## SERANGOON JUNIOR COLLEGE

## 2014 JC2 PRELIMINARY EXAMINATION

## MATHEMATICS

**Higher 2**                                                              **9740/1**

**Wednesday**                                                    **20 Aug 2014**

Additional materials:  Writing paper

List of Formulae (MF15)

**TIME  :**  3 hours

**READ THESE INSTRUCTIONS FIRST**

Write your name and class on the cover page and on all the work you hand in.
Write in dark or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets **[  ]** at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Total marks for this paper is 100 marks.

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**This question paper consists of 6 printed pages (inclusive of this page) and 2 blank pages.**

**Answer all questions [100 marks]**

1   A cubic polynomial has turning points at $A(-1, 13)$ and $B(2, -14)$.
   **(i)** Find the equation of this polynomial. [3]
   **(ii)** Hence find the coordinates of the point $C$ on the graph of this polynomial such
   that $AC$ is parallel to the $x$-axis. [2]

2   SRBank introduces the UniSave Bank Account to encourage young parents to
   save up for their child's university education. This account consists of two
   independent components – Ordinary Account and Birthday Account. The bank
   will provide an interest rate of 1% of the total amount in the Ordinary Account at
   the end of every year. As a bonus, the bank will deposit an amount equivalent to
   ten times the age of the child into the Birthday Account on the child's birthday
   each year, with the last deposit on the $18^{\text{th}}$ birthday.
   Mr and Mrs Hon intend to save up for their child's university education by
   depositing a fixed amount of $3000 into the UniSave's Ordinary Account at the
   start of every year from the year their child turns one year old.
   **(i)** Show that the total amount in the UniSave Bank Account at the end of the
   year when their child is $n$ years old, where $n \geq 19,$ is given by

   $$\$303000\left(1.01^n\right) - 301290.$$ [3]

   **(ii)** Given that $n \geq 19,$ find the least $n$ such that the total amount in the UniSave
   Bank account will exceed $70000. [2]

3   A sequence of real numbers $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

   $$u_{n+1} = \frac{u_n}{u_n + 3}, \quad n \in \square^+.$$

   Given that $u_1 = 1$, and by considering $\dfrac{2}{u_n} + 1$ for $n = 1, 2, 3,$ make a suitable

   conjecture for $u_n$ in the form of $\dfrac{k}{a^n - b}$, where $a, b, k \in \square^+$. [2]
   Prove the conjecture by Mathematical Induction. [4]

4   **(a)** A graph with equation $y = f(x)$ undergoes in succession, the following
       transformations:
           $A$ : A translation of 3 units in the direction of the negative $x$-axis
           $B$ : A reflection about the $y$-axis
           $C$ : A scaling parallel to the $x$-axis by a factor of 2

       The equation of the resulting curve is given by $y = \dfrac{x-4}{2x-18}$.

       Find the equation $y = f(x)$. [3]

**4** **(b)** The graphs of $y = |g(x)|$ and $y = -\sqrt{g(x)}$ are shown below.
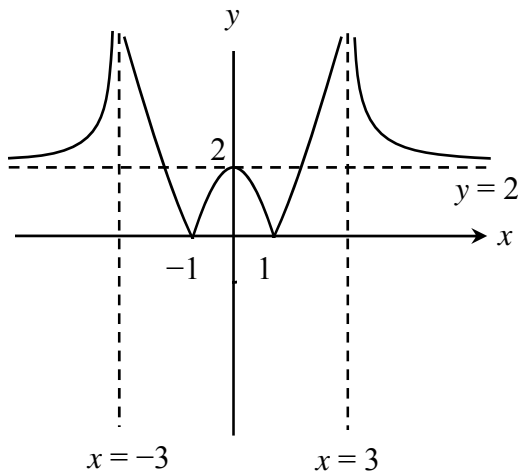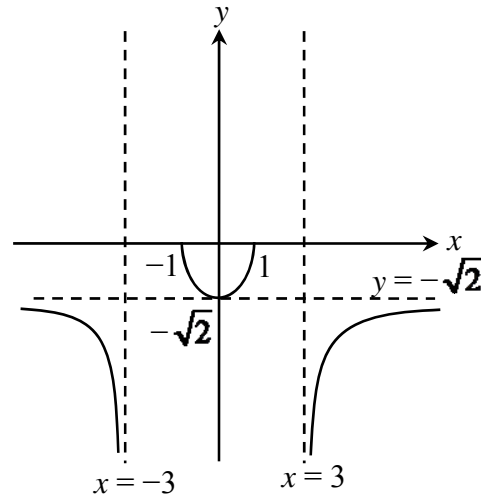


Graph of $y = |g(x)|$         Graph of $y = -\sqrt{g(x)}$

Sketch the graph of $y = g(x)$, showing clearly the equations of any asymptote and intercepts with the axes. [3]

**5** The curve $C$ has equation given by $y = \dfrac{x^2 + x + 1}{x + 1}$, $x \in \square$, $x \neq -1$.

   **(i)** Without using a calculator, find the set of values that $y$ can take. [3]

   **(ii)** Sketch the graph of $C$, indicating clearly the equations of any asymptotes and the coordinates of any turning points of the curve. [2]

   **(iii)** Show that $(-1, -1)$ lies on $y = kx + k - 1$, where $k \in \square$.

   Hence, find the range of values of $k$ where $kx - 1 = \dfrac{(x-1)^2 + x}{x}$ has two real roots. [2]

**6** The number of people infected with virus A is $x$. The rate at which $x$ is varying at any time $t$ is proportional to the difference between the number of infected people and the number of death due to virus A. At time $t$, it is also known that the number of deaths due to virus A is proportional to the square of the number of people infected with virus A. Initially there were 10 people infected and the number infected remains constant when it reaches 100.

Show that $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{k}{100}\left(100x - x^2\right)$, where $k$ is a constant. Hence find $x$ in terms of

$k$ and $t$ in the form $x = \dfrac{p}{1 + qe^{-kt}}$ where $p$ and $q$ are constants to be determined. [7]

**7** The equations of three planes $p_1, p_{2,} p_3$ are

$$x + y + z = 3,$$
$$x - z = 3,$$
$$3x + \lambda y - 2z = \mu,$$

respectively, where $\lambda$ and $\mu$ are constants.

A line $l$ passes through the origin and the point, $A(1, 2, -1)$.

**(i)** Find the coordinates of $B$, the point of intersection between the line $l$ and plane $p_1$. [3]

**(ii)** Find the sine of the acute angle between the line $l$ and the plane $p_1$. Hence find the exact shortest distance from point $A$ to the plane $p_1$. [2]

**(iii)** Find the conditions satisfied by $\lambda$ and $\mu$ given that there is no point in common among the three planes. [3]

**8** A curve has the parametric equations

$$x = \cos^2 t, \ y = \sin^3 t, \ \text{for } 0 \le t \le \frac{\pi}{2}.$$

**(i)** Sketch the curve, indicating clearly the axial intercepts. [1]

**(ii)** Find the equations of the tangent and normal to the curve at the point $P(\cos^2 \theta, \sin^3 \theta)$, where $0 < \theta < \frac{\pi}{2}$. [4]

**(iii)** The tangent to the curve at $P$ meets the $x$-axis at $A$ and the normal to the curve at $P$ meets the $x$-axis at $B$ respectively. Show that the area of triangle $PBA = \frac{1}{12} \sin^5 \theta \ (4 + 9\sin^2 \theta )$. [3]

**9** The functions f and g are defined as follows:

$$\text{f}: x \mapsto \left|4x - x^2\right|, \quad x < k$$
$$\text{g}: x \mapsto \sqrt{x+4}, \quad x > -4.$$

**(i)** State the largest value of $k$ for the inverse function f to exist. Hence, find $\text{f}^{-1}$ in similar form. [4]

**(ii)** Using the value of $k$ found in (i), explain why the composite function gf exists. State the range of gf. Find the rule of gf in the form $bx + a$, where $a, b \in \square$ , stating clearly its domain. [5]

**10 (a)** The curve $C$ is defined by the parametric equations

$$x = \ln t, \quad y = \frac{t^3 + t}{t + 1}, \qquad \text{where } t > 0.$$

Another curve $L$ is defined by the equation $y = e^{2x}$. The graphs of $C$ and $L$ are shown in the diagram below.
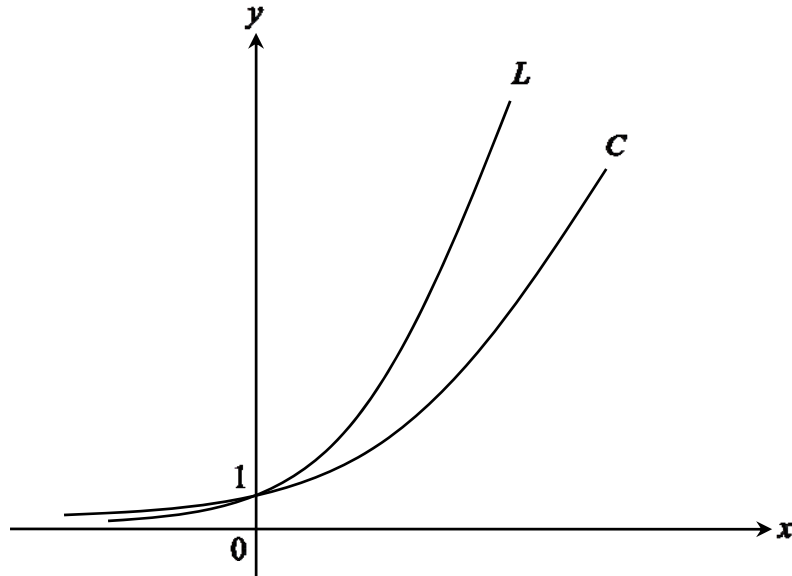


Find the exact area of the region bounded by $C$, $L$ and the line $x = \ln 2$, giving your answer in the form $\ln b$ where $b$ is a constant to be determined. [5]

**(b)** The curves $V$ and $W$ have equations $2y = (x-1)^2 + 4$ and $y = 2x^2$ respectively. The region in the first quadrant enclosed by the curves and the $y$-axis is denoted by $S$.

Find the exact volume of the solid generated when the region $S$ is rotated through $2\pi$ radians about the $y$-axis. [4]

**11** It is given that $\dfrac{1-N}{(N+1)(N+2)(N+3)} = \dfrac{1}{N+1} - \dfrac{3}{N+2} + \dfrac{2}{N+3}$.

**(i)** Find $S_n$ in terms of $n$, where $S_n = \displaystyle\sum_{r=1}^{n} \frac{4r-4}{(r+1)(r+2)(r+3)}$. [3]

**(ii)** Hence find $S_\infty$, stating clearly the reason. [2]

**(iii)** Using the result in part (i),

**(a)** find $\displaystyle\sum_{r=1}^{n} \frac{4r}{(r+2)(r+3)(r+4)}$ in terms of $n$, [2]

**(b)** deduce that $\displaystyle\sum_{r=11}^{n} \frac{r-1}{(r+2)^3} < \frac{11}{156}$. [3]

**12**  Sketch on a single Argand diagram the loci of $|z+3+4\mathrm{i}|=5$ and

$|z+3+4\mathrm{i}|=|z+6|$.  [3]

**(i)**  Hence indicate clearly on the Argand diagram the locus of $z$ that satisfies the

relation $|z+3+4\mathrm{i}|\leq 5$ and $|z+3+4\mathrm{i}|=|z+6|$.  [1]

**(ii)**  Given that $-\pi < \arg(z+3+4\mathrm{i}) \leq \pi$, find the exact range of values of

$\arg(z+3+4\mathrm{i})$.  [4]

**(iii)** Find the exact value of $z$ where $|\arg(z+3+4\mathrm{i})|$ is as large as possible.  [2]

**13**  A curve is defined by the equation $y^2 = \ln(\mathrm{e} - x)$, $y > 0$ and $x < \mathrm{e}$.

**(i)**  Show that $2y\dfrac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{-y^2} = 0$.  [2]

**(ii)**  By further differentiation, show that $(2+4y^2)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$.  [2]

**(iii)** Hence, find the Maclaurin's series of $y$, up to and including the term in $x^3$, giving your answers in exact form.  [3]

**(iv)** Find the Maclaurin's series of the curve defined by the equation

$$[y(2-x)]^2 = \ln(1-x) + 1.$$  [3]

**END OF PAPER**