\frac{\pi}{3}+\frac{\pi}{6}$ $= \sqrt{3}-1-\frac{\pi}{6}$ (exact)

5(i)

$$\sum_{r=1}^{6} \mathrm{P}(X = r) = 1$$

$k + 3k + 5k + 7k + 9k + 11k = 1$

$k = \dfrac{1}{36}$

5(ii)

$\mathrm{E}(X) = 1(k) + 2(3k) + 3(5k) + 4(7k) + 5(9k) + 6(11k)$

$= 161k$

$= \dfrac{161}{36}$

5(iii)

Required Probability

$= \mathrm{P}(\{6,6,4\}) + \mathrm{P}(\{6,5,5\})$

$= \left(\dfrac{11}{36}\right)^2 \left(\dfrac{7}{36}\right) \dfrac{3!}{2!} + \left(\dfrac{11}{36}\right)\left(\dfrac{9}{36}\right)^2 \dfrac{3!}{2!}$

$= 0.112$ (3 s.f.) Accept: $\dfrac{1738}{15552} = \dfrac{869}{7776}$

6(i)

Let X be the number of rocks containing fossils out of 20 rocks.

$X \sim \mathrm{B}(20,\ 0.07)$

$\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X \leq 2)$

$= 0.161$ (3 s.f.)

6(ii)

Let Y be the number of rocks containing fossils out of 20 rocks.

$Y \sim \mathrm{B}(n,\ 0.07)$

$\mathrm{P}(Y \geq 3) \geq 0.8$

Method 1a: Using GC Table

n	$\mathrm{P}(Y \geq 3)$	
59	0.79085	< 0.8
60	0.80023	> 0.8
61	0.80925	> 0.8

Hence, least $n = 60$.

Method 1b: Using GC Table

$\mathrm{P}(Y \leq 2) \leq 0.2$

n	$\mathrm{P}(Y \leq 2)$	
59	0.20915	> 0.2
60	0.19977	< 0.2
61	0.19075	< 0.2

Hence, least $n = 60$.

Method 2: Using the binomial distribution function

$\mathrm{P}(Y \leq 2) \leq 0.2$

$\mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) \leq 0.2$

$0.93^n + n(0.07)(0.93)^{n-1} + \dfrac{n(n-1)}{2}(0.07^2)(0.93)^{n-2} \leq 0.2$

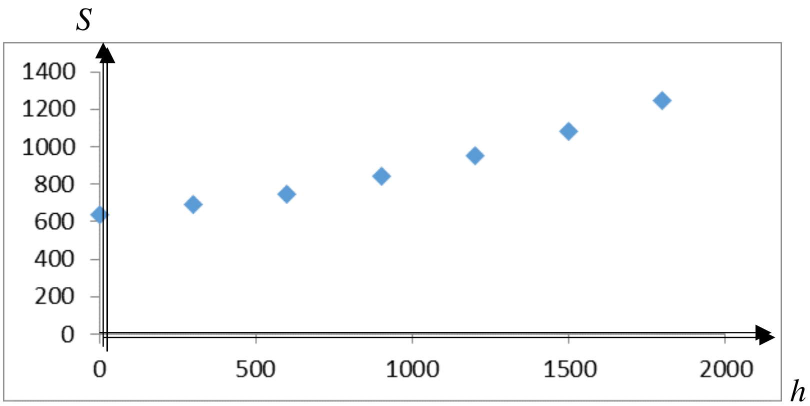
Using GC to sketch the graph:

Hence, least $n = 60$.

6(iii)

Let W be the number of fossils of *zilantophis schuberti* in a random sample of 10 rocks.

$W \sim \mathrm{B}(10,\ p)$

	$\mathrm{P}(W=3) > \mathrm{P}(W=2)$ $\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{2!8!}p^2(1-p)^8$ $120p^3(1-p)^7 > 45p^2(1-p)^8$ $8p > 3(1-p)$ (Since $0<p<1$) $\frac{8}{3}p > 1-p$ $p > \frac{3}{11}$ $\mathrm{P}(W=3) > \mathrm{P}(W=4)$ $\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{4!6!}p^4(1-p)^6$ $120p^3(1-p)^7 > 210p^4(1-p)^6$ $4(1-p) > 7p$ (Since $0<p<1$) $1-p > \frac{7}{4}p$ $p < \frac{4}{11}$ $\therefore \frac{3}{11} < p < \frac{4}{11}$
7(i)	
7(ii)	**(a)** $r = 0.980867 \approx 0.9809$ (4 d.p.) **(b)** $r = 0.996039 \approx 0.9960$ (4 d.p.)
7(iii)	The scatter diagram shows that <u>S increases at an increasing rate as h increases</u>, and for $S = ch^2 + d$, <u>$r \approx 0.9960$ which is closer to 1</u>, so the model $S = ch^2 + d$ is a better model.
7 (iv)	The equation of regression line is $S = 0.0001822853073h^2 + 671.7261905$ i.e. $S = 0.000182h^2 + 672$ (3 s.f.)
7 (v)	$S = 0.00018229(2200)^2 + 671.73$ $= 1554.0136$ $= 1550$ metres (3 s.f.)

	Estimate for when h = 2200 metres is <u>not</u> reliable since h = 2200 metres is outside the range of the given data and <u>extrapolation</u> is not a good practice.
8(i)	$X \sim \mathrm{N}(296,\ 8^2)$ $\quad Y \sim \mathrm{N}(290,\ 12^2)$ Required probability = [P(X > 300)] = 0.30854 (5 s.f.) (0.3085375322) = 0.309 (3 s.f.)
8(ii)	Let W be the number of days in which Albert processes more than 300 kg of raw material on that day out of 15 days. $W \sim \mathrm{B}(15, 0.30854)$ $\mathrm{P}(W=4) = 0.214$ (3 s.f.)
8(iii)	Let $S = Y_1 + Y_2 - 2X$ $\mathrm{E}(S) = \mathrm{E}(Y_1) + \mathrm{E}(Y_2) - 2\mathrm{E}(X) = 2\times290-2\times296=-12$ $\mathrm{Var}(S) = 2\mathrm{Var}(Y) + 2^2\,\mathrm{Var}(X) = 2\times12^2+2^2\times8^2=544$ Hence, $S \sim \mathrm{N}(-12,\ 544)$ $\mathrm{P}(S > 0) = 0.303$ (3 s.f.) (0.3034526994)
8(iv)	$X \sim \mathrm{N}(\mu,\ 8^2)$ $\mathrm{P}(X>300)=\mathrm{P}\left(Z>\frac{300-\mu}{8}\right)\geq 0.95$ $\mathrm{P}\left(Z\leq\frac{300-\mu}{8}\right)\leq 0.05$ $\frac{300-\mu}{8}\leq -1.6449$ $\mu \geq 313.1592$ Least value of $\mu = 314$ kg (3 s.f.)
9(i)	Every dustbin has **<u>an equal probability of being selected</u>** and the selections of each dustbin are **<u>made independently</u>**.
9(ii)	Since $n=50$ is <u>large</u>, by <u>Central Limit Theorem</u>, the **<u>mean</u>** mass of rubbish in dustbins will be approximately normally distributed.
9 (iii)	Unbiased estimate of population mean, $\bar{x}=\frac{924.5}{50}=18.49$ Unbiased estimate of population variance, $s^2=\frac{1}{49}\left[18249.2-\frac{924.5^2}{50}\right]=23.575$ (5 s.f.) $=23.6$ (3 s.f.) Let μ be the population mean mass of rubbish, in kg, in a domestic dustbin. To test: $\mathrm{H_0}$: $\mu=20$ against $\mathrm{H_1}$: $\mu<20$ at 1% level of significance Since $n=50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(20,\ \frac{23.575}{50}\right)$ approximately under $\mathrm{H_0}$. Test Statistic: $Z=\frac{\overline{X}-20}{\sqrt{23.575/50}} \sim \mathrm{N}(0,1)$ approximately under $\mathrm{H_0}$.

	Using GC, $[\bar{x}=18.49, s^2=23.575, n=50]$ $z_{test}=-2.199$, p-value = 0.013937 (5 s.f.) Since p-value = 0.013937 > 0.01, we do **not** reject H_0 and conclude that there is **insufficient** evidence at 1% level of significance to claim that there has been a reduction in the mass of rubbish in dustbins.
9 (iv)	For H_0 to be rejected, $z_{test}=\frac{18.49-20}{\sqrt{23.575}}\times\sqrt{n}<-2.3263$ $n>55.954$ Range of values of n is $n\geq 56,\ n\in\mathbb{Z}^+$ [Also Accept: $n>55,\ n\in\mathbb{Z}$ (or equivalent form)]
10(i)	Required probability $=\frac{30}{100}\times\frac{45}{120}\times\frac{25}{100}+\frac{30}{100}\times\frac{15}{120}\times\frac{45}{100}+\frac{20}{100}\times\frac{45}{120}\times\frac{45}{100}$ $=\frac{63}{800}$
10(ii)	Required probability $=\frac{(0.2)(0.5)(0.75)+(0.8)(0.5)(0.25)}{(0.2)(0.875)(0.75)+(0.8)(0.125)(0.75)+(0.8)(0.875)(0.25)}$ $=\frac{28}{61}$
10 (iii)	Number of different possible codes $={}^9\mathrm{C}_2\times 2!\ \times{}^7\mathrm{C}_1$ $=504$
10 (iv)	**<u>Method 1: Complementary Method</u>** Number of possible arrangements $=\left[{}^4\mathrm{C}_3\times{}^5\mathrm{C}_2\times 5!\right]-\left[\left({}^4\mathrm{C}_3\times 3!\right)\times{}^5\mathrm{C}_2\times 3!\right]$ $=3360$ **<u>Method 2: List by Cases</u>** <u>Case 1: All the even digits are separated</u> ${}^4\mathrm{C}_3\times{}^5\mathrm{C}_2\times 2!\ \times 3!=480$ <u>Case 2: Exactly two even digits are next to each other (and the third even digit is separated)</u> ${}^4\mathrm{C}_3\times\left({}^3\mathrm{C}_2\times 2!\right)\times{}^5\mathrm{C}_2\times 3!\ \times{}^2\mathrm{C}_1=2880$ Number of possible arrangements $=480+2880$ $=3360$

End of Paper