National Junior College
2016 – 2017 H2 Further Mathematics
Topic F6: Numerical Methods (Tutorial)

Basic Mastery Questions

Interval Bisection, Linear Interpolation and Newton-Raphson's Method

1 Estimate the positive root of the equation $x = \ln(x+2)$, giving your answer to four significant figures. J73/P1/14

2 Use linear interpolation to find an approximation to the root of equation $x - \ln(4\sin x) = 0$ which lies between 1 and 2. Give your answer correct to three decimal places.

3 Show that the equation $e^x + x - 2 = 0$ has one and only one real root. Use the Newton-Raphson process to find this root correct to three decimal places. J75/P1/14

4 Sketch the following graphs on a single diagram, stating the x-coordinates of all intersections with the x-axis and the equations of any asymptotes.

(i) $y = x(x^2-4) = x(x-2)(x+2)$

(ii) $y = \dfrac{x+1}{(x-1)^2}$

Use linear interpolation once on the interval $[-1, 0]$ to obtain an approximation to a root of the equation $x(x^2-4) = \dfrac{x+1}{(x-1)^2}$.

The Newton-Raphson method is to be used to find an approximation to another root of the equation. Use the method, with $x = 2$ as a first approximation, to obtain a second approximation to this root, giving your answer correct to 2 places of decimal.

N2000/P2/12

Iterations involving recurrence relations of the form $x_{n+1} = \mathrm{F}(x_n)$

5 The equation $x^3 - 12x + 1 = 0$ has two positive roots, α and β, $(\alpha < \beta)$ and one negative root.

(i) Prove that $0 < \alpha < 1$ and $3 < \beta < 4$.

(ii) Use the iterative formula $x_{n+1} = (12x_n - 1)^{\frac{1}{3}}$, $n \geq 1$, with 3.5 as a starting value to approximate β correct to two decimal places.

(iii) With the aid of a graph, show that the iterative formula in (ii) will converge to β for some starting values more than α.

Trapezium Rule and Simpson's Rule

6 Estimate the values of the following definite integrals, taking the number of ordinates in each case, using (a) the trapezium rule, (b) Simpson's Rule.

(i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\,dx$, 3 ordinates (ii) $\int_0^{0.4} \sqrt{1-x^2}\,dx$, 5 ordinates

Euler's Method and Improved Euler's Method

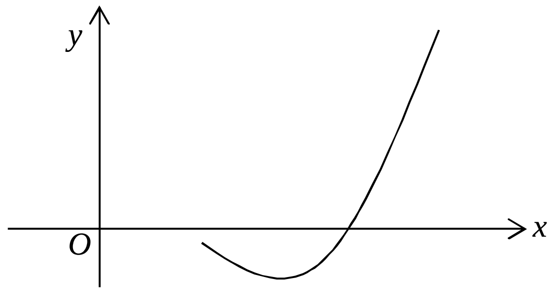
7 Apply Euler's method with step size of 0.1 to obtain an approximation to the given initial value problems at $y(0.5)$.

(a) $\frac{dy}{dt} = 1 - y^2, \quad y(0) = 0.5$ (b) $\frac{dy}{dt} = t + \sqrt{ty}, \quad y(0) = 1$

8 Apply improved Euler's method with step size of 0.1 to the initial value problems given in **Question 7** to obtain an approximation to the value at $y(0.5)$.

Practice Questions

9 By considering the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, show that the equation $x + 4\cos x = 0$ has one negative root and two positive roots.
Use linear interpolation, once only, on the interval to find an approximation to the negative root of the equation $x + 4\cos x = 0$, giving 2 decimal places in your answer.



The diagram shows part of the graph $y = x + 4\cos x$ near the larger positive root, α, of the equation $x + 4\cos x = 0$. Explain why, when using Newton-Raphson method to find α, an initial approximation which is smaller than α may not be satisfactory.
Use the Newton-Raphson method to find α correct to 2 significant figures.

N94/P2/13

10(a) A function f is such that $\mathrm{f}(4)=1.158$ and $\mathrm{f}(5)=-3.381$, correct to 3 decimal places in each case. Assuming that there is a value of x between 4 and 5 for which $\mathrm{f}(x)=0$, use linear interpolation to estimate this value.

For the case where $\mathrm{f}(x)=\tan x$, and x is measured in radians, the values of $\mathrm{f}(4)$ and $\mathrm{f}(5)$ are as given above. Explain with the aid of a sketch why linear interpolation using these values does not give an approximation to a solution of the equation $\tan x=0$.

(b) Show, by means of a graphical argument that the equation $\ln(x-1)=-2x$ has exactly one real root, and show that this root lies between 1 and 2.

The equation may be written in the form $\ln(x-1)+2x=0$. Show that neither $x=1$ nor $x=2$ is a suitable initial value for the Newton-Raphson method in this case.

The equation may also be written in the form $x-1-\mathrm{e}^{-2x}=0$. For this form, use two applications of the Newton-Raphson method, starting with $x=1$, to obtain an approximation to the root, giving three decimal places in your answer.

N95/P2/13

11 The equation $x^3-5x^2+2=0$ has one negative root and two positive roots.

(i) Find the integer n such that the smaller positive root α , denoted by, lies in $(n,n+1)$.

(ii) Show that the graph of $y=x^3-5x^2+2$ is strictly increasing for $x<0$.

(iii) If β_0 is a first approximation to the negative root, β , obtained by linear interpolation method, explain why $\beta_0>\beta$.

(iv) An iteration formula for finding β is given by $x_{n+1}=-\sqrt{\dfrac{2}{5-x_n}}$.

Taking $x_1=-0.6$, apply this formula to find β , correct to 3 significant figures.

12 The area of the finite region bounded by the curve $y=\dfrac{\ln x}{x^2}$, the x-axis and the line $x=2$ is given by A.

(i) Evaluate A, leaving your answer in terms of natural logarithms.

(ii) Use the trapezium rule with two strips to obtain an approximation A_1 to the value of A, giving your answer to three significant figures.

With the aid of a diagram, explain whether A_1 is an underestimation or overestimation.

13 Using the Simpson's Rule with 5 ordinates, find an approximation to

$$\int_0^{0.5}\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x .$$

Hence use your result to obtain an estimated value of π . Give all your answers correct to 2 decimal places.

Without evaluating the approximation to $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$ using Trapezium Rule, state with reason which approximation, using Simpson's Rule or Trapezium Rule, is a better estimation.

14 (a) Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t + y, \qquad y(0) = 1 \qquad \text{------} \quad (1)$$

Apply Euler's method with step size $\Delta t = 0.1$, to find the approximate value of y when $t = 1$, correct to 4 decimal places. Compare the approximated results with the exact solution.

(b) Use improved Euler's method with step size $\Delta t = 0.1$ to find the approximate value of y when $t = 1$, correct to 4 decimal places. Compare the approximated results with the exact solution and that obtained from the Euler's method.

15 Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} - y = -\frac{1}{2}\mathrm{e}^{\frac{t}{2}}\sin(5t) + 5\mathrm{e}^{\frac{t}{2}}\cos(5t), \quad y(0) = 0$$

Use Euler's method and the step sizes of $\Delta t = 0.1$, $\Delta t = 0.05$, $\Delta t = 0.01$, $\Delta t = 0.005$, and $\Delta t = 0.001$ to find the approximations to the solution at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. How does changing the values of t affect the accuracy of the approximations? Justify your answer.

16 Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 2 - \mathrm{e}^{-4t}, \qquad y(0) = 1$$

Use Euler's method and the step sizes of $\Delta t = 0.1$, $\Delta t = 0.05$, $\Delta t = 0.01$, $\Delta t = 0.005$, and $\Delta t = 0.001$ to obtain approximations at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. How does changing the values of Δt affect the accuracy of the approximations?

17 Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 - 2y + 1, \qquad y(0) = 2$$

Use improved Euler's method with $\Delta t = 0.5$ to find the approximate solutions for $0 \le t \le 2$. Your answer should include a table of approximate values of the dependable variable and compare your values obtained from Euler's method.

Numerical Answers to Numerical Methods Tutorial

Basic Mastery Questions

1.	1.146	2.	1.365
3.	0.443	4.	2.23
5.	3.42	6.	(i) 1.05 ; 1.01 (ii) 0.389 ; 0.389
7.	(a) 0.795 (b) 1.30	8.	(a) 0.781 (b) 1.37

Practice Questions

9. 3.6
10. (a) 4.255 (b) 1.109
11. (i) $n = 0$ (iv) -0.598
12. (i) $\frac{1}{2}-\frac{1}{2}\ln 2$ (ii) 0.133
13. 0.52; 3.14