CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

MATHEMATICS

9740/01

Paper 1

24 Aug 2016

3 hours

Additional Materials: List of Formulae (MF15)

Name: ______________________________ **Class:** ________________

READ THESE INSTRUCTIONS FIRST

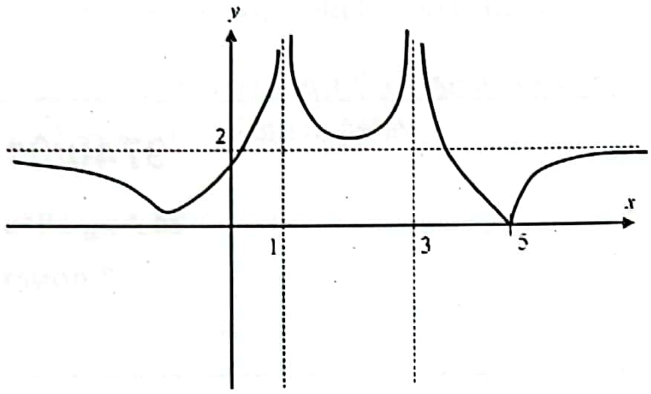
Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

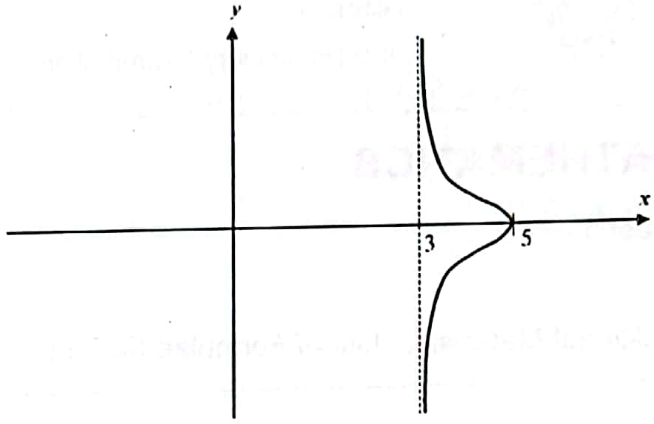
At the end of the examination, arrange your answers in NUMERICAL ORDER.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 **[In this question, sketches of the given graphs are not drawn to scale]**
The graphs of $y=|f(x)|$ and $y^2=f(x)$ are given below.



$y=|f(x)|$



$y^2=f(x)$

On separate diagrams, draw sketches of the graphs of

(a) $y^2=f(-2x)$, [2]

(b) $y=f(x)$, [3]

stating the equations of any asymptotes and the coordinates of any points of intersection with the axes.

2 The vectors **a** and **b** are given by

$$\mathbf{a}=4\mathbf{i}+6p\mathbf{j}-8\mathbf{k} \text{ and } \mathbf{b}=2\mathbf{i}-3\mathbf{j}+4p\mathbf{k}, \text{ where } p>0.$$

It is given that $|\mathbf{a}|=2|\mathbf{b}|$.

(i) Find p. [2]

(ii) Give a geometrical interpretation of $\frac{1}{|\mathbf{b}|}|\mathbf{b}\cdot\mathbf{a}|$. [1]

(iii) Using the value of p found in part (i), find the exact value of $\frac{1}{|\mathbf{b}|}|\mathbf{b}\cdot\mathbf{a}|$. [2]

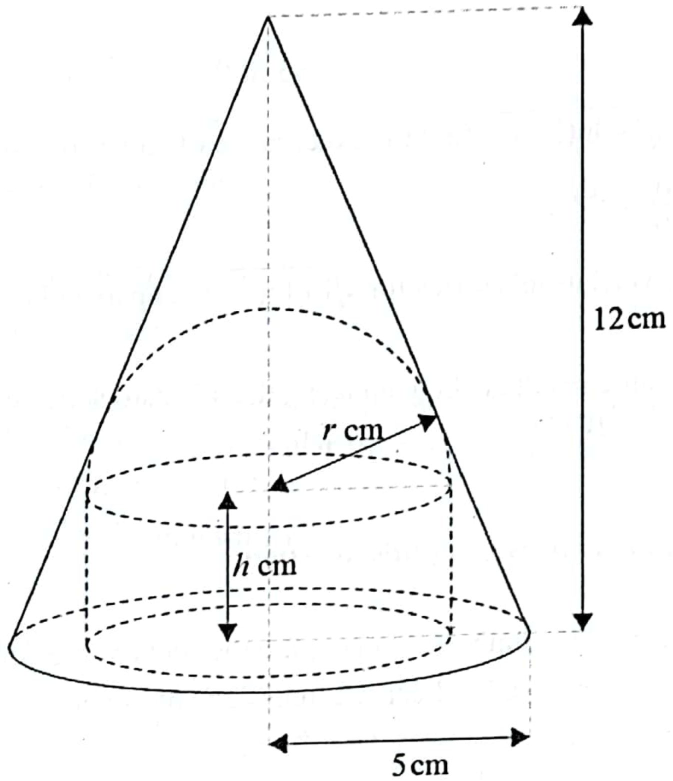
3 The cubic equation $x^3+ax^2+bx+c=0$, where a, b and c are constants, has roots $3+\mathrm{i}$ and 2.

(i) One JC2 student remarked that the third root is $3-\mathrm{i}$. State a necessary assumption the student made in order that the remark is true. [1]

(ii) Given that the assumption in part (i) holds, find the values of a, b and c. [4]

4 A closed container is made up of a cylinder of base radius r cm and height h cm, and a hemispherical top with the same radius r.

It is inscribed within a fixed right circular cone of base radius 5 cm and height 12 cm, as shown in the diagram below.



(i) By using similar triangles, show that $h = 12 - \frac{13}{5}r$.

Determine the exact range of possible values of length r. **[3]**

(ii) Find the total volume V of the closed container in terms of r.
By differentiation, find the exact value of r that produces the maximum container volume V, as r varies.

[Volume of a sphere with radius R is $\frac{4}{3}\pi R^3$.] **[4]**

5 A sequence $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation $u_n = \frac{n}{(n-1)^2}u_{n-1}$, for $n \geq 2$.

(a) Given that $u_1 = 2$, use the method of mathematical induction to prove that $u_n = \frac{2n}{(n-1)!}$, for $n \geq 1$. **[4]**

(b) Given that $u_1 = a$, where a is any constant. Write down u_2, u_3 and u_4 in terms of a.

Hence or otherwise, find u_n in terms of a. **[3]**

6 (i) Given that $1-2r = A(r+1)+Br$, find the constants A and B. [1]

(ii) Use the method of differences to find $\sum_{r=1}^{n} \frac{1-2r}{3^r}$. [3]

(iii) Hence find the value of $\sum_{r=1}^{\infty} \frac{2-2r}{3^r}$. [4]

7 (i) Given that $y=\sqrt{1+\ln(1+x)}$, find the exact range of values of x for y to be well defined. [1]

(ii) Show that $2y\frac{dy}{dx}=e^{1-y^2}$. [2]

(iii) Hence, find the Maclaurin's series for $\sqrt{1+\ln(1+x)}$, up to and including the term in x^2. [3]

(iv) Verify that the same result is obtained using the standard series expansions given in the List of Formulae (MF15). [3]

8 **Do not use a calculator in answering this question.**

(i) It is given that complex numbers z_1 and z_2 are the roots of the equation $z^2-6z+36=0$ such that $\arg(z_1) > \arg(z_2)$. Find exact expressions of z_1 and z_2 in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4]

(ii) Find the complex number $\frac{z_1^{\ 4}}{iz_2}$ in exact polar form. [3]

(iii) Find the smallest positive integer n such that $z_2^{\ n}$ is a positive real number. [2]

9 (i) By using the substitution $u=\sqrt{x+1}$, find $\int \frac{\sqrt{x+1}}{x-1}\,dx$. [5]

(ii) The region R is bounded by the curve $y=\frac{\sqrt{x+1}}{x-1}$ and the lines $x=8$ and $y=1$.

Find

(a) the exact area of R, simplifying your answer in the form $A-\sqrt{2}\ln\left(\frac{B-\sqrt{2}}{B+\sqrt{2}}\right)$ where A and B are integers to be determined, [5]

(b) the volume of the solid generated when R is rotated 2π radians about the x-axis, giving your answer correct to 2 decimal places. [3]

10 The plane p passes through the points A, B and C with coordinates $(0, 1, 1)$, $(2,-1,4)$ and $(-2,-1,0)$ respectively.

(i) Show that a cartesian equation of the plane p is $2x - y - 2z = -3$. **[3]**

The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$.

(ii) Find the acute angle between l and p. **[2]**

The point Q has position vector $5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(iii) Show that Q lies on the line l. **[1]**

(iv) It is given that a variable point R lies on the plane p and is at a distance of $\sqrt{45}$ from the point Q. Find the foot of perpendicular from the point Q to the plane p and hence describe geometrically the locus of R. **[6]**

(v) Find a vector equation of the line which is a reflection of the line l in plane p. **[3]**

11 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{2x+k}{x-2}, x \in \mathbb{R},\ x \neq 2,$$

where k is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the x and y axes. **[3]**

(ii) Describe fully a sequence of transformations which would transform the curve $y = \frac{1}{x}$ onto $y = \mathrm{f}(x)$. **[4]**

(iii) Find f^{-1} in a similar form and write down the range of f^{-1}. **[3]**

(iv) Hence or otherwise, find f^2.

Find the value of $\mathrm{f}^{2017}\left(\frac{1}{2}\right)$, leaving your answer in terms of k. **[4]**

The function g is defined by

$$\mathrm{g} : x \mapsto a + \sqrt{x-3}, \quad x \in \mathbb{R},\ x > 3,$$

where a is a real constant.

(v) Given that fg exists, write down an inequality for a and explain why gf does not exist. **[3]**

— THE END —