Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 10 Integration Techniques
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 10 Integration Techniques
Core Concept Notes

Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Integrate derivatives to obtain the anti-derivatives / integrals of a function (i.e. indefinite integration is the reverse process of differentiation) ☐ Evaluate definite integrals using graphing calculator ☐ Integrate the following standard functions: ➢ Constant, a ➢ x^n, $n \in \mathbb{R} \setminus \{-1\}$ ➢ x^{-1} ➢ e^x ➢ a^x ☐ Recognise and integrate integrands of the form $f'(x)[f(x)]^n$, where $n \in \mathbb{R}$ ☐ Recognise and integrate integrands of the form $f'(x)e^{f(x)}$ ☐ Integrate basic trigonometric functions	☐ Evaluate definite integrals using anti-derivatives: $\int_a^b f(x)\,dx = F(b) - F(a)$, where $\frac{d}{dx}F(x) = f(x)$ ☐ Integrate powers of basic trigonometric functions ☐ Use MF27 integral formulas to integrate functions of the following standard forms: **(a)** $\frac{1}{a^2+x^2}$ **(b)** $\frac{1}{\sqrt{a^2-x^2}}$ **(c)** $\frac{1}{a^2-x^2}$, $(\lvert x \rvert < a)$ **(d)** $\frac{1}{x^2-a^2}$, $(\lvert x \rvert > a)$ ☐ Recognise and integrate integrands of the form $\int \frac{f(x)}{g(x)}\,dx$ or $\int \frac{f(x)}{\sqrt{g(x)}}\,dx$	☐ Use a given substitution to simplify and integrate an expression ☐ Use integration by parts to integrate an expression

§1 Indefinite Integrals

If two functions F(x) and f(x) are related as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{F}(x)\right)=\mathrm{f}(x),$$

then f(x) is the **derivative** of F(x) and F(x) is called an **anti-derivative** or **integral** of f(x).

Illustration:

Since $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1)=2x$, x^2+1 is an anti-derivative of $2x$.

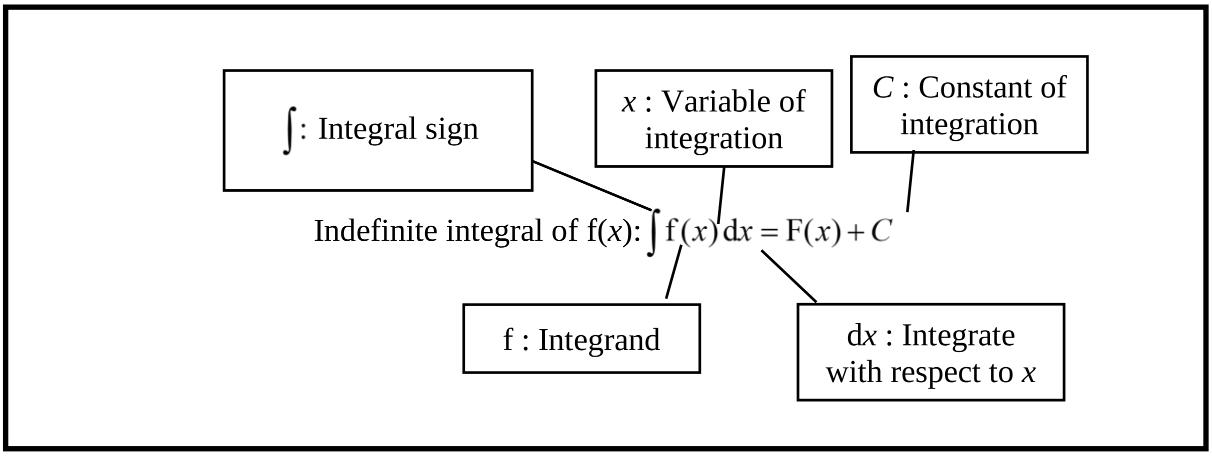
Since $\frac{\mathrm{d}}{\mathrm{d}x}(x^2-7)=2x$, x^2-7 is an anti-derivative of $2x$.

Since $\frac{\mathrm{d}}{\mathrm{d}x}(x^2)=2x$, x^2 is also an anti-derivative of $2x$.

Observe that $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{F}(x)+C\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{F}(x)\right)+\frac{\mathrm{d}}{\mathrm{d}x}C=\mathrm{f}(x)+0=\mathrm{f}(x)$ where C is any arbitrary real constant.

Recall that the process of finding $\int \mathrm{f}(x)\ \mathrm{d}x$ for a given function f is called **integration**.

We know that $\int \mathrm{f}(x)\ \mathrm{d}x$ is actually equivalent to F(x) + C, the collection of all anti-derivatives of f(x), and we write:



Note:

Integration is the **"reverse" of differentiation.**

i.e.: $\int \mathrm{f}(x)\,\mathrm{d}x=\mathrm{F}(x)+C$ because $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{F}(x)+C\right)=\mathrm{f}(x)$

Example:

(1) $\int \cos x\,\mathrm{d}x=\sin x+C$ because $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x+C\right)=\cos x$

(2) $\int (x-5)^9\mathrm{d}x=\frac{1}{10}(x-5)^{10}+C$ because $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{10}(x-5)^{10}+C\right]=\frac{10}{10}(x-5)^9=(x-5)^9$

(3) $\int \mathrm{e}^{5x+1}\ \mathrm{d}x=\frac{1}{5}\mathrm{e}^{5x+1}+C$ because $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{5}\mathrm{e}^{5x+1}+C\right)=\frac{5}{5}\mathrm{e}^{5x+1}=\mathrm{e}^{5x+1}$

§2 Basic Properties

1. $\int [\mathrm{f}(x)+\mathrm{g}(x)]\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x + \int \mathrm{g}(x)\,\mathrm{d}x$
2. $\int \mathrm{f}(x)-\mathrm{g}(x)\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x - \int \mathrm{g}(x)\,\mathrm{d}x$
3. $\int k\mathrm{f}(x)\,\mathrm{d}x = k\int \mathrm{f}(x)\,\mathrm{d}x$, where k is any non-zero real constant.

Quick Check: Are the following True/ False?

1. Is $\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\,\mathrm{d}x = \frac{\int \mathrm{f}(x)\,\mathrm{d}x}{\int \mathrm{g}(x)\,\mathrm{d}x}$?	E.g: $\int \frac{x+1}{x}\,\mathrm{d}x = \frac{\int x+1\,\mathrm{d}x}{\int x\,\mathrm{d}x}$?	T / F
2. Is $\int \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x \int \mathrm{g}(x)\,\mathrm{d}x$?	E.g: $\int x(x+1)\,\mathrm{d}x = \int x\,\mathrm{d}x \int x+1\,\mathrm{d}x$?	T / F
3. Is $\int \mathrm{g}(x)\mathrm{f}(x)\,\mathrm{d}x = \mathrm{g}(x)\int \mathrm{f}(x)\,\mathrm{d}x$?	Eg: $\int x(x+1)\,\mathrm{d}x = x\int (x+1)\,\mathrm{d}x$?	T / F

Example 1

Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^{2x}\right)$. Hence, find $\int x\mathrm{e}^{2x}(x+1)\,\mathrm{d}x$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^{2x}\right) = 2x\mathrm{e}^{2x} + 2x^2\mathrm{e}^{2x} = 2x\mathrm{e}^{2x}(x+1)$$

$$\int 2x\mathrm{e}^{2x}(x+1)\,\mathrm{d}x = x^2\mathrm{e}^{2x} + C$$

$$\therefore \int x\mathrm{e}^{2x}(x+1)\,\mathrm{d}x = \frac{1}{2}x^2\mathrm{e}^{2x} + B, \text{ where } B = \frac{C}{2}$$

Learning Point: Integration is the reverse process of Differentiation.

§3 Computation of Definite Integrals

The definite integral from *a* to *b* of f(*x*), where f(*x*) is continuous on the interval [a, b], is given by :

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \left[\ \mathrm{F}(x)\ \right]_a^b = \mathrm{F}(b) - \mathrm{F}(a), \quad \text{where } \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x) = \mathrm{f}(x)$$

a and *b* are called the ***limits of integration***, where ***a*** is the **lower limit** and ***b*** is the **upper limit.**

$\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is called the **definite integral from *a* to *b* of f(*x*) w.r.t *x*.**

Note: the constant of integration *C* is eliminated in the subtraction.

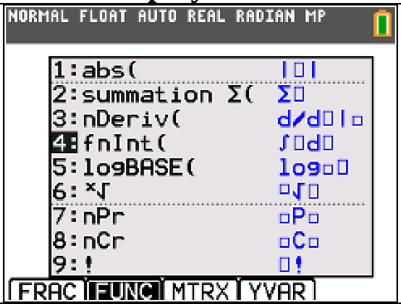
Proof:
$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \left[\mathrm{F}(x) + C\right]_a^b$$
$$= \left[\mathrm{F}(b) + C\right] - \left[\mathrm{F}(a) + C\right]$$
$$= \mathrm{F}(b) - \mathrm{F}(a)$$

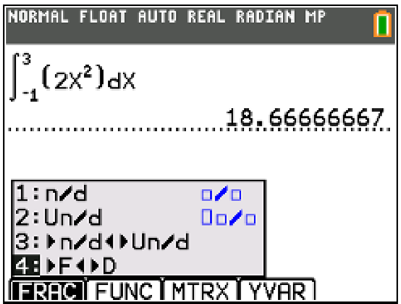
Some Important Results

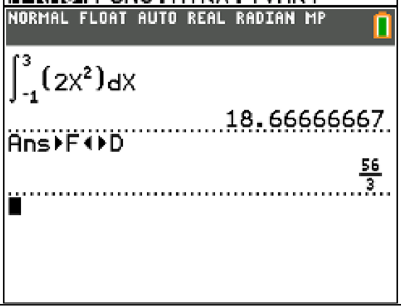
- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$ e.g. : $\int_0^{-2} \mathrm{f}(x)\,\mathrm{d}x = -\int_{-2}^0 \mathrm{f}(x)\,\mathrm{d}x$
- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x$ e.g. : $\int_1^2 \mathrm{f}(x)\,\mathrm{d}x + \int_2^3 \mathrm{f}(x)\,\mathrm{d}x = \int_1^3 \mathrm{f}(x)\,\mathrm{d}x$

§4 Computation of Definite Integrals using Graphing Calculator

Example 2 Evaluate $\int_{-1}^{3} 2x^2 \, dx$.

<table>
<tr><th>Steps/Keystrokes/Explanations</th><th>Screen Display</th></tr>
<tr><td>1. Press [alpha][window] and select 4: fnInt(.</td><td>
</td></tr>
<tr><td>2. Key in the lower and upper limits, integrand and variable of integration and press [enter]</td><td>
</td></tr>
<tr><td>3. To convert answer to exact form, press [alpha][y=] and select 4:►F◄►D to switch between fraction and decimal. Press [enter]

<u>Alternative</u>
Press [math] and select 1:►Frac to convert to fraction. Press [enter].</td><td>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Solution:

Using GC, $\int_{-1}^{3} 2x^2 \, dx = \frac{56}{3}$

§5 Integrals of Standard Functions [Not in MF27]

	'O' Level
1	$\int a\ \mathrm{d}x = ax + C$
2	For $n \in \mathbb{R}$, a) $n \neq -1$, $\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$
	b) $n = -1$, $\int x^{-1}\ \mathrm{d}x = \int \dfrac{1}{x}\ \mathrm{d}x = \ln\lvert x\rvert + C$ *There is a need to put **modulus** here to ensure the existence of the integral.
3	$\int \mathrm{e}^x\ \mathrm{d}x = \mathrm{e}^x + C$
4	$\int a^x\ \mathrm{d}x = \dfrac{a^x}{\ln a} + C$

*****VERY Important Result*****:

In general, if $\int \mathrm{f}(x)\mathrm{d}x = \mathrm{F}(x) + C$, then we have

$$\int \mathrm{f}(px+q)\mathrm{d}x = \frac{1}{p}\mathrm{F}(px+q) + C.$$

(Result can be proven using integration by substitution.)

Golden Rule:
When x is replaced by a **linear form** $(px+q)$, we divide the answer by coefficient of x.

Example 3

(a)	$\int 3x^5 + \dfrac{2}{x} - \dfrac{5}{x^3}\,\mathrm{d}x$ $= \dfrac{1}{2}x^6 + 2\ln\lvert x\rvert + \dfrac{5}{2x^2} + C$	**(b)**	$\int \dfrac{1}{4x+3}\ \mathrm{d}x$ $= \dfrac{1}{4}\ln\lvert 4x+3\rvert + C$ **Golden Rule:** Replace x by $(4x+3)$, so we need to divide by coefficient of x which is 4. **Recall:** $\int \dfrac{1}{x}\ \mathrm{d}x = \ln\lvert x\rvert + C$

(c)	$\int (2x-1)^5 \, dx$ $= \frac{1}{2} \cdot \frac{(2x-1)^6}{6} + C$ $= \frac{1}{12}(2x-1)^6 + C$ **Golden Rule:** Replace x by $(2x-1)$, so we need to divide by coefficient of x which is 2. **Recall:** $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$	**(d)**	$\int e^{3x-4} dx$ $= \frac{1}{3} e^{3x-4} + C$ **Golden Rule:** Replace x by $(3x-4)$, so we need to divide by coefficient of x which is 3. **Recall:** $\int e^x \, dx = e^x + C$

§6 <u>Integration involving the function f (x) and its derivative f $'(x)$</u>

1	For $n \in \mathbb{R},\ n \neq -1$, $\int f'(x)\left[f(x)\right]^n dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + C$	<u>**Proof**</u>: Since $\frac{d}{dx}[f(x)]^{n+1} = (n+1)[f(x)]^n f'(x)$ and integration is a reverse process of differentiation, $\int (n+1)[f(x)]^n f'(x) \, dx = [f(x)]^{n+1}$ $\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{(n+1)} + C$
2	For $n \in \mathbb{R}, n = -1$, $\int f'(x)[f(x)]^{-1} dx = \int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x)\rvert + C$	<u>**Proof:**</u> Since $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$ and integration is a reverse process of differentiation $\int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x)\rvert + C$
3	$\int f'(x) e^{f(x)} \, dx = e^{f(x)} + C$	<u>**Proof:**</u> Since $\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$ and integration is a reverse process of differentiation, $\int e^{f(x)} f'(x) \, dx = e^{f(x)} + C$

Example 4 **Concept:** For $n \in \mathbb{R},\ n \neq -1,\ \int \mathrm{f}'(x)[\mathrm{f}(x)]^n \, dx = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + C$

(a)

$$\int x(1+x^2)^3 dx$$
$$= \frac{1}{2}\int 2x(1+x^2)^3 dx$$
$$= \frac{1}{2}\frac{(1+x^2)^4}{4} + C$$
$$= \frac{1}{8}(1+x^2)^4 + C$$

Compensate with **constant** $\frac{1}{2}$

Thinking Process:
Let $\mathrm{f}(x) = 1 + x^2$
$\mathrm{f}'(x) = 2x$ and $n = 3$

(b)

$$\int \frac{x}{\sqrt{1-x^2}} dx$$
$$= \int x(1-x^2)^{-\frac{1}{2}} dx$$

Good Strategy

$$= -\frac{1}{2}\int -2x(1-x^2)^{-\frac{1}{2}} dx$$
$$= -\frac{1}{2}\frac{1}{\left(\frac{1}{2}\right)}(1-x^2)^{\frac{1}{2}} + C$$
$$= -(1-x^2)^{\frac{1}{2}} + C$$

Compensate with **constant** $-\frac{1}{2}$

Thinking Process:
Let $\mathrm{f}(x) = 1 - x^2$
$\mathrm{f}'(x) = -2x$ and $n = -\frac{1}{2}$

(c)

$$\int \frac{(\ln x)^5}{x} dx$$
$$= \int \frac{1}{x}(\ln x)^5 dx$$
$$= \frac{(\ln x)^6}{6} + C$$

Good Strategy

Thinking Process:
Let $\mathrm{f}(x) = \ln x$
$\mathrm{f}'(x) = \frac{1}{x}$ and $n = 5$

(d)

$$\int \cos x \sin^3 x \, dx$$
$$= \int \cos x (\sin x)^3 dx$$
$$= \frac{(\sin x)^4}{4} + C$$

Good Strategy

Thinking Process:
Let $\mathrm{f}(x) = \sin x$
$\mathrm{f}'(x) = \cos x$ and $n = 3$

Example 5 **Concept:** $\int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^{-1}\mathrm{d}x = \int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\left|\mathrm{f}(x)\right| + C$

(a)

$$\int \frac{x}{3-2x^2}\,\mathrm{d}x$$

$$= -\frac{1}{4}\int \frac{-4x}{3-2x^2}\,\mathrm{d}x$$

$$= -\frac{1}{4}\ln\left|3-2x^2\right| + C$$

Compensate with **constant** $-\frac{1}{4}$

Thinking Process:
Let $\mathrm{f}(x) = 3-2x^2$
$\mathrm{f}'(x) = -4x$

(b)

$$\int \frac{\mathrm{e}^{-2x}}{1+\mathrm{e}^{-2x}}\,\mathrm{d}x$$

$$= -\frac{1}{2}\int \frac{-2\mathrm{e}^{-2x}}{1+\mathrm{e}^{-2x}}\,\mathrm{d}x$$

$$= -\frac{1}{2}\ln\left|1+\mathrm{e}^{-2x}\right| + C$$

$$= -\frac{1}{2}\ln\left(1+\mathrm{e}^{-2x}\right) + C, \text{ since } 1+\mathrm{e}^{-2x} > 0 \ \ \forall x \in \mathbb{R}$$

Compensate with **constant** $-\frac{1}{2}$

Thinking Process:
Let $\mathrm{f}(x) = 1+\mathrm{e}^{-2x}$
$\mathrm{f}'(x) = -2\mathrm{e}^{-2x}$

Example 6 **Concept:** $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + C$

(a)

$$\int x\mathrm{e}^{x^2+1}\,\mathrm{d}x$$

$$= \frac{1}{2}\int 2x\mathrm{e}^{x^2+1}\,\mathrm{d}x$$

$$= \frac{1}{2}\mathrm{e}^{x^2+1} + C$$

Compensate with **constant** $\frac{1}{2}$

Thinking Process:
Let $\mathrm{f}(x) = x^2+1$
$\mathrm{f}'(x) = 2x$

(b)

$$\int \frac{\mathrm{e}^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

$$= \int \frac{1}{\sqrt{x}}\mathrm{e}^{\sqrt{x}}\,\mathrm{d}x$$

Good Strategy

$$= 2\int \frac{1}{2\sqrt{x}}\mathrm{e}^{\sqrt{x}}\,\mathrm{d}x$$

$$= 2\mathrm{e}^{\sqrt{x}} + C$$

Compensate with **constant** 2

Thinking Process:
Let $\mathrm{f}(x) = \mathrm{e}^{\sqrt{x}}$
$\mathrm{f}'(x) = \frac{1}{2\sqrt{x}}$

§7 Integration of Algebgraic Functions of the standard form (MF27)

$\dfrac{1}{a^2+x^2}$, $\dfrac{1}{\sqrt{a^2-x^2}}$, $\dfrac{1}{a^2-x^2}$, $\dfrac{1}{x^2-a^2}$.

Golden Rule:
When x is replaced by a linear form $px+q$, we divide the answer by coefficient of x.

$f(x)$	$\int f(x)\,dx$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$	

Taken from MF27 (Pg 4): ***Integrals*

Note: **The restrictions on x in MF27 is to ensure that the function within the ln( ) is positive. However, if we convert the ( ) to modulus | |, we will not have to worry about the range of values of x.

Example 7

(a)	$\displaystyle\int \frac{1}{x^2+25}\,dx = \int \frac{1}{x^2+5^2}\,dx$ $= \dfrac{1}{5}\tan^{-1}\left(\dfrac{x}{5}\right)+C$	**(b)**	$\displaystyle\int \frac{3}{5-x^2}\,dx = 3\int \frac{1}{\left(\sqrt{5}\right)^2-x^2}\,dx$ $= \dfrac{3}{2\left(\sqrt{5}\right)}\ln\left\lvert\dfrac{\sqrt{5}+x}{\sqrt{5}-x}\right\rvert+C$ **Formula in MF27:** Put modulus for ln function
(c)	$\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx = \int \frac{1}{\sqrt{3^2-(2x)^2}}\,dx$ $= \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{3}\right)+C$ **Golden Rule:** Replace x by $(2x)$, so we need to divide by coefficient of x which is 2.		

§8 Integration of Algebraic Functions of the form $\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\,\mathrm{d}x$ or $\int \frac{\mathrm{f}(x)}{\sqrt{\mathrm{g}(x)}}\,\mathrm{d}x$

Given $\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\,\mathrm{d}x$ or $\int \frac{\mathrm{f}(x)}{\sqrt{\mathrm{g}(x)}}\,\mathrm{d}x$, where $\mathrm{f}(x)$ and $\mathrm{g}(x)$ **are polynomials** in x,

1. Check whether $\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ is **proper** [i.e. degree of $\mathrm{f}(x)$ is less than degree of $\mathrm{g}(x)$]

2. If $\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ is improper, use long division or 'juggling' method to express
$\frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \text{quotient} + \frac{\text{remainder}}{\mathrm{g}(x)}$.

3. Consider to change $\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\,\mathrm{d}x$ or $\int \frac{\mathrm{f}(x)}{\sqrt{\mathrm{g}(x)}}\,\mathrm{d}x$ using the following techniques (in the following order)

Section 8.1 Using $\int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)}\,\mathrm{d}x$ or $\int \mathrm{g}'(x)\left[\mathrm{g}(x)\right]^n\,\mathrm{d}x$

Section 8.2 Using Completing the square for denominator and MF27 (4 formulas)

Section 8.3 Using Partial Fractions if $\mathrm{g}(x)$ can be factorised completely

Section 8.4 Using splitting the numerator

8.1 Using $\int \frac{g'(x)}{g(x)}\,dx$ or $\int g'(x)[g(x)]^n\,dx$

If the algebraic function involves $g(x)$ and $g'(x)$, use

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C \quad\text{or}\quad \int g'(x)[g(x)]^n\,dx = \frac{[g(x)]^{n+1}}{n+1} + C$$

Example 8

(a)

$$\int \frac{x}{2+x^2}\,dx = \frac{1}{2}\int \frac{2x}{2+x^2}\,dx$$

$$= \frac{1}{2}\ln|2+x^2| + C$$

$$= \frac{1}{2}\ln(2+x^2) + C \text{ since } 2+x^2 > 0 \text{ for all } x \in \mathbb{R}$$

(b)

$$\int \frac{x^4+8x^2+x+4}{x^3+2x+1}\,dx$$

Observation: $6x^2+4$ can be expressed as a scalar multiple of derivative of x^3+2x+1.

Note: $\frac{x^4+8x^2+x+4}{x^3+2x+1}$ is **not proper fraction** $\Rightarrow$ perform long division.

$$= \int x + \frac{6x^2+4}{x^3+2x+1}\,dx$$

$$= \frac{x^2}{2} + 2\int \frac{3x^2+2}{x^3+2x+1}\,dx$$

$$= \frac{x^2}{2} + 2\ln|x^3+2x+1| + C$$

Compensate with **constant** 2

Thinking Process:

Let $g(x) = x^3+2x+1$

$g'(x) = 3x^2+2$

(c)

Observation: $4x^2+1$ can be expressed as a scalar multiple of derivative of $4x^3+3x$.

$$\int \frac{4x^2+1}{\sqrt{4x^3+3x}}\,dx$$

$$= \frac{1}{3}\int (12x^2+3)(4x^3+3x)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\frac{(4x^3+3x)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}(4x^3+3x)^{\frac{1}{2}} + C$$

Compensate with **constant** $\frac{1}{3}$

Thinking Process:

Let $g(x) = 4x^3+3x$

$g'(x) = 12x^2+3$ and $n = -\frac{1}{2}$

8.2 Using Completing the Square and MF27

For $\frac{f(x)}{g(x)}$ or $\int \frac{f(x)}{\sqrt{g(x)}}\,dx$ where $f(x)$ is a constant and $g(x)$ is a quadratic function, use complete the square for $g(x)$ and use MF27 formulas where appropriate.

Example 9

(a)

$$\int \frac{x^2+2x+7}{x^2+2x+4}\,dx$$

Note: $\frac{x^2+2x+7}{x^2+2x+4}$ is **not proper fraction** $\Rightarrow$ perform long division.

$$= \int 1+\frac{3}{x^2+2x+4}\,dx$$

Observation: $\frac{3}{x^2+2x+4}$ is $\frac{\text{constant}}{\text{quadratic}} \Rightarrow$ MF27

$$= \int 1+\frac{3}{(x+1)^2+\left(\sqrt{3}\right)^2}\,dx$$

Complete the square.

MF27 Pg 4
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= x+\frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

$$= x+\sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

(b)

$$\int \frac{3}{\sqrt{7-25x^2-30x}}\,dx$$

Observation: $\frac{3}{\sqrt{7-25x^2-30x}}$ is $\frac{\text{constant}}{\sqrt{\text{quadratic}}} \Rightarrow$ MF27

$$= \int \frac{3}{\sqrt{4^2-(5x+3)^2}}\,dx$$

Complete the square.
$7-25x^2-30x$
$= -\left[(5x)^2+2(5x)(3)+3^2-3^2\right]+7$
$= -\left[(5x+3)^2-9\right]+7$
$= 16-(5x+3)^2$

MF27 Pg 4
$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{3}{5}\sin^{-1}\left(\frac{5x+3}{4}\right)+C$$

Apply Golden Rule

(c)

$$\int \frac{4}{-x^2+2x+3}\,dx$$

$$= \int \frac{4}{2^2-(x-1)^2}\,dx$$

Observation: $\dfrac{4}{-x^2+2x+3}$ is $\dfrac{\text{constant}}{\text{quadratic}} \Rightarrow$ MF27

Complete the square.

$$-x^2+2x+3$$

$$= -\left[x^2-2(x)(1)+1^2-1^2\right]+3$$

$$= -\left[(x-1)^2-1\right]+3$$

$$= 4-(x-1)^2$$

MF27 Pg 4

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$$

$$= \frac{4}{2(2)}\ln\left|\frac{2+(x-1)}{2-(x-1)}\right| + C$$

$$= \ln\left|\frac{1+x}{3-x}\right| + C$$

Formula in MF27:
Put modulus for ln function

Alternative Method (using Partial Fractions in Section 8.3)

$$\int \frac{4}{-x^2+2x+3}\,dx$$

Factorise $g(x) = -x^2+2x+3$

$$= \int \frac{4}{(1+x)(3-x)}\,dx$$

Partial Fractions

$$= \int \frac{1}{1+x} + \frac{1}{3-x}\,dx$$

$$= \ln|1+x| - \ln|3-x| + C$$

$$= \ln\left|\frac{1+x}{3-x}\right| + C$$

8.3 Using Partial Fractions

Check $\frac{f(x)}{g(x)}$ is **proper**. If $g(x)$ can be fully factorised, express $\frac{f(x)}{g(x)}$ in partial fractions then use MF27 formulas where appropriate.

Example 10

$$\frac{3x-x^3}{(x+1)^2(x^2+1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+1}$$

Solving, $A=-1,\ B=-1,\ C=0,\ D=2$

$$\int\frac{3x-x^3}{(x+1)^2(x^2+1)}\,dx$$

$$=\int-\frac{1}{x+1}-\frac{1}{(x+1)^2}+\frac{2}{x^2+1}\,dx$$

$\int\frac{1}{(x+1)^2}\,dx$ can be written as $\int(x+1)^{-2}\,dx$

$\int\frac{1}{x}\,dx=\ln|x|$

$\int x^n\,dx=\frac{x^{n+1}}{n+1}$

MF27 Pg 4

$\int\frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$=-\ln|x+1|-\frac{1}{(-1)(x+1)}+2\tan^{-1}x+C$$

$$=-\ln|x+1|+\frac{1}{x+1}+2\tan^{-1}x+C$$

From MF27:

Partial fractions decomposition

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)}=\frac{A}{(ax+b)}+\frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2}=\frac{A}{(ax+b)}+\frac{B}{(cx+d)}+\frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)}=\frac{A}{(ax+b)}+\frac{Bx+C}{(x^2+c^2)}$$

8.4 <u>Using splitting the numerator</u>

(1) For $\int \frac{px+q}{ax^2+bx+c}\,dx$, where $g(x)=ax^2+bx+c$ cannot be factorised into real linear factors:

Method: Express $px+q$ in terms of $g'(x)$. I.e Find constants A and B such that $px+q=A(2ax+b)+B$ where $g'(x)=2ax+b$ and then use the "splitting the numerator method".

Example 11

By first expressing $x+1$ as $A(2x+4)+B$, find $\int \frac{x+1}{x^2+4x+6}\,dx$.

$x+1=A(2x+4)+B$

$\quad =2Ax+(4A+B)$

Note: $g(x)=x^2+4x+6 \Rightarrow g'(x)=2x+4$

Compare coefficient:

$1=2A \Rightarrow A=\frac{1}{2}$

$1=4A+B \Rightarrow B=-1$

$x+1=\frac{1}{2}(2x+4)-1$

$$\int \frac{x+1}{x^2+4x+6}\,dx=\int \frac{\frac{1}{2}(2x+4)-1}{x^2+4x+6}\,dx$$

Splitting the numerator.

$$=\frac{1}{2}\int \frac{2x+4}{x^2+4x+6}\,dx-\int \frac{1}{x^2+4x+6}\,dx$$

Complete the square.

$$=\frac{1}{2}\int \frac{2x+4}{x^2+4x+6}\,dx-\int \frac{1}{(x+2)^2+(\sqrt{2})^2}\,dx$$

Thinking Process:

Let $g(x)=x^2+4x+6$

$g'(x)=2x+4$

$\int \frac{g'(x)}{g(x)}\,dx=\ln|g(x)|$

MF27 Pg 4

$\int \frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$=\frac{1}{2}\ln|x^2+4x+6|-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+2}{\sqrt{2}}+C$$

(2) For $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x$ where $\mathrm{g}(x)=ax^2+bx+c$:

Method: Express $px+q$ in terms of $\mathrm{g}'(x)$. I.e Find constants A and B such that $px+q=A(2ax+b)+B$ where $\mathrm{g}'(x)=2ax+b$ and then use the "splitting the numerator method".

Example 12

Find $\int \frac{3+x}{\sqrt{2-2x-x^2}}\,\mathrm{d}x$.

Let $\mathrm{g}(x)=2-2x-x^2 \Rightarrow \mathrm{g}'(x)=-2-2x$

Therefore, $3+x=A(-2-2x)+B$

Comparing coefficient of x: $1=-2A \Rightarrow A=-\frac{1}{2}$

Comparing constant: $3=-2A+B \Rightarrow B=3+2\left(-\frac{1}{2}\right)=2$

$\therefore 3+x=-\frac{1}{2}(-2-2x)+2$

$$\int \frac{3+x}{\sqrt{2-2x-x^2}}\,\mathrm{d}x=\int \frac{-\frac{1}{2}(-2-2x)+2}{\sqrt{2-2x-x^2}}\,\mathrm{d}x$$

$$=-\frac{1}{2}\int \frac{-2-2x}{\sqrt{2-2x-x^2}}\,\mathrm{d}x+\int \frac{2}{\sqrt{2-2x-x^2}}\,\mathrm{d}x$$

Splitting the numerator.

$$=-\frac{1}{2}\int (-2-2x)(2-2x-x^2)^{-\frac{1}{2}}\,\mathrm{d}x+2\int \frac{1}{\sqrt{(\sqrt{3})^2-(x+1)^2}}\,\mathrm{d}x$$

Thinking Process:
Let $\mathrm{g}(x)=2-2x-x^2$

$\mathrm{g}'(x)=-2-2x$ and $n=-\frac{1}{2}$

$\int \mathrm{g}'(x)[\mathrm{g}(x)]^n\,\mathrm{d}x=\frac{[\mathrm{g}(x)]^{n+1}}{n+1}$

MF27 Pg 4
$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x}{a}\right)$

Complete the square.
$2-2x-x^2$
$=2-(x^2+2x+1^2-1^2)$
$=2-\left[(x+1)^2-1\right]$
$=3-(x+1)^2$

$$=-\frac{1}{2}\frac{(2-2x-x^2)^{\frac{1}{2}}}{\frac{1}{2}}+2\sin^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

$$=-(2-2x-x^2)^{\frac{1}{2}}+2\sin^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

§9 Integration of Trigonometric Functions

9.1 Basic Trigonometric Functions

$f(x)$	$\int f(x)\,dx$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert$

Golden Rule: When x is replaced by a linear form $(px+q)$, we divide the answer by coefficient of x.

Taken from MF27 (Pg 4): ***Integrals*

Note: **The restrictions on x in MF27 is to ensure that the function within the ln( ) is positive. However, if we convert the ( ) to modulus | |, we will not have to worry about the range of values of x.

Example 13

(a) $\int 5\sin 2x\,dx$

$=-\frac{5}{2}\cos 2x+C$

Apply Golden Rule

(b) $\int 3\cos\left(4x-\frac{\pi}{6}\right)dx$

$=\frac{3}{4}\sin\left(4x-\frac{\pi}{6}\right)+C$

Golden Rule:
Replace x by $\left(4x-\frac{\pi}{6}\right)$, so we need to divide by coefficient of x which is 4.

(c) $\int \operatorname{cosec}5x\,dx$

$=-\frac{1}{5}\ln\lvert\operatorname{cosec}5x+\cot 5x\rvert+C$

Apply Golden Rule

Formula in MF27:
Put modulus for ln function

(d) $\int \sec(1-x)\,dx$

$=-\ln\lvert\sec(1-x)+\tan(1-x)\rvert+C$

Apply Golden Rule

Formula in MF27:
Put modulus for ln function

9.2 Powers of Basic Trigonometric Functions

Golden Rule:
When x is replaced by a linear form $(px+q)$, we divide the answer by coefficient of x.

i) $\int \sec^2 x \, dx = \tan x + C$ $\because \frac{d}{dx}\tan x = \sec^2 x$

ii) $\int \text{cosec}^2 x \, dx = -\cot x + C$ $\because \frac{d}{dx}\cot x = -\text{cosec}^2 x$

Integration is the reverse process of Differentiation.

iii) $\int \cot^2 x \, dx$

iv) $\int \tan^2 x \, dx$

Make use of trigonometry identities
$1+\cot^2 A = \text{cosec}^2 A$
$1+\tan^2 A = \sec^2 A$

v) $\int \sin^2 x \, dx$

vi) $\int \cos^2 x \, dx$

Make use of double angle formula (in MF27)
$\cos 2A = 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$

Example 14

(a)

$\int \sec^2(3x-5)\, dx$

$= \frac{1}{3}\tan(3x-5) + C$

Golden Rule:
Replace x by $(3x-5)$, so we need to divide by coefficient of x which is 3.

(b)

$\int \cot^2 3x \, dx = \int (\text{cos ec}^2 3x - 1)\, dx$

$= -\frac{1}{3}\cot 3x - x + C$

Apply Golden Rule

(c)

$\int \cos^2\theta \, d\theta = \frac{1}{2}\int 1+\cos 2\theta \, d\theta$

$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$

Double angle formula (in MF 27):
$\cos 2A = 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$

Apply Golden Rule

9.3 Integration of Functions of the form sec x tan$x$ and cosec x cot x

Golden Rule:
When x is replaced by a linear form $(px+q)$, we divide the answer by coefficient of x.

$f(x)$	$f'(x)$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Taken from MF27 (Pg 3): ***Derivatives***

Example 15

(a)	$\int \sec(2x+1)\tan(2x+1)\,dx$ $=\frac{1}{2}\sec(2x+1)+C$ **Golden Rule:** Replace x by $(2x+1)$, so we need to divide by coefficient of x which is 2.	**(b)**	$\int \operatorname{cosec}(3x-1)\cot(3x-1)\,dx$ $=-\frac{1}{3}\operatorname{cosec}(3x-1)+C$ **Golden Rule:** Replace x by $(3x-1)$, so we need to divide by coefficient of x which is 3.

§10 Integration by Substitution

When the integration of $\mathrm{f}(x)$ cannot be obtained directly, we can apply the method of integration by substitution. A suitable substitution is introduced so that the $\mathrm{f}(x)$ can be reduced into one which is **similar to one of the standard forms**.

Let $I = \int \mathrm{f}(x)\,\mathrm{d}x$

Differentiate **w.r.t. *x*:**

Since differentiation is a reverse process of integration.

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \mathrm{f}(x).$$

Assume x is a function of u (a new variable) i.e. $x = \mathrm{g}(u)$. By applying chain rule, we have

$$\frac{\mathrm{d}I}{\mathrm{d}u} = \frac{\mathrm{d}I}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}u} = \mathrm{f}(x)\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Integrate **w.r.t. *u***:

Since integration is a reverse process of differentiation.

$$I = \int \mathrm{f}(x)\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

and we obtain

$$\int \mathrm{f}(x)\,\mathrm{d}x = \int \mathrm{f}(\mathrm{g}(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

Example 16

(a) Using the substitution $x = \frac{1}{u}$, find $\int \frac{1}{x\sqrt{x^2-2}}\,\mathrm{d}x$, where $x > 0$.

Solution:

$$\int \frac{1}{x\sqrt{x^2-2}}\,\mathrm{d}x = \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-2}}\left(-\frac{1}{u^2}\right)\mathrm{d}u$$

$$= \int \frac{-1}{u\sqrt{\frac{1-2u^2}{u^2}}}\,\mathrm{d}u$$

$$= \int \frac{-1}{\sqrt{1-2u^2}}\,\mathrm{d}u$$

$$= -\int \frac{1}{\sqrt{1^2-\left(\sqrt{2}u\right)^2}}\,\mathrm{d}u$$

MF27 Pg 4

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}u\right) + C$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}}{x}\right) + C$$

Since $x > 0$, $u > 0$ as $x = \frac{1}{u}$.
Therefore, $\sqrt{u^2}$ can be simplified to u instead of $|u|$.

Steps:

(1) Using $x = \frac{1}{u}$

$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}$

$\mathrm{d}x = -\frac{1}{u^2}\,\mathrm{d}u$

(2) Replace expression using $x = \frac{1}{u}$

$\frac{1}{x\sqrt{x^2-2}} = \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-2}}$

(3) Replace variable u back to original variable x using $x = \frac{1}{u}$

(b)

Using the substitution $u = x^3$, evaluate $\int_0^3 \frac{x^2}{1+x^6}\,dx$.

Solution:

$$\int_0^3 \frac{x^2}{1+x^6}\,dx$$

$$= \int_0^{27} \frac{1}{1+u^2}\left(\frac{1}{3}\right)du$$

$$= \frac{1}{3}\int_0^{27} \frac{1}{1+u^2}\,du$$

MF27 Pg 4

$$= \frac{1}{3}\left[\tan^{-1} u\right]_0^{27}$$

$$= \frac{1}{3}\tan^{-1}(27)$$

Steps:

(1) Using $u = x^3$

$$\frac{du}{dx} = 3x^2$$

$$dx = \frac{1}{3x^2}\,du$$

(2) Replace expression using $u = x^3$

$$\frac{1}{1+x^6} = \frac{1}{1+u^2}$$

(3) Replace limit using $u = x^3$

When $x = 3$, $u = 3^3 = 27$

When $x = 0$, $u = 0^3 = 0$

(c)

Using the substitution $x = \sin\theta$, find $\int \sqrt{1-x^2}\,dx$ for $0 \le \theta \le \frac{\pi}{2}$.

Solution:

$$\int \sqrt{1-x^2}\,dx$$

$$= \int \sqrt{1-\sin^2\theta}\,(\cos\theta)\,d\theta$$

$$= \int \sqrt{\cos^2\theta}\,(\cos\theta)\,d\theta$$

$$= \int (\cos\theta)(\cos\theta)\,d\theta \quad \because 0 \le \theta \le \frac{\pi}{2}$$

$$= \int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int (\cos 2\theta + 1)\,d\theta$$

Double-angle formula (MF27 Pg 3)

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C$$

$$= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C$$

$$= \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + C$$

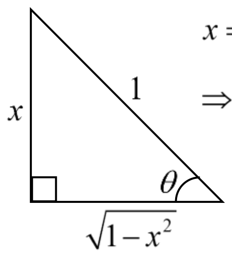
(1) Using $x = \sin\theta$

$$\Rightarrow \frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta\,d\theta$$

(2): Replace expression using $x = \sin\theta$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$$



$x = \sin\theta$

$\Rightarrow \sin\theta = \frac{x}{1}$

(3) Replace the variable back to x.
From the triangle,

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{\sqrt{1-x^2}}{1}$$

Useful strategy: Use right angle triangle to obtain exact trigo ratio.

§11 <u>Integration by Parts</u>

This method is usually used to integrate (i) a single function or (ii) product of 2 functions which ***cannot be integrated using any of the standard forms or using substitution.***

For functions of the form $\mathrm{f}(x) = u.v$ where u and v are non-zero functions of x, we use integration by parts:

$$\int uv\ \mathrm{d}x = u\left(\int v\ \mathrm{d}x\right) - \int \frac{\mathrm{d}u}{\mathrm{d}x}\left(\int v\ \mathrm{d}x\right)\mathrm{d}x$$ (**K**eep **I**ntegrate $-\int$ **D**ifferentiate **I**ntegrate)

Note:

1. When integrating a product of 2 functions, the function that cannot be directly integrated, is chosen as 'u' and the other which can be integrated as 'v'.
   e.g. $\int x\tan^{-1}x\,\mathrm{d}x$; $\int \sqrt{x}\ln 2x\ \mathrm{d}x$

2. This method is useful in finding integrals of single functions which are differentiable but cannot be directly integrated. The integrand is chosen as 'u' and unity as 'v'.
   e.g. $\int \ln x\ \mathrm{d}x$; $\int \tan^{-1}x\,\mathrm{d}x$; $\int \sin^{-1}x\,\mathrm{d}x$

As a general rule, choose u (the one to keep) in the following order:

L - logarithmic functions	e.g. :	$\ln x$, $\ln(2x-3)$
I - inverse trigonometric functions	e.g. :	$\sin^{-1}(x+1)$, $\tan^{-1}x$
A - algebraic functions	e.g. :	3, $4x^2+1$
T - trigonometric functions	e.g. :	$\sin x$, $\cos 2x$
E - exponential functions	e.g. :	e^x, e^{-2x}

Note: there are times when this rule can be relaxed e.g. when integrating $\mathrm{e}^x\sin x$.

Example 17

(a)

$$\int x\mathrm{e}^{-2x}\,\mathrm{d}x$$

$\int \mathrm{e}^{-2x}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-2x}$

Keep Integrate $-\int$ Differentiate Integrate

$$= x\left(-\frac{1}{2}\mathrm{e}^{-2x}\right) - \int (1)\cdot\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\mathrm{d}x$$

$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$

$$= -\frac{1}{2}x\mathrm{e}^{-2x} + \frac{1}{2}\int \mathrm{e}^{-2x}\,\mathrm{d}x$$

$$= -\frac{1}{2}x\mathrm{e}^{-2x} - \frac{1}{4}\mathrm{e}^{-2x} + C$$

Key Concept:
Apply Integration by parts using LIATE.

x Keep

e^{-2x}

(b)

$$\int x\cos x\,\mathrm{d}x$$

$\int \cos x\,\mathrm{d}x = \sin x$

Keep Integrate $-\int$ Differentiate Integrate

$$= x\sin x - \int (1)\sin x\,\mathrm{d}x$$

$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$

$$= x\sin x + \cos x + C$$

Key Concept:
Apply Integration by parts using LIATE.

x Keep

$\cos x$

(c)

$$\int \tan^{-1}x\,\mathrm{d}x$$

$$= \int (1)\tan^{-1}x\,\mathrm{d}x$$

$\int 1\,\mathrm{d}x = x$

Keep Integrate $-\int$ Differentiate Integrate

$$= \left(\tan^{-1}x\right)(x) - \int \frac{1}{1+x^2}\cdot(x)\,\mathrm{d}x$$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

$$= x\tan^{-1}x - \frac{1}{2}\int\left(\frac{2x}{1+x^2}\right)\mathrm{d}x$$

$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\left|\mathrm{f}(x)\right|$

$$= x\tan^{-1}x - \frac{1}{2}\ln\left|1+x^2\right| + C$$

$$= x\tan^{-1}x - \frac{1}{2}\ln\left(1+x^2\right) + C \qquad (\because 1+x^2 > 0)$$

Key Concept:
Introduce 1 and apply Integration by parts using LIATE.

$\tan^{-1}x$ Keep

1

(d) $\int_1^e x\ln x \, dx$

Strategy:
Consider solving the question without limits first.

Consider $\int x\ln x\,dx$ first:

$\int x\ln x \, dx$

$\int x\,dx = \frac{1}{2}x^2$

Key Concept:
Apply Integration by parts **(with limits)** using LIATE.

$\ln x$ Keep

x

Keep Integrate $- \int$ **Differentiate Integrate**

$= (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right) dx$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

$\therefore \int_1^e x\ln x\,dx$

$= \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^e$

$= \left(\frac{1}{2}e^2\ln e - \frac{1}{4}e^2\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right)$

$= \frac{1}{4}e^2 + \frac{1}{4}$

Strategy:
Apply the limits only after solving the question.

Alternatively: Incorporate the limits into integration by parts formula.

$\therefore \int_1^e x\ln x\,dx$

$= \left[\ln x\left(\frac{x^2}{2}\right)\right]_1^e - \int_1^e \frac{1}{x}\left(\frac{x^2}{2}\right)dx$

$= \left(\frac{e^2}{2}\ln e - 0\right) - \frac{1}{2}\int_1^e x\,dx$

$= \frac{e^2}{2} - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$

$= \frac{e^2}{2} - \frac{1}{2}\left[\frac{e^2}{2} - \frac{1}{2}\right]$

$= \frac{1}{4}e^2 + \frac{1}{4}$

Take note of how the limits is applied in this case.

(e)

$\int x^2 \sin x \, dx$

$\int \sin x \, dx = -\cos x$

Keep Integrate $- \int$ **Differentiate Integrate**

$= -x^2 \cos x - \int -(2x)\cos x \, dx$

$\frac{d}{dx}(x^2) = 2x$

$= -x^2 \cos x + 2\int x \cos x \, dx$

$= -x^2 \cos x + 2\left[x \sin x - \int (1)\sin x \, dx \right]$

$= -x^2 \cos x + 2x \sin x + 2\cos x + C$

Apply Integration by parts formula again using LIATE.

Key Concept:
Apply Integration by parts **TWICE** using LIATE.aa

(f)

$\int e^x \sin x \, dx$

$\int e^x \, dx = e^x$

Keep Integrate $- \int$ **Differentiate Integrate**

$= e^x \sin x - \int (\cos x) e^x \, dx$

$\frac{d}{dx}(\sin x) = \cos x$

$= e^x \sin x - \left[\cos x (e^x) - \int (-\sin x) e^x \, dx \right]$

$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

Apply Integration by parts formula again using LIATE.

Key Concept:
Apply Integration by parts **TWICE** using LIATE and looping (original integral appears in the result) occurs.

$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

Strategy:
Group the original integral together.

$\Rightarrow 2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x + A$

$\Rightarrow \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C, \quad \text{where } C = \frac{A}{2}$

Annex:

Proving of importance result in page 6:

In general, if $\int \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(x) + C$, then we have

$$\int \mathrm{f}(px+q)\,\mathrm{d}x = \frac{1}{p}\mathrm{F}(px+q) + C.$$

$\int \mathrm{f}(px+q)\,\mathrm{d}x$

$= \int \mathrm{f}(u)\frac{1}{p}\,\mathrm{d}u$

$= \frac{1}{p}\mathrm{F}(u) + C$

$= \frac{1}{p}\mathrm{F}(px+q) + C$

(1) let $u = px + q$

$\frac{\mathrm{d}u}{\mathrm{d}x} = p$

(2) Replace expression $\mathrm{f}(px+q) = \mathrm{f}(u)$

Since $\int \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(x) + C$

(3) Replace variable back to x

Derivation of results in MF 27

Derivation of results marked with * is required in the A Level syllabus.
(a denotes a positive constant.)

$$*\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Proof:

Let $y = \tan^{-1}\left(\frac{x}{a}\right)$

$\therefore \frac{x}{a} = \tan y$

$\frac{1}{a} = \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\sec^2 y}$

$= \frac{1}{a\left(\left(\frac{x}{a}\right)^2 + 1\right)}$ since $\tan^2 y + 1 = \sec^2 y$ and $\frac{x}{a} = \tan y$

$= \frac{1}{a\left(\frac{x^2+a^2}{a^2}\right)} = \frac{a}{x^2+a^2}$

Hence $\int \frac{a}{x^2+a^2}\,\mathrm{d}x = \tan^{-1}\left(\frac{x}{a}\right) + C$

$\Rightarrow \int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

In particular, when $a = 1$, we have $\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + C$.

$$*\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right)+C,\ |x|<a$$

Proof:

Let $y=\sin^{-1}\left(\frac{x}{a}\right)$

$\therefore \frac{x}{a}=\sin y$

$\frac{1}{a}=\cos y\frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{a\cos y}$

$=\frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}$ since $\sin^2 y+\cos^2 y=1$ and $\frac{x}{a}=\sin y$

$=\frac{1}{a\sqrt{\frac{a^2-x^2}{a^2}}}$

$=\frac{1}{\sqrt{a^2-x^2}}$

Hence $\int\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x}{a}\right)+C$.

In particular, when a = 1, we have $\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\sin^{-1}x+C$.

For the rest of the formulae: Use the result $\int\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x=\ln|\mathrm{f}(x)|+C$.

$$*\int\frac{1}{x^2-a^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C,\ x>a$$

Proof:

$$\begin{aligned}\int\frac{1}{x^2-a^2}\,\mathrm{d}x &= \int\frac{1}{(x+a)(x-a)}\,\mathrm{d}x\\ &=\frac{1}{2a}\int\left[\frac{1}{(x-a)}-\frac{1}{(x+a)}\right]\mathrm{d}x\\ &=\frac{1}{2a}\left[\ln|x-a|-\ln|x+a|\right]+C\\ &=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C\end{aligned}$$

$$*\int \frac{1}{a^2-x^2}\ \mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C,\ \ |x|<a$$

Proof:

$$\int \frac{1}{a^2-x^2}\ \mathrm{d}x = \int \frac{1}{(a+x)(a-x)}\ \mathrm{d}x$$

$$= \frac{1}{2a}\int\left[\frac{1}{(a+x)}+\frac{1}{(a-x)}\right]\mathrm{d}x$$

$$= \frac{1}{2a}\left[\ln|a+x| - \ln|a-x|\right] + C = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$\int \tan x\,\mathrm{d}x = \ln|\sec x| + C,\ \ |x|<\frac{1}{2}\pi$$

Proof:

$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x$$

$$= -\int \frac{-\sin x}{\cos x}\mathrm{d}x$$

$$= -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$$

$$\int \cot x\ \mathrm{d}x = \ln|\sin x| + C,\ \ 0<x<\pi$$

Proof:

$$\int \cot x\ \mathrm{d}x = \int \frac{\cos x}{\sin x}\mathrm{d}x = \ln|\sin x| + C$$

$$\int \operatorname{cosec} x\ \mathrm{d}x = -\ln\ |\operatorname{cosec} x + \cot x| + C,\ \ 0<x<\pi$$

Proof:

$$\int \operatorname{cosec} x\ \mathrm{d}x = -\int -\operatorname{cosec} x\,\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x}\mathrm{d}x$$

$$= -\int \frac{-\operatorname{cosec} x\cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x}\mathrm{d}x$$

$$= -\ln\ |\operatorname{cosec} x + \cot x| + C$$

$$\int \sec x\ \mathrm{d}x = \ln\ |\sec x + \tan x| + C,\ \ |x|<\frac{1}{2}\pi$$

Proof:

$$\int \sec x\ \mathrm{d}x = \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\mathrm{d}x$$

$$= \int \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}\mathrm{d}x = \ln\ |\sec x + \tan x| + C$$

H2 Mathematics (9758)
Chapter 10 Integration Techniques
Discussion Questions

Level 1

Integration – Reverse of Differentiation

1 Find $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\theta\cos\theta)$. Hence, find $\displaystyle\int \theta\sin\theta \,\mathrm{d}\theta$.

Integration of Standard Functions

2 Find the following integrals:

(a) $\displaystyle\int\left(2\sqrt{\mathrm{e}^{x}}+3\mathrm{e}^{5-3x}\right)\mathrm{d}x$

(b) $\displaystyle\int_k^1\left(1+\frac{2}{x}\right)^2 \mathrm{d}x$, $k>0$

(c) $\displaystyle\int\frac{(2x-5)(x+2)}{\sqrt{x}}\mathrm{d}x$

Integration involving the function and its derivative

Formula to memorise (not in MF27) and apply

(1) For $n\in\mathbb{R},\ n\neq -1,\ \displaystyle\int \mathrm{f}\,'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x=\frac{[\mathrm{f}(x)]^{n+1}}{n+1}+C$

(2) For $n\in\mathbb{R},\ n=-1,\ \displaystyle\int \mathrm{f}\,'(x)\left[\mathrm{f}(x)\right]^{-1}\mathrm{d}x=\int\frac{\mathrm{f}\,'(x)}{\mathrm{f}(x)}\,\mathrm{d}x=\ln\left|\mathrm{f}(x)\right|+C$

(3) $\displaystyle\int \mathrm{f}\,'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x=\mathrm{e}^{\mathrm{f}(x)}+C$

3 Find the following integrals:

(a) $\displaystyle\int x\sqrt{3-7x^2}\ \mathrm{d}x$

(b) $\displaystyle\int\frac{x}{2x^2-6}\,\mathrm{d}x$

(c) $\displaystyle\int x^2\mathrm{e}^{x^3+1}\mathrm{d}x$

Integration of Rational Algebraic Functions (including MF27)

4 Find the following integrals:

(a) $\int \frac{1}{3-4t^2}\,\mathrm{d}t$ **(b)** $\int \frac{1}{(x+3)(x+4)}\,\mathrm{d}x$

(c) $\int \frac{10}{x^2-2x+11}\,\mathrm{d}x$

Integration of Trigonometric Functions

5 **(a)** $\int \sin^3 x \cos x \,\mathrm{d}x$ **(b)** $\int \sin^2 x \,\mathrm{d}x$

Integration by substitution

6 Using the substitution $u=\sqrt{x}$ to find $\int \frac{1}{(1-x)\sqrt{x}}\,\mathrm{d}x$.

Integration by Parts

7 Find the following integrals:

(a) $\int (x+1)\mathrm{e}^{-x}\mathrm{d}x$

(b) $\int x\sin 2x\,\mathrm{d}x$

(c) $\int_1^{\mathrm{e}} x\ln x\,\mathrm{d}x$

Level 2

Integration – Reverse of Differentiation

8 Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \mathrm{e}^{x+1}\right)$. Hence, find $\int x\mathrm{e}^x (x+2)\,\mathrm{d}x$.

Integration involving the function and its derivative

Formula to memorise (not in MF27) and apply

(1) For $n \in \mathbb{R},\ n \neq -1,\ \int \mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + C$

(2) For $n \in \mathbb{R}, n = -1, \int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^{-1}\mathrm{d}x = \int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\left|\mathrm{f}(x)\right| + C$

(3) $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + C$

9 Find the following integrals:

(a) $\int \sin 2\theta \cos 2\theta \,\mathrm{d}\theta$

(b) $\int \mathrm{e}^{\cos\frac{x}{6}} \sin\frac{x}{6}\,\mathrm{d}x$

(c) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,\mathrm{d}x$

(d) $\int \frac{1}{x(1+\ln 3x)}\,\mathrm{d}x$

(e) $\int \frac{\mathrm{e}^{-3x}}{\left(2-\mathrm{e}^{-3x}\right)^3}\,\mathrm{d}x$

(f) $\int \frac{7x+3}{7x^2+6x}\,\mathrm{d}x$

Golden Rule:
When x is replaced by a linear form $(px+q)$, we divide by coefficient of x.

Integration of Rational Algebraic Functions (including MF27)

10 **N2009/I/2**

Find the exact value of p such that $\int_0^1 \frac{1}{4-x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1-p^2x^2}}\,\mathrm{d}x$. [5]

11 Find the following integrals:

(a) $\int \frac{6x^3}{3x^2+1}\,\mathrm{d}x$

(b) $\int \frac{3x^2+2}{\sqrt{\left(x^3+2x-8\right)}}\,\mathrm{d}x$

(c) $\int \frac{x+35}{x^2-25}\,\mathrm{d}x$

(d) $\int \frac{x-4}{x^2+6x+11}\,\mathrm{d}x$

12 **N2014/II/2**

Using partial fractions, find

$$\int_0^2 \frac{9x^2+x-13}{(2x-5)\left(x^2+9\right)}\,\mathrm{d}x$$

Give your answer in the form $a\ln b + c\tan^{-1} d$, where a, b, c and d are rational numbers to be determined. [9]

Integration of Trigonometric Functions

13 Find the following integrals:

(a) $\int \sec^2 x + 2\operatorname{cosec}^2\left(4x-\frac{\pi}{3}\right)\mathrm{d}x$

(b) $\int \cos^2 2x + \tan^2 2x\ \mathrm{d}x$

(c) $\int -\frac{1}{2}\sec\left(\frac{\pi}{6}-x\right)\tan\left(x-\frac{\pi}{6}\right)\mathrm{d}x$

(d) $\int \frac{1}{1+\cos 4x}\,\mathrm{d}x$

Integration by substitution

14 Using the suggested substitution, find:

(a) $\int \tan^3 x\ \mathrm{d}x$, let $u = \tan x$

(b) $\int \frac{1}{x^2\sqrt{25-x^2}}\,\mathrm{d}x$, let $x = 5\cos\theta$

(c) $\int \frac{1}{\mathrm{e}^x + 2\mathrm{e}^{-x}}\,\mathrm{d}x$, let $u = \mathrm{e}^x$

(d) $\int_{\pi/2}^{\pi} \frac{\sin\theta}{1+\cos^2\theta}\,\mathrm{d}\theta$, let $x = \cos\theta$

Integration by parts

15 Find the following integrals:

(a) $\int x^2\cos x\ \mathrm{d}x$

(b) $\int_0^{\frac{1}{\sqrt{2}}} x\sin^{-1}\left(x^2\right)\mathrm{d}x$

(c) $\int \mathrm{e}^{2x}\sin x\ \mathrm{d}x$

Level 3

16 **2009/MJC/II/1**

(i) Differentiate $\mathrm{e}^{\cos x}$ with respect to x. [1]

(ii) Find $\int \mathrm{e}^{\cos x} \sin 2x \, \mathrm{d}x$. [3]

17 **N2019/II/1**

You are given that $I = \int x(1-x)^{\frac{1}{2}} \, \mathrm{d}x$.

(i) Use integration by parts to find an expression for I. [2]

(ii) Use the substitution $u^2 = 1 - x$ to find another expression for I. [2]

(iii) Show algebraically that your answers to parts **(i)** and **(ii)** differ by a constant. [2]

Answer Key

1	$\sin\theta - \theta\cos\theta + C'$ OR $\sin\theta - \theta\cos\theta - C$
2	**(a)** $4\mathrm{e}^{\frac{x}{2}} - \mathrm{e}^{5-3x} + C$ **(b)** $\frac{4}{k} - k - 4\ln k - 3 \quad (k > 0)$ **(c)** $\frac{4}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 20\sqrt{x} + C$
3	**(a)** $-\frac{1}{21}(3-7x^2)^{\frac{3}{2}} + C$ (b) $\frac{1}{4}\ln\left|2x^2 - 6\right| + C$ (c) $\frac{1}{3}\mathrm{e}^{x^3+1} + C$
4	**(a)** $\frac{\sqrt{3}}{12}\ln\left|\frac{\sqrt{3}+2t}{\sqrt{3}-2t}\right| + C$ **(b)** $\ln\left|\frac{x+3}{x+4}\right| + C$ **(c)** $\sqrt{10}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right) + C$
5	**(a)** $\frac{\sin^4 x}{4} + C$ **(b)** $\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$
6	$\ln\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right| + C$

7	**(a)** $-\frac{x+2}{e^x}+C$ **(b)** $\frac{1}{4}\sin 2x-\frac{1}{2}x\cos 2x+C$ **(c)** $\frac{1}{4}\left(e^2+1\right)$
8	$\frac{d}{dx}\left(x^2e^{x+1}\right)=xe^{x+1}\left(2+x\right)$; $\frac{1}{e}\left(x^2e^{x+1}\right)+\frac{C}{e}$ OR x^2e^x+B where $B=\frac{C}{e}$
9	**(a)** $\frac{1}{4}\sin^2 2\theta+C$ OR $-\frac{1}{8}\cos 4\theta+C$ **(b)** $-6e^{\cos\frac{x}{6}}+C$ **(c)** $\frac{\left(\sin^{-1}x\right)^2}{2}+C$ **(d)** $\ln\left\|1+\ln 3x\right\|+C$ **(e)** $-\frac{1}{6}\left(2-e^{-3x}\right)^{-2}+C$ **(f)** $\frac{1}{2}\ln\left\|7x^2+6x\right\|+C$
10	$p=\frac{2\pi}{3\ln 3}$
11	**(a)** $x^2-\frac{1}{3}\ln\left(3x^2+1\right)+C$ **(b)** $2\left(x^3+2x-8\right)^{\frac{1}{2}}+C$ **(c)** $4\ln\left\|x-5\right\|-3\ln\left\|x+5\right\|+C$ OR $\frac{1}{2}\ln\left\|(x-5)(x+5)\right\|+\frac{7}{2}\ln\left\|\frac{x-5}{x+5}\right\|+C$ **(d)** $\frac{1}{2}\ln\left(x^2+6x+11\right)-\frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C$
12	$\frac{3}{2}\ln\left(\frac{13}{45}\right)+\frac{8}{3}\tan^{-1}\left(\frac{2}{3}\right)$ $\therefore a=\frac{3}{2},b=\frac{13}{45},c=\frac{8}{3},d=\frac{2}{3}$
13	**(a)** $\tan x-\frac{1}{2}\cot\left(4x-\frac{\pi}{3}\right)+C$ **(b)** $\frac{1}{8}\sin 4x+\frac{1}{2}\tan 2x-\frac{x}{2}+C$ **(c)** $-\frac{1}{2}\sec\left(\frac{\pi}{6}-x\right)+C$ **(d)** $\frac{1}{4}\tan 2x+C$

14	**(a)** $\dfrac{\tan^2 x}{2}+\ln\lvert\cos x\rvert+C$ **(b)** $-\dfrac{\sqrt{25-x^2}}{25x}+C$ **(c)** $\dfrac{\sqrt{2}}{2}\tan^{-1}\left(\dfrac{e^x}{\sqrt{2}}\right)+C$ **(d)** $\dfrac{\pi}{4}$
15	**(a)** $x^2\sin x+2x\cos x-2\sin x+C$ **(b)** $\dfrac{\pi}{24}+\dfrac{\sqrt{3}}{4}-\dfrac{1}{2}$ **(c)** $\dfrac{2}{5}e^{2x}\left(\sin x-\dfrac{1}{2}\cos x\right)+C$
16	**(i)** $-e^{\cos x}\sin x$ **(ii)** $-2e^{\cos x}\cos x+2e^{\cos x}+C$
17	**(i)** $\dfrac{-2x}{3}(1-x)^{\frac{3}{2}}-\dfrac{4(1-x)^{\frac{5}{2}}}{15}+C$ **(ii)** $2\left[\dfrac{(1-x)^{\frac{5}{2}}}{5}-\dfrac{(1-x)^{\frac{3}{2}}}{3}\right]+D$

H2 Mathematics (9758)
Chapter 10 Integration Techniques
Extra Practice Questions

1 **2018/ACJC Prelim/1/6**

Find

(a) $\int \left(\sin^{-1} 2x\right)\frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$. [4]

(b) $\int \frac{x-1}{x^2+2x+6}\,\mathrm{d}x$. [4]

2 **2011/CJC Prelim/2/2 (modified)**

Use partial fractions to evaluate $\int_0^1 \frac{2+10x}{(1+3x)(1+3x^2)}\mathrm{d}x$, giving your answer in an exact form. [5]

3 **2011/DHS Prelim/1/8 (modified)**

(a) Express $\frac{x}{1-2x+x^2}$ in partial fractions. Hence, find $\int \frac{x}{1-2x+x^2}\,\mathrm{d}x$. [3]

(b) Find

(i) $\int \sin^{-1}x\,\mathrm{d}x$, [3]

(ii) $\int \frac{x^2}{x^2-2x+3}\,\mathrm{d}x$. [4]

4 **2011/IJC Prelim/1/3**

Using the substitution $x=\frac{1}{2}\mathrm{e}^u$, find $\int \frac{\left[\ln(2x)\right]^2}{x\left\{25-2\left[\ln(2x)\right]^2\right\}}\,\mathrm{d}x$. [5]

5 **2015/MI Prelim/1/2**

Find

(i) $\int \frac{\sin x}{1+2\cos x}\,\mathrm{d}x$, , [2]

(ii) $\int_0^{\frac{\pi}{2}} \mathrm{e}^x \cos 2x\,\mathrm{d}x$. [4]

6 **2015/ACJC Prelim/1/1**

Use the substitution $u = 3 - x^2$ to find $\int x^3 \sqrt{\left(3 - x^2\right)}\, \mathrm{d}x$. [3]

7 **2015/NJC Prelim/2/1**

(a) Use the substitution $x = 3\tan\theta$ to find the exact value of $\int_{\sqrt{3}}^{3} \frac{1}{x^2\sqrt{x^2+9}}\, \mathrm{d}x$ [4]

(b) Using integration by parts, find $\int \ln\left(x^2 + 4\right) \mathrm{d}x$. [4]

8 **2015/PJC Prelim/1/9**

(a) **(i)** By considering the derivative of e^{x^2}, find $\int x\mathrm{e}^{x^2}\, \mathrm{d}x$. [2]

(ii) Hence, find $\int x^3\mathrm{e}^{x^2}\, \mathrm{d}x$. [3]

(b) Use the substitution $u = \sin^2 x$ to find $\int \sqrt{\frac{1-u}{u}}\, \mathrm{d}u$. [5]

9 **2017/JJC Prelim/1/2 modified**

(a) Find $\int \sin(3\theta)\cos(3\theta)\, \mathrm{d}\theta$. [2]

(b) Use the substitution $\theta = \sqrt{x}$ to find the exact value of $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right) \mathrm{d}\theta$. [5]

10 **2017/NYJC Prelim/1/4**

(i) By using the substitution $x - 1 = 3\tan\theta$, find $\int \frac{1}{\sqrt{x^2 - 2x + 10}}\, \mathrm{d}x$. [5]

(ii) By expressing $x + 3 = A(2x - 2) + B$, find $\int \frac{x+3}{\sqrt{x^2 - 2x + 10}}\, \mathrm{d}x$. [3]

Answer Key

1	(a) $\left[-\frac{1}{4}\left(\sin^{-1}2x\right)\sqrt{1-4x^2}\right]+\frac{1}{2}x+C$ (b) $\frac{1}{2}\ln\left\|x^2+2x+6\right\|-\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+C$
2	(a) $-\frac{1}{6}\ln 4+\frac{\pi}{3}$
3	(a) $\ln\|1-x\|+\frac{1}{(1-x)}+C$ (b)(i) $x\sin^{-1}x+\sqrt{1-x^2}+C$ (b)(ii) $x+\ln\left\|x^2-2x+3\right\|-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-1}{\sqrt{2}}+C$
4	$-\frac{1}{2}\left[\ln(2x)-\frac{5}{2\sqrt{2}}\ln\left\|\frac{5+\sqrt{2}\ln(2x)}{5-\sqrt{2}\ln(2x)}\right\|\right]+c$
5	(i) $\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+c$ (ii) $-\frac{1}{5}\left(e^{\frac{\pi}{2}}+1\right)$
6	$\frac{1}{5}\left(3-x^2\right)^{\frac{5}{2}}-\left(3-x^2\right)^{\frac{3}{2}}+c$
7	(a) $\frac{2-\sqrt{2}}{9}$ (b) $x\ln\left(x^2+4\right)-2x+4\tan^{-1}\left(\frac{x}{2}\right)+c$
8	(a)(i) $\frac{1}{2}e^{x^2}+C$ (a)(ii) $\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C$ (b) $\sqrt{u-u^2}+\sin^{-1}\sqrt{u}+C$
9	(a) $-\frac{1}{12}\cos 6\theta+C$ (b) $-\frac{1}{2}-\frac{\pi}{4}$
10	(i) $\ln\left\|\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\|+C$ (ii) $\sqrt{x^2-2x+10}+4\ln\left\|\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\|+C$