Chapter 1 MEASUREMENT

Content

- Physical quantities and SI units
- Scalars and vectors
- Errors and Uncertainty

Learning Outcomes

Candidates should be able to:

(a) recall the following base quantities and their SI units: mass (kg), length (m), time (s), current (A), temperature (K), amount of substance (mol).

(b) express derived units as products or quotients of the base units and use the named units listed in 'Summary of Key Quantities, Symbols and Units" as appropriate. (Refer to pg 34-35 of the 9749 Syllabus by SEAB.)

(c) use SI base units to check the homogeneity of physical equations.

(d) show an understanding of and use the conventions for labelling graph axes and table columns as set out in the ASE publication *Signs, Symbols and Systematics: The ASE Companion to 16-19 Science (2000)*. (Taught in lab.)

(e) use the following prefixes and their symbols to indicate decimal sub-multiples or multiples of both base and derived units: pico (p), nano (n), micro (μ), milli (m), centi (c), deci (d), kilo (k), mega (M), giga (G), tera (T).

(f) make reasonable estimates of physical quantities included within the syllabus.

(g) distinguish between scalar and vector quantities and give examples of each.

(h) add and subtract coplanar vectors.

(i) represent a vector as two perpendicular components.

(j) show an understanding of the distinction between systematic errors (including zero errors) and random errors.

(k) show an understanding of the distinction between precision and accuracy.

(l) assess the uncertainty in a derived quantity by addition of actual, fractional, percentage uncertainties or by numerical substitution (a rigorous statistical treatment is not required).

1.1 Introduction

Links

Physics is an experimental science. Precise and accurate measurements enable the collection of useful experimental data that can be tested against theoretical predictions to refine the development of physical theories. Experimental evidence is the ultimate authority in discriminating between competing physical theories. Scientific knowledge continues to evolve as data from new or improved measurements helps us to better understand and explain physical phenomena.

Measurements are subject to uncertainties, and it is important to estimate these to understand the reliability of the measurements. Error analysis involves estimating the uncertainties in measurements and finding ways to reduce them if necessary. In an experiment, the record of measurements made should include the estimated uncertainties and an analysis of the possible sources of errors with a discussion of steps taken to reduce the uncertainties should be documented. Doing this enables better conclusions to be drawn from the experimental data.

The act of measurement affects the object being measured due to the interaction between the measuring device and the object. Common examples of this include measurements made using a thermometer, voltmeter or ammeter. Thus, improving the accuracy of measurements often requires the use of better instruments and enhanced experimental techniques.

Applications

Physicists are very serious about measurements, and the other sciences and society as a whole have benefitted from the spill-over effects of the invention of many amazing measuring devices and techniques. Modern engineering also depends heavily on accurate measurements in areas such as design, construction, optimization and communication. Precise measurements have made many advanced technological applications possible; examples include the study and manipulation of materials, and breakthroughs in fields as diverse as geophysics and biology. Measurements using sophisticated devices like magnetic resonance imaging (MRI) scanners are important in the medical industry as it provides a wealth of data that aids in clinical diagnosis and influences decisions with regards to treatment.

1.2 Quantities and Units

Physical Quantities

The laws of Physics are commonly expressed as mathematical relationships among physical quantities and are verified by conducting experiments which involved the measurements of these quantities.

All physical quantities consist of a numerical magnitude and a unit. For "a force of five newtons", force 'F' is the physical quantity, '5' is the numerical magnitude and newtons 'N' is the unit.

unit

$$F = 5\ \mathrm{N}$$

physical quantity

numerical magnitude

S.I. Base Quantities and Units

Base quantities are physical quantities that are the most fundamental and they are used to define other physical quantities. Scientists, in the interest of simplicity, chose seven base quantities that give a full description of the physical world.

The *Système Internationale d'Unités* (International System of Units or SI units) is based on the seven base quantities and their corresponding base units are listed below. In 2019, some of SI base units were redefined. (Use the Internet to find out what they are.)

Base quantity	Base unit	Symbol
time	second	s
length	metre	m
mass	kilogram	kg
current	ampere	A
temperature	kelvin	K
amount of substance	mole	mol
luminous intensity (not in syllabus)	candela	cd

S.I. Derived Quantities and Units

Derived quantities are physical quantities formed by combining base quantities according to algebraic relations involving products and/or quotients. Hence, **derived units** are defined in terms of base units and are expressed as products and/or quotients of base units.

Derived units are obtained from the base units according to a **defining equation** that relates the physical quantities.

For example, the defining equation for speed v is given by $v = \frac{s}{t}$, where s is distance and t is time. Hence, the unit of speed is metre per second i.e. m s^{-1}.

Note

There is a space between the m and the s^{-1}. Otherwise, it may be misread as per millisecond (ms^{-1}). Units should always be expressed in indices form, e.g. m s^{-2} instead of m/s^2.

Other examples of derived units are given below.

Derived quantity	Defining equation	Base units	Derived unit	Symbol of derived unit
force	force = mass × acceleration	(kg) × (m ÷ s ÷ s) = kg m s^{-2}	newton	N
work done	work done = force × displacement	(kg m s^{-2}) × (m) = kg m^2 s^{-2}	joule	J

Homogeneity of Equations

In order for a physical equation to be operational or meaningful, each term in the equation must have the **same base units** (or dimensions). Only quantities with the same base units can be added, subtracted or equated. In other words, each term separated by "+", "-" or "=" sign must have the same base units.

When each of the terms in a physical equation has the same base units, the equation is said to be **homogeneous** or **dimensionally consistent**.

Example 1 Analyse whether the equation $v = u + at$ is homogeneous.

Unit of v = m s^{-1}

Unit of u = m s^{-1}

Unit of $at = (\text{m s}^{-2})(\text{s}) = \text{m s}^{-1}$

Since every term in the equation has the same base units, the equation $v = u + at$ is homogeneous.

Example 2 Analyse whether the equation $v^2 = u^2 + 2as^2$ is homogeneous.

Unit of $v^2 = (\text{m s}^{-1})^2 = \text{m}^2\ \text{s}^{-2}$
Unit of $u^2 = (\text{m s}^{-1})^2 = \text{m}^2\ \text{s}^{-2}$
Unit of $2as^2 = (1)\ (\text{m s}^{-2})\ (\text{m})^2 = \text{m}^3\ \text{s}^{-2}$ (note that "2" is unitless)

Since not every term in the equation has the same units, the equation is **not homogenous.**

Checking the homogeneity of an equation using base units is a powerful way of establishing if the physical equation is **plausible**. See example 3.

Example 3 Analyse whether the equation $s = ut + at^2$ is homogeneous.

Unit of s = m
Unit of $ut = (\text{m s}^{-1}) \times (\text{s}) = \text{m}$
Unit of $at^2 = (\text{m s}^{-2}) \times (\text{s}^2) = \text{m}$

However, we know that through manipulating relevant kinematics equations, the equation $s = ut + \frac{1}{2}at^2$ is derived. Hence, the equation $s = ut + at^2$ is homogenous but is **physically wrong**!

Note

An equation which is found to be homogenous **need not be physically correct** due to either (i) wrong coefficients/signs or (ii) missing/additional terms.

Example 4 The period T of a simple pendulum is thought to depend on its length L, its mass m and the acceleration due to gravity g according to the equation:

$$T = k L^x m^y g^z$$

where k is **a dimensionless constant** (i.e. **a constant with no unit**).

Deduce the relation between T, L, m and g by finding the indices x, y and z.

Base units of LHS = units of T = s

Base units of RHS = units of $(k L^x m^y g^z)$

= (units of k) (units of L^x) (units of m^y) (units of g^z)

= (1) (m^x) (kg^y) $(\text{m s}^{-2})^z$

= $\text{m}^{x+z}\ \text{kg}^y\ \text{s}^{-2z}$

Comparing the indices of

kg : $y = 0$

s: $-2z = 1 \Rightarrow z = -1/2$

m: $x + z = 0 \Rightarrow x = -z = 1/2$

A plausible equation is:

$T = k L^{1/2} g^{-1/2} \Rightarrow T = k\sqrt{\frac{L}{g}}$

Note

It is not possible to deduce the value of any constant such as k through this analysis. Constants can only be determined through more rigorous derivations or experimentations.

1.3 Prefixes, Standard Form and Significant Figures

Decimal Sub-multiples or Multiples

The following prefixes and their symbols can be used to indicate decimal sub-multiples or multiples of both base and derived units:

Prefix	pico	nano	micro	milli	centi	deci	kilo	mega	giga	tera
Symbol	p	n	µ	m	c	d	k	M	G	T
Multiple	10^{-12}	10^{-9}	10^{-6}	10^{-3}	10^{-2}	10^{-1}	10^{3}	10^{6}	10^{9}	10^{12}

E.g. 0.00000123 J $= 1.23 \times 10^{-6}$ J $= 1.23$ μJ

4850000 V $= 4.85 \times 10^{6}$ V $= 4.85$ MV

Standard Form

The standard form expresses a number as $N \times 10^{n}$, where n is an integer, either positive or negative, and N is any number such that $1 \leq N < 10$.

E.g. 0.00000123 J $= 1.23 \times 10^{-6}$ J

4850000 V $= 4.85 \times 10^{6}$ V

Significant Figures

Zeroes on the left of the first non-zero digit are not significant. Zeroes on the right of the last non-zero digit are only significant after a decimal point.

E.g. 0.0600 g has 3 significant figures

6.080×10^{4} J has 4 significant figures

Note

A speed of 100 m s^{-1} can be considered to be 3 or 2 or even 1 s.f.. This is dependent on the precision of the instruments used to measure the speed directly or the measurements that lead to the calculation of this value. It also depends on the s.f. of the other quantities appearing within a question. If in doubt, assume 3 s.f..

1.4 Estimation

Order of Magnitude

Sometimes, due to incomplete information of available data, coupled with the need to quickly get an idea of the magnitude of a physical quantity, we compute an estimated value of the quantity. The estimated value can also serve as a reality check on a calculation to ensure that no blunders were made. In making an approximation, it is usually necessary to make certain simplifying assumptions.

We refer to an **order of magnitude** estimate of a physical quantity as the **power of ten** of the number that describes it. Attaching the prefix "kilo-" to a unit increases the size of the unit by three orders of magnitude, for example, in "kg".

If two numbers differ by one order of magnitude, one is about ten times larger than the other. Two numbers have the same order of magnitude if the larger value is less than ten times the smaller value. When bacteria in a flask have multiplied from some hundreds to some millions, the population of the bacteria has increased by four orders of magnitude (or 10^4 times).

Physical quantity	Estimate	Order of Magnitude
Mass of a car	1300 kg ~ 10^3 kg	3
Power of a domestic bulb	100 W ~ 10^2 W	2
Speed of a car along an expressway	25 m s^{-1} ~ 10^1 m s^{-1}	1

Example 5 Estimate the volume of a wooden half-metre rule found in the school laboratory. State the order of magnitude in S.I. base units.

Volume of half-metre rule
= (L x B x t)
= (0.500 x 0.030 x 0.005) m^3
= 7.5 x 10^{-5} m^3
= 8 x 10^{-5} m^3

Order of magnitude is -5 (or 10^{-5} at GCE A-Level)

1.5 Measurements and Uncertainties

Measuring Instruments and their Associated Uncertainty

All **measuring instruments** and **methods of measurements** have limitations, and hence it is important to express measurements together with their uncertainties.

The inherent **uncertainty** when reading off the scale of a measuring instrument is determined by the ~~**precision**~~ resolution of the instrument, i.e., the **smallest division** of the scale.

As the precision of a measurement increases, the number of significant figures quoted increases. For example, a measurement made using a micrometer screw gauge is more precise (and hence more sf) than one made using a metre rule.

Physical quantity	Measuring instrument	~~Precision~~ resolution.
Length	Metre rule	0.1 cm [1mm]
	Vernier calipers	0.01 cm [0.1 mm]
	Micrometer screw gauge	0.001 cm [0.01 mm]
Time	▵ Digital stopwatch	0.01 s
Mass	▵ Electronic balance	0.01 g
Voltage or current	▵ Digital multimeter	Depends on setting used

▵ For digital/electronic meters, measurements are recorded as displayed on the meters.

1.6 Systematic and Random Errors

Systematic Errors

In the process of taking measurements, we may encounter errors that give rise to incorrect readings. Errors are generally classified as either systematic or random.

Definition

Systematic errors result in all readings or measurements being either always above or always below the true value by a **fixed amount**.

A systematic error can be **eliminated** only if the source of the error is known and accounted for. It cannot be eliminated by repeating the measurements and averaging them.

Examples of systematic errors:

- Not accounting for zero-error in a measurement
  An instrument has a zero error if the scale reading is non-zero before a reading is taken. Instruments should be checked for zero error and the zero error must be accounted for in the measurement.

- Not accounting for background radiation
  Background radiation is constantly present in the environment and is emitted by a variety of natural and artificial sources. When measuring the activity of a radioactive source, the value of background radiation must be subtracted from the count rate recorded by the Geiger Muller counter.

Random Errors

Definition

Random errors result in readings or measurements being scattered about a mean value. These errors have equal probability of being positive or negative.

Random errors may be **reduced** by:

- repeating the measurements and finding the average value.
- plotting a graph and drawing a line of best fit for the plotted points.

Examples of random errors:

- variation in diameter of a piece of wire when measurements are taken along a piece of wire
- fluctuations in the count-rate of a radioactive decay

Example 6 Complete the table below.

Systematic or Random Error?	
The initial reading of the scale was not zero before taking measurement.	Systematic
Estimation in starting or stopping the stop-watch due to a lack of a reference marker.	Random
Starting or stopping the stop-watch with a delay between observation and tapping the stop-watch button.	Systematic
Simultaneously taking two different readings (which vary with time) from two different meters.	Random

Note

Errors in timings occur due to a variety of reasons. It is important to establish how the error occurs in order to determine which type of error it is and whether or not it can be eliminated.

1.7 Accuracy and Precision

Accuracy vs Precision

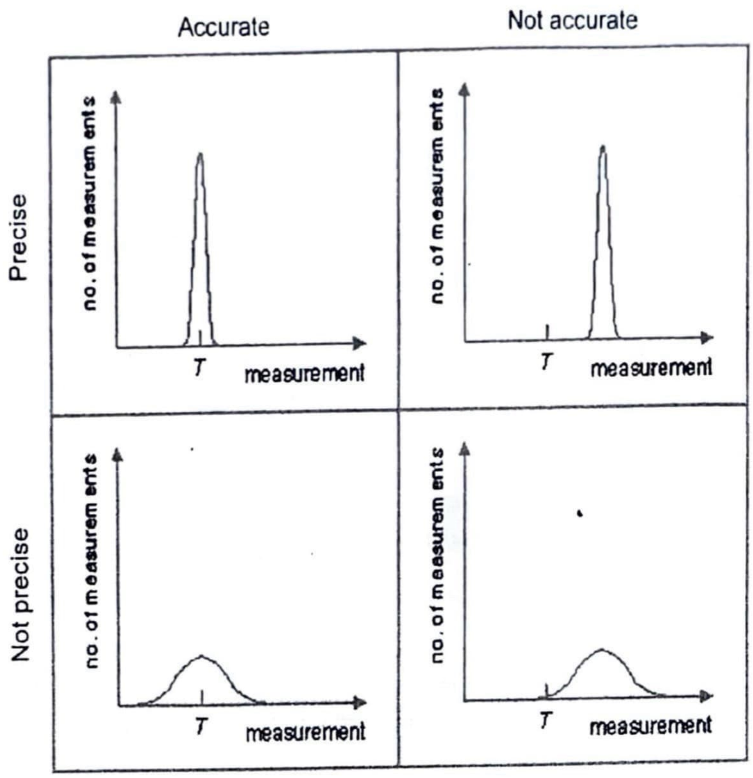
Definition

Accuracy is the degree of closeness of the mean value of the measurements to the true value. It is affected by systematic error.

Precision is the degree of agreement between repeated measurements of the same quantity. It is affected by random error.

Note

- "Accuracy" and "Precision" are comparative terms, i.e. we need to compare at least two measurements to decide which is more accurate or precise.
- When we discuss the precision of a measuring instrument, we are referring to the smallest unit it can measure. Hence, in Section 1.5, the micrometer screw gauge is more precise than the metre-rule.
- When a large number of repeated measurements are plotted in a distribution graph, a bell-shaped graph is obtained. Accuracy and precision are graphically interpreted below.



T denotes true value of the measured quantity.

Related Terminologies

1. **Responsiveness** of an instrument is the ability of the instrument to detect rapid variations in the quantity being measured.

2. **Sensitivity** of an instrument is the ability of the instrument to detect small changes in the quantity being measured.

3. **Reliability** of the measurement is a measure of the confidence that can be placed in a set of measurements, i.e. whether or not they are both accurate and precise.

4. **Reproducibility** of a measurement is the agreement between different experimenters performing the same measurement or the same experimenter performing repeated measurements of the same physical quantity.

1.8 Calculations of Uncertainties of Derived Quantities

Uncertainties of derived quantities

We can combine two or more quantities that we have measured to produce a derived quantity. For example:

- dividing distance by time to get speed,
- adding two lengths to get a total length.

Now that we have learned how to determine the uncertainties in directly-measured quantities, we need to learn how these uncertainties propagate to the uncertainty of the derived quantity. We assume that the two directly-measured quantities are x and y with uncertainties Δx and Δy, respectively.

Note

The measurements x and y must be <u>independent of each other</u>.

The Upper-Lower Bound Method of Uncertainty Propagation

The simplest way to determine the uncertainty of derived quantity is to use the **upper-lower bound method** of uncertainty propagation.

This method uses the uncertainty ranges of each measurement to calculate the **maximum** Q_{max} and **minimum** Q_{min} values of the derived quantity Q, and then using them to determine the **uncertainty** ΔQ of the derived quantity:

$$\Delta Q = \frac{Q_{max} - Q_{min}}{2}$$

<u>Worked Example:</u>

An angle θ is measured using a protractor, where $\theta = (25 \pm 1)°$, and you need to calculate $Q = \cos\theta$ and its uncertainty ΔQ. Hence,

$$Q = \cos 25° = 0.90631$$

$$Q_{min} = \cos 26° = 0.89879$$
$$Q_{max} = \cos 24° = 0.91355$$

Note

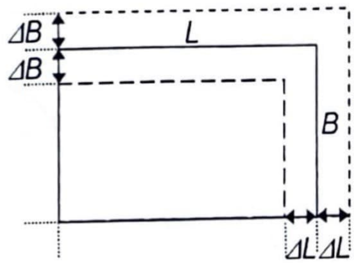
$$\Delta Q = \frac{Q_{max} - Q_{min}}{2} = \frac{0.91355 - 0.89879}{2} = 0.007 \text{ (1 s.f.)}$$

$\therefore Q = 0.906 \pm 0.007$ (both numbers must have the same d.p.)

This method is especially useful when the relationship between the variables is complex or is incomplete.

Example 7(i) A rectangle has length $L = (34.3 \pm 0.1)$ cm and breadth $B = (21.8 \pm 0.1)$ cm. Calculate the perimeter P and area A of the rectangle, and express their values with their absolute uncertainties.

Upper-Lower Bound Method



$A = L \times B = 34.3 \times 21.8 = 747.74 \text{ cm}^2$

$A_{max} = L_{max} \times B_{max} = 34.4 \times 21.9 = 753.36 \text{ cm}^2$

$A_{min} = L_{min} \times B_{min} = 34.2 \times 21.7 = 742.14 \text{ cm}^2$

$$\Delta A = \frac{A_{max} - A_{min}}{2} = \frac{753.36 - 742.12}{2} = 5.61 = 6 \text{ cm}^2 \text{ (1sf)}$$

$\therefore A = (748 \pm 6) \text{ cm}^2$ (no dp)

Uncertainty Formulae The formulae used to calculate the uncertainties of derived quantities are summarised in the table below. (They are derived using calculus.)

1.	Addition	$Q = aX + bY$	Absolute uncertainty: **(1 sf)**	$\Delta Q = \lvert a\rvert\, \Delta X + \lvert b\rvert\, \Delta Y$
2.	Subtraction	$Q = aX - bY$	Absolute uncertainty: **(1 sf)**	$\Delta Q = \lvert a\rvert\, \Delta X + \lvert b\rvert\, \Delta Y$
3.	Product	$Q = aX \times Y$	Fractional uncertainty: **(2 sf)**	$\frac{\Delta Q}{Q} = \frac{\Delta X}{X} + \frac{\Delta Y}{Y}$
4.	Division	$Q = a\frac{X}{Y}$	Fractional uncertainty: **(2 sf)**	$\frac{\Delta Q}{Q} = \frac{\Delta X}{X} + \frac{\Delta Y}{Y}$
5.	Product with powers	$Q = aX^m \times Y^n$	Fractional uncertainty: **(2 sf)**	$\frac{\Delta Q}{Q} = \lvert m\rvert\frac{\Delta X}{X} + \lvert n\rvert\frac{\Delta Y}{Y}$
6.	Quotient with powers	$Q = a\frac{X^m}{Y^n}$	Fractional uncertainty: **(2 sf)**	$\frac{\Delta Q}{Q} = \lvert m\rvert\frac{\Delta X}{X} + \lvert n\rvert\frac{\Delta Y}{Y}$

Note

- *a*, *b*, *m*, *n* are numerical constants.
- Uncertainties do not cancel out one another. They always add up to maximise the uncertainty.
- The **absolute uncertainty** of a quantity is expressed to **1 s.f.** The **quantity** is then expressed to the **same d.p.** as the absolute uncertainty.
- The **fractional uncertainty** is the value of the absolute uncertainty divided by the value of the quantity ie. Δx/x. The fractional uncertainty multiplied by 100 is the percentage error,

$$\text{i.e. \% uncertainty of } x = \frac{\Delta x}{x} \times 100\%$$

- **Always re-arrange a given equation such that only the quantity of interest appears on the left-hand-side.**

Example 7(ii)
Using Formulae List

$P = 2L + 2B = 2\times34.3 + 2\times21.8 = 112.2$ cm

$\Delta P = 2\Delta L + 2\Delta B = 2 \times (0.1 + 0.1) = 0.4$ cm (to 1 s.f.)

$\therefore P = (112.2 \pm 0.4)$ cm

$A = L \times B = 34.3 \times 21.8 = 747.74$ cm^2

$$\frac{\Delta A}{A} = \frac{\Delta L}{L} + \frac{\Delta B}{B} = \frac{0.1}{34.3} + \frac{0.1}{21.8} = 0.00750$$

$\Delta A = (\Delta A/A) \times A = 0.00750 \times 747.74 = 5.61$ cm^2 = 6 cm^2 (to 1 s.f.)

$\therefore A = (748 \pm 6)$ cm^2 **SIMILAR ANSWER TO LOWER-UPPER BOUND METHOD IN EG 7(i)!**

Example 8 A metal cube of side *L* has mass *M*. Its density ρ can be calculated from $M = \rho L^3$.

If $M = (0.065 \pm 0.001)$ kg and $L = (0.200 \pm 0.001)$ m, express ρ with its associated uncertainty.

Making ρ the subject : $\rho = \dfrac{M}{L^3} = \dfrac{0.065}{0.200^3} = 8.125\,\text{kg}\,\text{m}^{-3}$

Fractional uncertainty : $\dfrac{\Delta\rho}{\rho} = \left(\dfrac{\Delta M}{M}\right) + 3\left(\dfrac{\Delta L}{L}\right) = \left(\dfrac{0.001}{0.065}\right) + 3\left(\dfrac{0.001}{0.200}\right) = 0.0304$

Absolute uncertainty : $\Delta\rho = 0.0304 \times 8.125$
= 0.247 kg m^{-3}
= 0.2 kg m^{-3} (absolute uncertainty to 1sf)

Therefore, density $\rho = (8.1 \pm 0.2)$ kg m^{-3} (answer to same dp as absolute uncertainty)

Example 9
N2012/P1/Q2

The equation connecting object distance u, image distance v and focal length f for a lens is

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}.$$

A student measures values of u and v, with their associated uncertainties, which are

$$u = 50 \text{ mm} \pm 3 \text{ mm},$$
$$v = 200 \text{ mm} \pm 5 \text{ mm}.$$

Determine values f and its associated uncertainty Δf.

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$
$$= \frac{1}{50} + \frac{1}{200} \qquad \Rightarrow f = 40 \text{ mm}$$
$$= \frac{1}{40}$$

$$\frac{1}{f_{min}} = \frac{1}{u_{min}} + \frac{1}{v_{min}}$$
$$= \frac{1}{47} + \frac{1}{155} \qquad \Rightarrow f_{min} = 37.87 \text{ mm}$$

$$\frac{1}{f_{max}} = \frac{1}{u_{max}} + \frac{1}{v_{max}}$$
$$= \frac{1}{53} + \frac{1}{205} \qquad \Rightarrow f_{max} = 42.11 \text{ mm}$$

$$\Delta f = \frac{42.11 - 37.87}{2} = 2.12$$
$$= 2 \text{ mm (1sf)}$$

$$\therefore f = (40 \pm 2) \text{ mm}$$

1.9 Scalars and Vectors

Scalar and Vectors

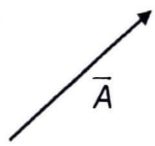
All physical quantities can be divided into *scalar* and *vector* quantities.
A ***scalar*** quantity has a magnitude only.
A ***vector*** quantity has both a magnitude and a direction.

Scalar Quantity	Vector Quantity
distance	displacement
mass	weight
speed	velocity
temperature	momentum
time	acceleration
energy	moment

Vector Representation

Scalars are represented only by a number representing its magnitude and a unit.
Vectors are represented by a number, a unit and a direction.

A vector is denoted by $\vec{A}$, $\underset{\sim}{A}$ or **A**.



A vector is represented by an arrow:

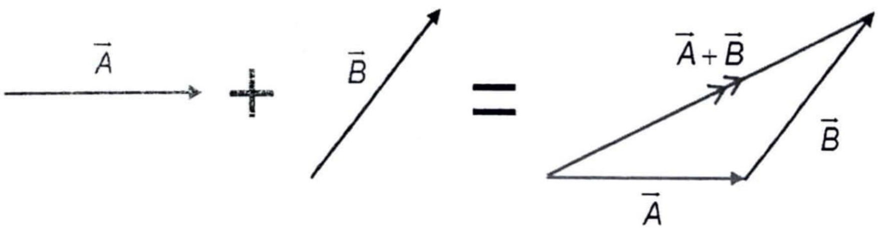
- the length of the arrow represents the magnitude of the vector.
- the direction of the arrow represents the direction of the vector.

Vector Addition

Two or more vectors may be added to form a resultant vector.

Polygon Law (for any number of vectors)

Consider the vector addition of two vectors $\vec{A}$ and $\vec{B}$ by joining the head of one vector to the tail of another:

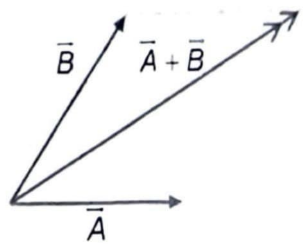


The resultant vector $\vec{A}+\vec{B}$ is a straight line represented with two arrow-heads that points from the tail of the first vector to the head of the last vector.

Parallelogram Law (for only two vectors)

It can be seen that if vectors $\vec{A}$ and $\vec{B}$ form the sides of a parallelogram by starting or ending at the same point, then the resultant vector $\vec{A}+\vec{B}$ is the diagonal of the parallelogram.

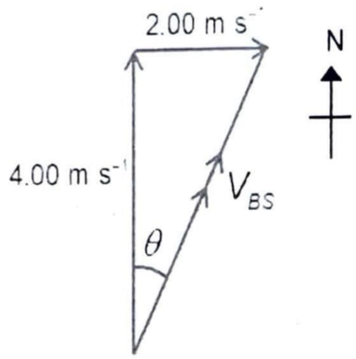
This is called the parallelogram law of vector addition and it is equivalent to the polygon law.



Example 10 A boat is travelling due north with a velocity V_{BW} of 4.00 m s^{-1} relative to the water. The current in the water is flowing at a velocity V_{WS} of 2.00 m s^{-1} in an easterly direction relative to the shore. Determine the velocity V_{BS} of the boat relative to the shore.

Using Pythagoras Theorem,
$v_{BS}^2 = 4.00^2 + 2.00^2$
$v_{BS} = 4.47$ m s^{-1}

$\tan\theta = 2.00/4.00$
$\theta = 26.6°$



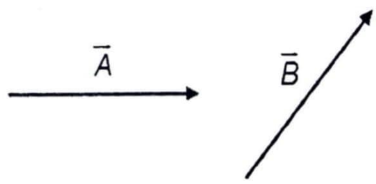
Velocity of the boat relative to the shore is 4.47 m s^{-1} in a direction 26.6° East of North.

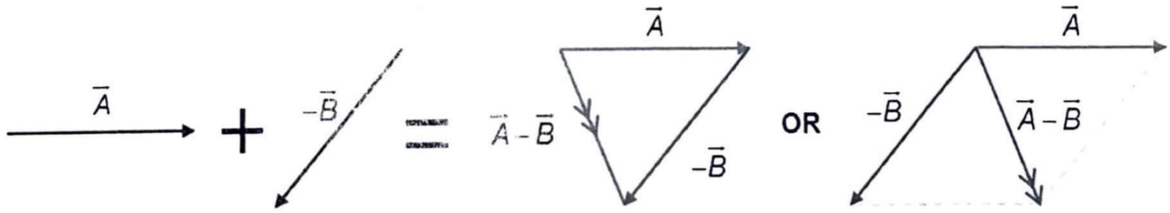
Vector Subtraction

Consider the same two vectors $\vec{A}$ and $\vec{B}$.

If vector $\vec{B}$ is to be subtracted from vector $\vec{A}$, the resultant vector is $\vec{A}-\vec{B}$.

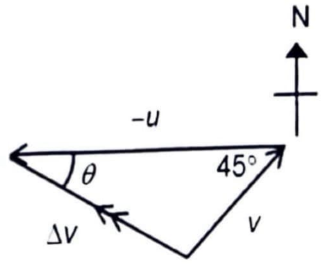
This can be rewritten as $\vec{A}+(-\vec{B})$, which is similar to vector addition with the second vector pointing opposite to its original direction.





Polygon Law Parallelogram Law

Example 11 An object was initially moving with a constant speed of 20 m s^{-1} towards the east. It changed direction and moved with a constant speed of 10 m s^{-1} in the north-easterly direction. Using vector analysis, calculate the change in velocity.



The magnitude is found using cosine rule:

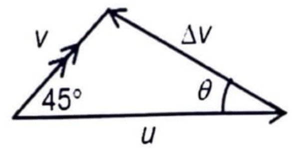
$$\Delta v = \sqrt{u^2 + v^2 - 2uv\cos 45^\circ}$$
$$= \sqrt{20^2 + 10^2 - 2\times 20\times 10\cos 45^\circ}$$
$$= 14.7 \text{ m s}^{-1}$$

The direction is found using sine rule:

$$\frac{\Delta v}{\sin 45^\circ} = \frac{v}{\sin\theta}$$
$$\sin\theta = \frac{10\sin 45^\circ}{14.74}$$
$$\theta = 28.7^\circ$$

The change in velocity is 14.7 m s^{-1}, 28.7° North of West.

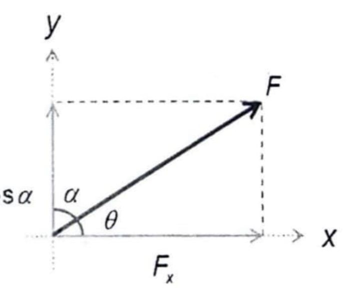
OR



The change in velocity is 15 m s^{-1}, 28.7° North of West.

Resolution of Vectors A vector can be resolved into two components that are perpendicular to each other. Usually, the chosen axes are along the *vertical* and *horizontal* directions. The directions of the axes can also be *parallel* and *perpendicular* to a slope.

Just as a vector can be "broken" into two components, these components can be "recombined" to give the magnitude and direction of the vector.



$F_y = F\sin\theta = F\cos\alpha$

$F_x = F\cos\theta = F\sin\alpha$

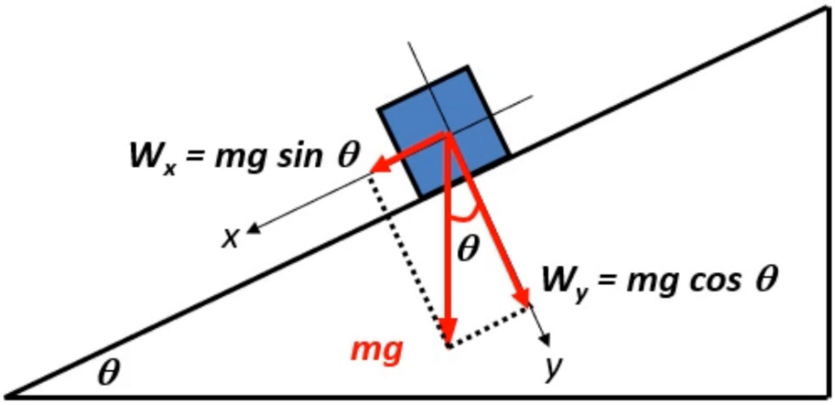
$$F = \sqrt{F_x^{\,2} + F_y^{\,2}}$$

$$\tan\theta = \frac{F_y}{F_x} \qquad \tan\alpha = \frac{F_x}{F_y}$$

Example 12 Indicate the component of the weight W of the object acting

(a) along the slope and

(b) perpendicular to the slope.



Strategies For Adding Multiple Vectors

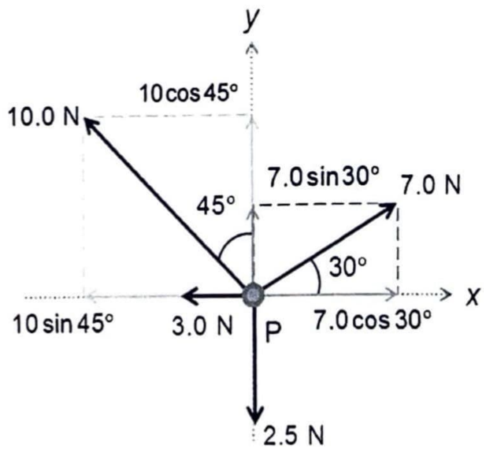
When two or more vectors are added, the following procedures can be applied to determine the resultant:

1. Choose a convenient orientation of the *x-y* coordinate system.
2. Draw all vectors starting from the origin with the appropriate lengths and directions.
3. Determine the *x*- and *y*-components of all vectors.
4. Determine the resultant components along the *x*- and y-axis.
5. Apply Pythagoras theorem to find the magnitude of the resultant vector.
6. Apply trigonometry to find the angle that the resultant vector makes with the *x*- or *y*-axis, or any other reference line.

Note: The original vector forms the hypotenuse of the right-angle triangle and the components are shorter than the vector (unless the vector lies along the axis). The resultant vector is independent of the orientation of the *x-y* axes.

Example 13 The figure below shows four forces acting at a point P. Calculate the resultant force on P by resolving each force in two perpendicular directions.

Taking upwards and rightwards as positive:

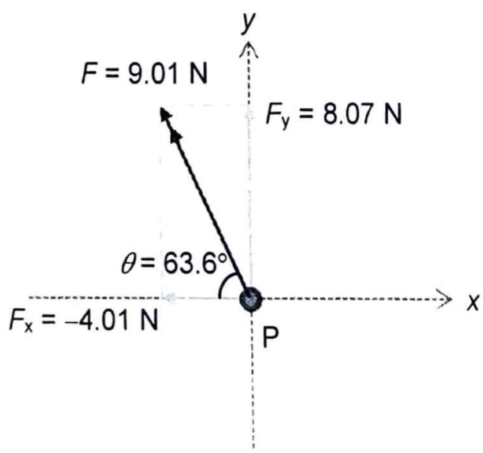


$$F_y = 7.0\sin 30° + 10.0\cos 45° - 2.5 = 8.07 \text{ N}$$

$$F_x = 7.0\cos 30° - 10.0\sin 45° - 3.0 = -4.01 \text{ N}$$

$$F = \sqrt{F_y^2 + F_x^2} = \sqrt{(8.07)^2 + (-4.01)^2}$$
$$= 9.01 \text{ N}$$

$$\tan\theta = \frac{F_y}{F_x} = \frac{8.07}{4.01}$$
$$\theta = 63.6°$$



Summary

SI Units

The seven **base units** are: metre (m), kilogram (kg), second (s), ampere (A), kelvin (K), mole (mol) and candela (cd).

Derived units are defined in terms of base units. They are expressed as products or quotients of base units.

An equation is said to be **homogeneous** or **dimensionally consistent** if every term on both sides of the equation has the same base units.

A homogeneous equation may not be physically correct.

Errors and Uncertainties

Item	Accuracy	Precision
Instrument	Calibration of instrument	Smallest division of instrument
Measurements	Closeness of mean to true value	Closeness of measurements to one another

Systematic errors result in all readings or measurements being either always above or always below the true value by a **fixed amount**.

Random errors result in readings or measurements being scattered about a mean value. These errors have equal probability of being positive or negative.

Accuracy is the degree of closeness of the mean value of the measurements to the true value. It is affected by systematic error.

Precision is the degree of agreement between repeated measurements of the same quantity. It is affected by random error.

Calculating uncertainties of derived quantities

If $Q = aX \pm bY$ then $\Delta Q = |a|\Delta X + |b|\Delta Y$

If $Q = (aX^m)(bY^n)$ OR $Q = \dfrac{aX^m}{bY^n}$ then $\dfrac{\Delta Q}{Q} = |m|\dfrac{\Delta X}{X} + |n|\dfrac{\Delta Y}{Y}$

Percentage uncertainty $= \dfrac{\Delta Q}{Q} \times 100\%$

The absolute uncertainty is expressed to 1 s.f. and the quantity is expressed to the same d.p. as that of the absolute uncertainty.
Fractional and percentage uncertainty can be expressed to 2 s.f.

Scalars & Vectors

A **scalar** quantity has a magnitude only.
A **vector** quantity has both a magnitude and a direction.

Tutorial 1 MEASUREMENT

Self-Check Questions

S1. List the SI base quantities and their SI units.

S2. Explain what is meant by *derived* units.

S3. State two ways in which an equation that is dimensionally consistent may be physically incorrect.

S4. Distinguish between *systematic error* and *random error.*

S5. Give examples of systematic errors and random errors, and explain how they can be eliminated / reduced.

S6. Distinguish between *accuracy* and *precision.*

S7. Distinguish between a *scalar* and a *vector* quantity.

S8. Explain how a vector can be resolved into its perpendicular components?

Self-Practice Questions

SP1 The drag force F experienced by a steel sphere of radius r dropping at speed v through a liquid is given by

$$F = arv,$$

where a is a constant.

What would be a suitable SI unit for a?

A N

B $N\ s^{-1}$

C $N\ m^2\ s^{-1}$

D $N\ m^{-2}\ s$

N02/I/1

SP2 An alternative form of the unit of resistance, the ohm (Ω), is VA^{-1}.

Which of the following examples shows a similar correct alternative form of unit?

	unit	alternative form
A	coulomb (C)	$A\ s^{-1}$
B	pascal (Pa)	$N\ m^{-2}$
C	volt (V)	J C
D	watt (W)	J s

N04/I/1

SP3 What is the order of magnitude of the energy of an electron when it hits the screen of a cathode-ray tube?

A 10^{-22} J

B 10^{-19} J

C 10^{-16} J

D 10^{-13} J

N09/I/2

SP4 Which estimate is realistic?

A The kinetic energy of a bus travelling on an expressway is 30 000 J.

B The power of a domestic light is 300 W.

C The temperature of a hot oven is 300 K.

D The volume of air in a car tyre is 0.03 m^3.

N08/I/2

SP5 An object of mass 1.000 kg is placed on four different balances. For each balance the reading is taken five times.

The table shows the values obtained together with the means.

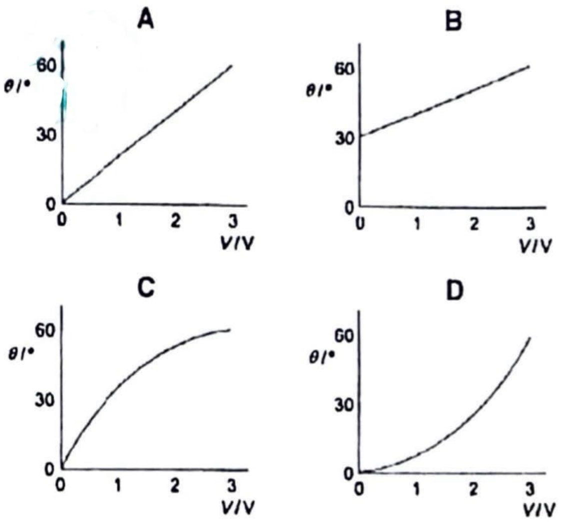
Which balance has the smallest systematic error but is not very precise?

balance	reading/kg					mean/kg
	1	2	3	4	5	
A	1.000	1.000	1.002	1.001	1.002	1.001
B	1.011	0.999	1.001	0.989	0.995	0.999
C	1.012	1.013	1.012	1.014	1.014	1.013
D	0.993	0.987	1.002	1.000	0.983	0.993

N02/I/2

SP6 Calibration curves showing the variation with potential difference V of the deflection θ of the pointer of four different voltmeters are given below.

Which meter gives the largest angular change per unit potential difference at 2.5 V?



N05/I/2

SP7 A wire of uniform circular cross-section has diameter d and length L. A potential difference V between the ends of the wire gives rise to a current I in the wire.

The resistivity ρ of the material of the wire is given by the expression

$$\rho = \frac{\pi d^2 V}{4LI}.$$

In one particular experiment, the following measurements are made.

$d = 1.20 + 0.01$ cm

$I = 1.50 + 0.05$ A

$L = 100 + 1$ cm

$V = 5.0 + 0.1$ V

Which measurement gives rise to the least uncertainty in the value for the resistivity?

A d

B I

C L

D V

N10/I/3

Discussion Questions

Units Analysis

D1 Boyle's Law governing the pressure P and volume V for a given mass of gas may be modified into the Van der Waals' form:

$$(P + \frac{a}{V^2})(V - b) = c$$

What are the base units of a, b and c? Present your working clearly. [3]

D2 The density ρ and the pressure P of a gas are related by the following expression:

$$c = \sqrt{\frac{\gamma P}{\rho}}$$, where c and γ are constants.

(a) Determine the base units of density ρ. [1]
(b) Show that the base units of pressure P are kg m^{-1} s^{-2}. [1]
(c) Given that the constant γ has no unit, determine the unit of c. [1]
(d) Suggest what quantity may be represented by the symbol c. [1]

D3 The displacement of an object moving under uniform acceleration is some function of the time t and the acceleration a. Suppose we write this displacement s as $s = k\,a^x\,t^y$, where k is a constant without unit.

(a) Find, by unit analysis, the values of indices x and y. [2]
(b) Hence, write down the equation relating s with k, a and t. [1]
(c) Can this analysis give the value of k? [1]

Estimation

D4 Carry out the following arithmetic operations and express the answers to the correct number of significant figures and/or decimal places.

(a) The sum of the numbers 756, 37.2, 0.83 and 2.5.

(b) The value of 1.6523 – 0.015.

(c) The product of 3.2 and 3.563.

(d) The product of 5.6 and π.

D5 Give **reasoned** estimates of the following quantities. In each case, give your answer in an SI unit.

(a) The area of the main island of Singapore

(b) The acceleration of a train on the Singapore rapid transit system.

(c) The power of a car travelling along an expressway.

[H1/N09/2/1]

Uncertainties

D6 The acceleration due to gravity, g, was calculated from one single timing of the period T of a simple pendulum using an electrical circuit connected to a stop-clock. The stop-clock, marked out in 0.01 s, reads 2.00 s exactly. The length L of the pendulum was 99.0 cm long, measured with a metre rule graduated in mm.

Express the value of g with its uncertainty, given the following relationship:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

[4]

D7 A student times the fall of a small metal ball. Data for the time t taken for the ball to fall a vertical distance h from rest are given below.
[Hint: You may apply the kinematic equation $s = ut + ½ at^2$.]

$h = 266 \pm 1$ cm

$t = 0.740 \pm 0.005$ s

(a) Use the data to determine

(i) a value to three significant figure, of the acceleration of free fall g, [1]

(ii) the percentage uncertainty, to two significant figures, of

1. the distance h, [1]

2. the time t. [1]

(b) Use your answers in **(a)** to determine the actual uncertainty in the value of g.
Hence give a statement of g, with its uncertainty, to an appropriate number of significant figures. [2]

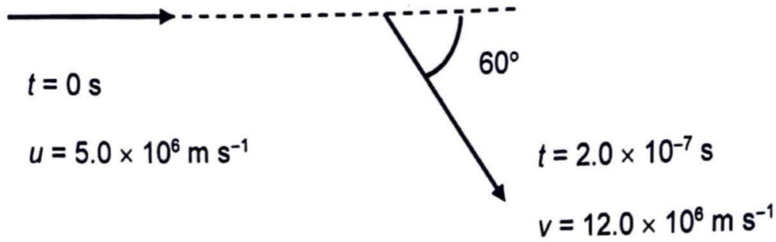
(c) Suggest two reasons why, in this experiment, although the value of t is precise, it may not be accurate. [2]

[H2/N07/2/1]

Vectors

D8 Determine the change in velocity of a tennis ball, if it approaches a tennis racquet at 25 m s^{-1} and leaves the racquet in the opposite direction at 35 m s^{-1}. [1]

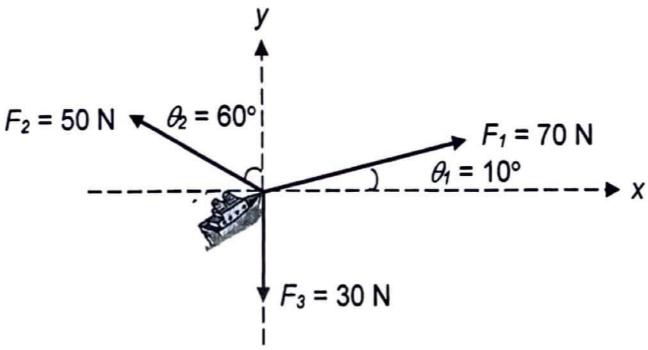
D9 At two instants, t = 0 s and t = 2.0 × 10^{-7} s, the velocity of an electron moving in a vacuum in the x–y plane is as indicated by the vectors shown in the figure below.



(a) Calculate the magnitude of the change in velocity which takes place over this time interval. [2]

(b) Hence, find the magnitude and direction of the average acceleration of the electron over this interval. [2]

D10 In the plan view below, three forces act on a boat. Determine the magnitude and direction of the resultant force on the boat. Give the answers to 2 significant figures. [4]



Challenging Questions

C1 (a) In the Système International d'Unitès, all derived mechanical quantities can be expressed as products or quotients of the base units of mass, length and time.

State with a reason whether it would be feasible to adopt alternative systems in which the *base* quantities were
(i) mass, area and density;
(ii) force, energy and power.

In this second system (the '2013 Raffles System'), the base units are the newton (N), joule (J) and watt (W) respectively.

Treating the SI base units as *derived* units in the Raffles System, express the metre, kilogram and second as products or quotients of appropriate Raffles base units.

(b) Suppose that the International Prototype kilogram (which, being made of platinum-iridium alloy, is a valuable object) has been stolen from the Standards Laboratory at Sèvres. It is suggested that the opportunity should be taken to replace the kilogram as an SI base unit by an entirely different base unit of mass: one-twelfth of the mass of an atom of carbon-12. (This unit already exists as the unified atomic mass constant.)

(i) Discuss whether the theft of the International Prototype would put the Système International at risk.
(ii) List the arguments for and against introducing the unified atomic mass constant as the base unit of mass.

Modified from GCE A-Level N91/0/Q1

Answers

D1 kg m^5 s^{-2} , kg m^2 s^{-2}

D2 (a) kg m^{-3} , (c) m s^{-1}

D3 (a) $x = 1$, $y = 2$, (b) $s = kat^2$

D4 (a) 797, (b) 1.637, (c) 11 or 11.4, (d) 18 or 17.6

D5 Approximate order of magnitude (a) 10^8 m^2 , (b) 10^0 m s^{-2} , (c) 10^4 W

D6 (9.8 ± 0.1) m s^{-2}

D7 (a)(i) 9.72 m s^{-2} , (ii) 0.38%, 0.68%, (b) $g = (9.7 \pm 0.2)$ m s^{-2}

D8 60 m s^{-1} away from the racquet

D9 (a) 10.4×10^6 m s^{-1} , (b) 5.22×10^{13} m s^{-2} , 84.5° clockwise from the original direction

D10 27 N, 16° above +x-axis

Tutorial 1 Measurement Suggested Solutions

S1 *Not in syllabus

SI base quantity	SI base unit
time	s
length	m
mass	kg
current	A
temperature	K
amount of substance	mol
luminous intensity*	cd*

S2 **Derived units** are defined in terms of base units and are expressed as products and/or quotients of base units.

S3 Wrong constants/signs (Eg: $s = ut - 2\,at^2$)
Missing.additional terms (Eg: $v = u + at + s/t$)

S4

Systematic errors	Random errors
All readings being above or below the accepted value by a fixed amount	Readings being scattered about a mean value
Can be eliminated if the source is known	Can be reduced by taking repeated measurements and finding average

S5

Systematic errors	Random errors
Background radiation	Fluctuation in radioactive decay
Zero-error in instruments	Variations in diameter of wire
Subtract from measurement	Take multiple measurements to find mean

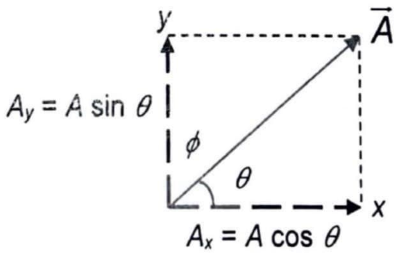
S6 ***Accuracy*** is the degree of closeness of the mean value of the measurements to the true value. It is affected by systematic error (eg correct calibration of instruments).

Precision is the degree of agreement between repeated measurements of the same quantity. It is affected by random error.

S7 A ***scalar*** quantity has a magnitude only.

A ***vector*** quantity has both a magnitude and a direction.

S8 Orientate the perpendicular x and y axes suitably and place the origin at the start of the vector. Project a line from the tip of the vector to each of the axis, such that a right-angle is formed on the axis. Apply sine or cosine function to determine the respective components.



SP1 From $F = arv$

$\Rightarrow a = \frac{F}{rv}$ (make the object of interest on the LHS)

$\therefore$ Unit of $a = \frac{\text{unit of } F}{\text{unit of } rv} = \frac{\text{N}}{(\text{m})(\text{m s}^{-1})} = \text{N m}^{-2}\ \text{s}^{1}$

Answer: D

Extension: Express a in terms of its base units.

unit of $a = \text{N m}^{-2}\ \text{s}$

$= (\text{kg m s}^{-2})\ \text{m}^{-2}\ \text{s}$

$= \text{kg m}^{-1}\ \text{s}^{-1}$

SP2 You need to know the following formulae to work out the correct answer:

Resistance R = (p.d. V) ÷ (current I)
Charge Q = (current I) × (time t)
Pressure P = (force F) ÷ (area A)
Potential difference V = (work done W) ÷ (charge Q)
Power P = (work done W) ÷ (unit t)

Given Unit	Quantity	Formulae	Alternative Units
Coulomb	Charge	$Q = I \times t$	$\text{A} \times \text{s} = \text{A s}$
Pascal	Pressure	$P = \frac{F}{\text{Area}}$	$\frac{\text{N}}{\text{m}^2} = \text{N m}^{-2}$
Volt	Potential Difference	$V = \frac{\text{Power}}{I} = \frac{\text{Energy}}{t} \times \frac{1}{I} = \frac{\text{Energy}}{\text{Charge}}$	$\frac{\text{J}}{\text{C}} = \text{J C}^{-1}$
Watt	Power	$P = \frac{\text{Energy}}{t}$	$\frac{\text{J}}{\text{s}} = \text{J s}^{-1}$

Answer: B

SP3 Electrons in cathode-ray tubes are normally accelerated through a potential difference of a few thousand volts: $\Delta V \approx 1000$ V.

Since the charge of an electron is e, the gain in kinetic energy is

$\Delta E_K = (e)(\Delta V) = (1.60 \times 10^{-19})(1000) = 1.60 \times 10^{-16}$ J

Answer: C

SP4 **A** Estimates: Mass of a bus $m \approx 10000$ kg; average speed on the highway $v \approx 60$ km h^{-1}

$$E_{K(BUS)} = \frac{1}{2}mv^2 = \frac{1}{2}(10000)\left(\frac{60\times10^3}{60\times60}\right)^2 = 1,390,000 \text{ J}$$

B Typical power of bulbs used at home is between 10 W to 110 W

C Temperature of an oven is about 250 °C or 523 K

D Estimates: outer diameter $2R \approx 50$ cm; inner diameter $2r \approx 30$ cm; width $w \approx 20$ cm

$$\text{Volume} = \left(\pi R^2 - \pi r^2\right)w = \pi\left((0.25)^2 - (0.15)^2\right)(0.20) = 0.025 \text{ m}^3$$

Hence, closest answer is D.

Answer: D

SP5

Balance	Reading/ kg					Mean/ kg	Deviation from **true** value	Biggest deviation from **mean** value	**Remarks**
	1	2	3	4	5				
A	1.000	1.000	1.002	1.001	1.002	1.001	0.001	0.001	accurate & precise
B	1.011	0.999	1.001	0.989	0.995	0.999	0.001	0.012	Somewhat accurate but not precise
C	1.012	1.013	1.012	1.014	1.014	1.013	0.013	0.001	Not accurate but precise
D	0.993	0.987	1.002	1.000	0.983	0.993	0.007	0.010	Not accurate nor precise

A and **B** give the smallest deviation from the true value: Most **accurate**
This also means that **A** and **B** have the smallest systematic error.
In the same vein, **C** gives the most inaccurate readings which means that the systematic error here is the largest.
A and **C** give readings with the least variation: Most **precise**
This means that **A** and **C** have the smallest random error.
B and **D** give the largest variations in readings. This means that the random error is very large.
To satisfy both *small systematic error* but *large random error*, answer is B.
Answer: B

SP6 The most sensitive voltmeter in the region of 2.5 V is the one that shows the greatest deflection for a small change in voltage, ie the steepest graph around 2.5 V.
Answer: D

SP7

$$\rho = \frac{\pi d^2 V}{4LI} \Rightarrow \frac{\Delta\rho}{\rho} = 2\frac{\Delta d}{d} + \frac{\Delta V}{V} + \frac{\Delta L}{L} + \frac{\Delta I}{I}$$

Contribution to uncertainty from d: $2\frac{\Delta d}{d} = 2\times\frac{0.01}{1.20} = 0.017$ (2 sf)

Contribution to uncertainty from V: $\frac{\Delta V}{V} = \frac{0.1}{5.0} = 0.020$ (2 sf)

Contribution to uncertainty from L: $\frac{\Delta L}{L} = \frac{1}{100} = 0.010$ (2 sf) **smallest contribution!**

Contribution to uncertainty from I: $\frac{\Delta I}{I} = \frac{0.05}{1.50} = 0.033$ (2 sf)

Answer: C