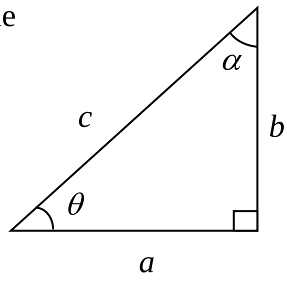
National Junior College
2016 – 2017 H2 Mathematics
Revision: Trigonometry **Tutorial Solutions**

Basic Mastery Questions

1. Consider the following right-angled triangle



Noting that $\alpha = \frac{\pi}{2} - \theta$, we have

(a) $\sin\alpha = \frac{a}{c} = \cos\theta,$

(b) $\cos\,\alpha = \frac{b}{c} = \sin\,\theta,$

(c) $\tan\alpha = \frac{a}{b} = \frac{1}{\left(\frac{b}{a}\right)} = \frac{1}{\tan\theta} = \cot\theta.$

Remark: Even though we have only justified the above results for an acute angle θ, these relationships in fact hold for ***all*** values of θ.

2. We use the ASTC diagram.

(a) $\sin(120°)$ is positive as it lies in the second quadrant. Therefore n = 1.

(b) The smallest positive integer n is 2, since

$n = 1$: $\tan(120°) < 0$ (2nd quad)

$n = 2$: $\tan(240°) > 0$ (3rd quad)

(c) The smallest positive integer n is 3. This is because the cosine function gives negative values in the 2nd and 3rd quadrants, and $\cos(360°) = 1$.

Remark: If the <u>smallest</u> positive integer is $n = k$, you must also explain why n cannot be 1, 2, 3, … , $k - 1$.

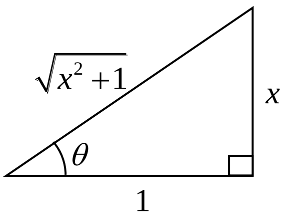
3. Since the principal value of $\tan^{-1}1$ is $\frac{\pi}{4}$, $\cos\left(\tan^{-1}1\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

4. (a) By considering the principal range for $\tan^{-1} x$, and the graph of the tangent (or inverse tangent) function,

$$0 < \theta < \frac{\pi}{2}.$$

(b)(i) $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}.$

(b)(ii)



From the above right-angled triangle,

$$\sec\theta = \frac{1}{\cos\theta} = \sqrt{x^2+1}.$$

Note: Do not give your solution as $\sec\left(\tan^{-1} x\right)$.

Alternatively, use the identity $\tan^2\theta + 1 = \sec^2\theta$ and the fact that $\sec\,x$ is positive when θ is an acute angle.

(b)(iii) By the same right-angled triangle,

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+1}}{x}.$$

Alternatively, use the identity

$$\cot\,\theta = \frac{\operatorname{cosec}\theta}{\sec\theta}.$$

(Analogy: $\tan\theta = \frac{\sin\theta}{\cos\theta}$)

5. $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$= \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\times\frac{1}{3}\right)}$

$= 1.$

$\tan(2A) = \dfrac{2\tan A}{1-\tan^2 A}$

$= \dfrac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}$

$= \dfrac{4}{3}.$

6. $R = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2.$

$\alpha = \tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right) = \dfrac{\pi}{3}.$

Therefore,

$\sin x+\sqrt{3}\cos x = 2\sin\left(x+\dfrac{\pi}{3}\right).$

Presentation: For this question, do not merely give the value of R and α, because this is not precisely what the question is asking for. Your solution should be an expression in the form $R\sin(x+\alpha)$.

Remark: R-formulae are related to the addition formulae. Observe that

$\sin x+\sqrt{3}\cos x$

$= \sqrt{1^2+\left(\sqrt{3}\right)^2}\left[(\sin x)\frac{1}{\sqrt{1^2+\left(\sqrt{3}\right)^2}}+(\cos x)\frac{\sqrt{3}}{\sqrt{1^2+\left(\sqrt{3}\right)^2}}\right]$

$= R\left(\sin x\cos\alpha+\cos x\sin\alpha\right)$

$= R\sin(x+\alpha).$

7. Basic angle = $\cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}.$

$\cos x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}.$

8. Basic angle = $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}.$

$\tan x = -\dfrac{1}{\sqrt{3}} \Rightarrow x = \dfrac{5\pi}{6}$ or $\dfrac{11\pi}{6}.$

9. Since $0\le x\le 2\pi$, we need to consider the interval $\dfrac{\pi}{4}\le\left(2x+\dfrac{\pi}{4}\right)\le 4\pi+\dfrac{\pi}{4}.$

Therefore, $\sin\left(2x+\dfrac{\pi}{4}\right)=0.$

Basic angle = 0

$\Rightarrow \left(2x+\dfrac{\pi}{4}\right) = \pi, 2\pi, 3\pi$ or 4π

$\Rightarrow x = \dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}$ or $\dfrac{15\pi}{8}.$

10. (a) By sine rule,

$\dfrac{\sin\angle ABC}{AC} = \dfrac{\sin\angle BAC}{BC}$

$\Rightarrow \sin\angle ABC = \dfrac{16}{20}\sin 30^\circ = \dfrac{2}{5}$

$\Rightarrow \angle ABC \approx 23.6^\circ$ or $156.4^\circ.$

(rejected)

The second solution is rejected because sum of all angles in a triangle must be 180°.

(b) By sine rule,

$\dfrac{DF}{\sin\angle DEF} = \dfrac{EF}{\sin\angle EDF}$

$\Rightarrow DF = 10\left(\dfrac{\sin 53.1^\circ}{\sin 30^\circ}\right)$

≈ 16.0 cm.

(c) By cosine rule,

$HI^2 = GH^2+GI^2-2(GH)(GI)\cos\angle HGI$

$\Rightarrow HI^2 = 12^2+10^2-2(12)(10)\cos 30^\circ$

≈ 36.154

$\Rightarrow HI = \sqrt{36.154...}$ (lengths are positive)

≈ 6.01 cm.

(d) By cosine rule,

$JK^2 = JL^2+KL^2-2(JL)(KL)\cos\angle JLK$

$\Rightarrow \cos\angle JLK = \dfrac{JL^2+KL^2-JK^2}{2(JL)(KL)}$

$= \dfrac{9^2+8^2-12^2}{2(9)(8)} = \dfrac{1}{144}$

$\Rightarrow \angle JLK \approx 89.6^\circ.$

Intermediate Level Questions

1. Since $\sin^2\theta+\cos^2\theta=1$ for all θ,

$$\sin A=\pm\sqrt{1-\cos^2 A}=\frac{3}{5}\ \Rightarrow\ \cos A=\pm\frac{4}{5}$$

$$\cos B=\pm\sqrt{1-\sin^2 B}=\frac{12}{13}\ \Rightarrow\ \sin B=\pm\frac{5}{13}$$

Using the identity
$\cos(A+B)=\cos A\cos B-\sin A\sin B$,

$\cos A$	$\sin B$	$\cos(A+B)$
$\frac{4}{5}$	$\frac{5}{13}$	$\frac{33}{65}$
$-\frac{4}{5}$	$\frac{5}{13}$	$-\frac{63}{65}$
$-\frac{4}{5}$	$-\frac{5}{13}$	$-\frac{33}{65}$
$\frac{4}{5}$	$-\frac{5}{13}$	$\frac{63}{65}$

Extension Question:
Identify which quadrant does the angle $A + B$ lies in for each of the possible values of $\cos(A+B)$.

2. (a) $$\text{LHS}=\frac{\sin A}{1-\cos 2A}=\frac{\sin A}{1-\left(1-2\sin^2 A\right)}=\frac{\sin A}{2\sin^2 A}=\text{RHS.}$$

(b) $$\text{LHS}=\sqrt{2+2\cos x}=\sqrt{2+2\left(2\cos^2\frac{x}{2}-1\right)}=\sqrt{4\cos^2\frac{x}{2}}=\text{RHS.}$$

(c) $$\text{LHS}=\cos^4\theta-\sin^4\theta$$
$$=\left(\cos^2\theta\right)^2-\left(\sin^2\theta\right)^2$$
$$=\left(\cos^2\theta+\sin^2\theta\right)\left(\cos^2\theta-\sin^2\theta\right)$$
$$=(1)(\cos\ 2\theta)$$
$$=\text{RHS.}$$

2. (d) $$\text{RHS}=\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}=\frac{\frac{\sin x}{\cos x}+1}{1-\frac{\sin x}{\cos x}}=\frac{\frac{\sin x+\cos x}{\cos x}}{\frac{\cos x-\sin x}{\cos x}}=\text{LHS.}$$

(e) $$\text{LHS}=\frac{\sin 5\theta+\sin\theta}{\cos 5\theta-\cos\theta}=\frac{2\sin 3\theta\cos 2\theta}{-2\sin 3\theta\sin 2\theta}=\text{RHS.}$$

(f) $$\text{LHS}=\sin\left(3x+\frac{\pi}{4}\right)\cos\left(3x-\frac{\pi}{4}\right)=\frac{1}{2}\left[\sin(6x)+\sin\left(\frac{\pi}{2}\right)\right]=\text{RHS.}$$

3. **<u>Method 1</u>: Double angle Formula**

$\sin 2x=\sin x$, where $0\le x\le 2\pi$

$$\Rightarrow\ 2\sin x\cos x=\sin x$$
$$\Rightarrow\ (\sin x)(2\cos x-1)=0$$
$$\Rightarrow\ \sin x=0\ \text{ or }\ \cos x=\frac{1}{2}$$
$$\Rightarrow\ x=0,\ \pi,\ 2\pi,\ \frac{\pi}{3},\ \text{or}\ \frac{5\pi}{3}.$$

<u>Method 2</u>: Factor Formula

$\sin 2x-\sin x=0$, where $0\le x\le 2\pi$

$$\Rightarrow\ 2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)=0$$
$$\Rightarrow\ \cos\left(\frac{3x}{2}\right)=0\ \text{ or }\ \sin\left(\frac{x}{2}\right)=0$$
$$\Rightarrow\ \frac{3x}{2}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}\quad\text{or}\quad\frac{x}{2}=0,\ \pi$$
$$\Rightarrow\ x=\frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 0,\ \text{or}\ 2\pi.$$

4. This is a quadratic equation in terms of $\sin 2x$.

$$\sin^2 2x - \sin 2x = 2, \text{ where } 0 \le x \le 2\pi$$
$$\Rightarrow \sin^2 2x - \sin 2x - 2 = 0$$
$$\Rightarrow (\sin 2x - 2)(\sin 2x + 1) = 0$$
$$\Rightarrow \sin 2x = -1$$
$$\text{or} \quad \sin 2x = 2$$
$$(\text{rejected}, \because -1 \le \sin 2x \le 1).$$

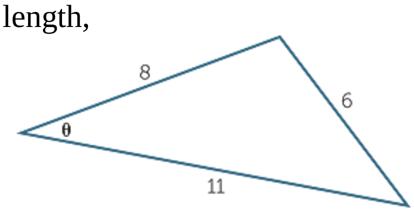
Since basic angle for $\sin 2x = -1$ is $\frac{\pi}{2}$,

$$\Rightarrow 2x = \frac{3\pi}{2} \text{ or } \frac{7\pi}{2} \qquad (\text{as } 0 \le 2x \le 4\pi)$$
$$\Rightarrow x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}.$$

5. $\angle XYZ = 180° - (72° + 55°) = 53°$.
By Sine rule,

$$\frac{13.2}{\sin 53°} = \frac{XY}{\sin 72°} = \frac{YZ}{\sin 55°}$$
$$\Rightarrow \text{XY} = 15.7 \text{ cm} \quad (\text{to 3 s.f.}), \text{ and}$$
$$\text{YZ} = 13.5 \text{ cm} \quad (\text{to 3 s.f.}).$$

6. By EITHER
using cosine rule three times to find the smallest angle among all three angles,
OR
observing* that the smallest angle of the triangle must be opposite the side of the triangle with the smallest length,



$$6^2 = 8^2 + 11^2 - 2(8)(11)\cos\theta$$
$$\Rightarrow \theta \approx 32.2°.$$

* This observation is proved in Q8.

7. (a)
$$\begin{aligned}\text{LHS} &= 2\cos\theta - \cos 3\theta - \cos 5\theta \\ &= 2\cos\theta - (\cos 3\theta + \cos 5\theta) \\ &= 2\cos\theta - 2\cos(-\theta)\cos 4\theta \\ &= 2\cos\theta[1 - \cos 4\theta] \\ &= 2(\cos\theta)(2\sin^2 2\theta) \\ &= 4\cos\theta(2\sin\theta\cos\theta)^2 \\ &= \text{RHS.}\end{aligned}$$

Alternatively,
$$\begin{aligned} &= (\cos\theta - \cos 3\theta) + (\cos\theta - \cos 5\theta) \\ &= -2\sin 2\theta \sin(-\theta) - 2\sin 3\theta \sin(-2\theta) \\ &= 2(\sin 2\theta)(\sin\theta + \sin 3\theta) \\ &= 2(\sin 2\theta)(2\sin 2\theta\cos(-\theta)) \\ &= 4\sin^2 2\theta\cos\theta \\ &= 4(2\sin\theta\cos\theta)^2\cos\theta \\ &= \text{RHS.}\end{aligned}$$

(b)
$$\begin{aligned}\text{f}(\theta) &= \cos^2\theta + \frac{1}{2}\sin 2\theta - 1 \\ &= \frac{1+\cos 2\theta}{2} + \frac{1}{2}\sin 2\theta - 1 \\ &= \frac{1}{2}(\cos 2\theta + \sin 2\theta) - \frac{1}{2} \\ &= \frac{\sqrt{2}}{2}\left[(\cos 2\theta)\frac{1}{\sqrt{2}} + (\sin 2\theta)\frac{1}{\sqrt{2}}\right] - \frac{1}{2} \\ &= \frac{\sqrt{2}}{2}\cos\left(2\theta - \frac{\pi}{4}\right) - \frac{1}{2} \quad (\text{R-formula}).\end{aligned}$$

(c) Since $\underbrace{\frac{-\sqrt{2}-1}{2}}_{-\text{ve valued}} \le [\text{f}(\theta)] \le \frac{\sqrt{2}-1}{2}$,

we have

$$0 \le [\text{f}(\theta)]^2 \le \max\left\{\left(\frac{\sqrt{2}-1}{2}\right)^2, \left(\frac{-\sqrt{2}-1}{2}\right)^2\right\}$$
$$\Rightarrow 0 \le [\text{f}(\theta)]^2 \le \frac{3+2\sqrt{2}}{4}.$$

Remark: In the above solution, we define $\max\{a, b\}$ to be the maximum of the two numbers a and b. Thus if $a > b$, then $\max\{a, b\} = a$. If $a < b$, then $\max\{a, b\} = b$.

Illustration: $\max\{-3, -4\} = -3$.

8. (i) From sine rule

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\text{constant},$$

$$a>b>c\Rightarrow \sin A>\sin B>\sin C.$$

(ii) Given that $0<A,B,C<90°$, we use the fact that the graph $y = \sin x$ is increasing in this interval to conclude that $A > B > C$.

(iii) Since $A=180°-(B+C)$ and $B + C$ is acute with $B + C > B$ ($> C$ respectively),

$$\sin A=\sin\left(180°-(B+C)\right)$$
$$=\sin(B+C)$$
$$>\sin B \quad (>\sin C \text{ resp.}).$$

Observe that $A > B + C > B$.
In addition from (ii), $B > C$.
Therefore $A > B > C$.

9. $\angle ATB=31°-18°=13°.$
Using sine rule on triangle ΔATB,

$$\frac{20}{\sin 13°}=\frac{TB}{\sin 18°}$$

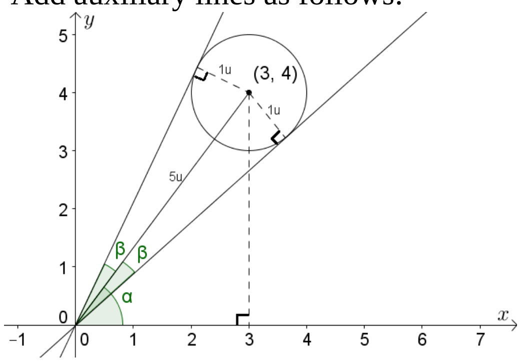
$$\Rightarrow TB=\frac{20\sin 18°}{\sin 13°}.$$

Using sine rule on triangle ΔBTP (where point P is the base of the tower),

$$\frac{TB}{\sin 90°}=\frac{h}{\sin 31°}$$

$$\Rightarrow h=TB\sin 31°.$$
$$=\frac{20\sin 18°}{\sin 13°}\times\sin 31°$$
$$\approx 14.2 \text{ metres}.$$

10. Add auxiliary lines as follows:



10. **(Continued)**

By Pythagoras theorem, distance from origin to centre of circle = 5 units.

Therefore required answer

$$=\alpha\pm\beta = \tan^{-1}\left(\frac{4}{3}\right)\pm\sin^{-1}\left(\frac{1}{5}\right)$$

Smallest acute angle = 0.726 rad.
Largest acute angle = 1.129 rad.

Challenging Questions

1. Let $y=\cos^{-1}x$ so that using the identity proved in BMQ Q1(a),

$$x=\cos y=\sin\left(\frac{\pi}{2}-y\right)$$

$$\Rightarrow \sin^{-1}x=\frac{\pi}{2}-y.$$

Therefore,

$$\text{LHS}=\sin^{-1}x+\cos^{-1}x$$
$$=\left(\frac{\pi}{2}-y\right)+y$$
$$=\text{RHS}.$$

2. Squaring both sides gives

$$\sin^2 x+2\sin x\cos x+\cos^2 x=1$$
$$\Rightarrow 2\sin x\cos x=0$$
$$\Rightarrow \sin x=0 \text{ or } \cos x=0$$
$$\Rightarrow x=0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ \text{or } 2\pi.$$

However, we have introduced incorrect solutions after squaring both sides, and thus we need to eliminate these incorrect solutions:

x	$\sin x+\cos x$	Correct?
0	1	Yes
$\pi/2$	1	Yes
π	− 1	No
$3\pi/2$	− 1	No
2π	1	Yes

Therefore, $x=0,\ \frac{\pi}{2}$ or 2π.

Remark: Alternative method is to use R-formula.