2021 Year 6 H2 Math Preliminary Paper 2: Solutions with Comments

Section A: Pure Mathematics [40 marks]

1 Functions f and g are defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^{(x-1)^2}, \quad x \in \mathbb{R},$$

$$\mathrm{g} : x \mapsto \frac{1}{2-x}, \quad x \in \mathbb{R}, \quad 1 \le x < 2.$$

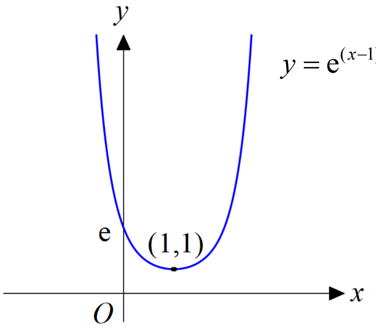
(i) Sketch the graph of $y = \mathrm{f}(x)$. [1]

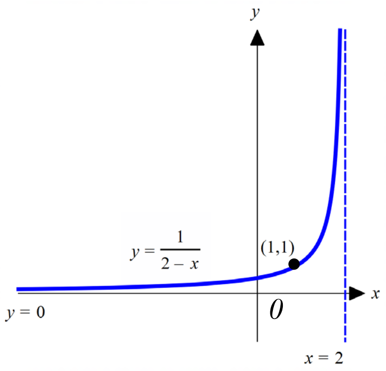
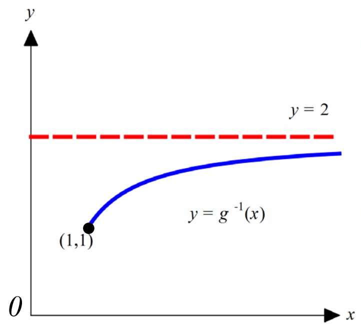
(ii) If the domain of f is restricted to $x \ge k$, state with a reason the least value of k for which the function f^{-1} exists. [2]

In the rest of the question, the domain of f is $x \ge k$, using the value of k found in part **(ii)**.

(iii) Find $\mathrm{g}^{-1}(x)$ and show that the composite function $\mathrm{g}^{-1}\mathrm{f}^{-1}$ exists. [4]

(iv) Find the range of $\mathrm{g}^{-1}\mathrm{f}^{-1}$. [1]

Solutions		**Comments**
(i) **[1]**		The graph and its properties can be easily obtained from the GC. However, a good number of students did not label either or both the y-intercept and minimum point. Students should also note that the graph is symmetrical about the line $x = 1$.
(ii) **[2]**	For f^{-1} to exist, f must be a one-one function. Least value of $k = 1$.	Many students elaborated the horizontal line test instead of stating that the condition is for f to be a 1-1 function. In this case, students have to take note of the precise phrasing "**every** horizontal line $y = k,\ k \in [1, \infty)$, cuts the graph of **f** at **one and only one point**."

(iii) **[4]**	Let $y=\dfrac{1}{2-x}$ $2-x=\dfrac{1}{y}$ $x=2-\dfrac{1}{y}$ $\mathrm{g}^{-1}(x)=2-\dfrac{1}{x}$ For $\mathrm{g}^{-1}\mathrm{f}^{-1}$ to exist, $\mathrm{R}_{\mathrm{f}^{-1}}\subseteq \mathrm{D}_{\mathrm{g}^{-1}}$. $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=[1,\infty)$ $\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{R}_{\mathrm{g}}=[1,\infty)$ Since $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{g}^{-1}}$, $\therefore$ $\mathrm{g}^{-1}\mathrm{f}^{-1}$ exists. 	Most students managed to find $\mathrm{g}^{-1}(x)$ correctly, with a very small number leaving their answers as $\mathrm{g}^{-1}(x)=2-\dfrac{1}{y}$, which is incorrect. Most students could remember the condition to check for $\mathrm{g}^{-1}\mathrm{f}^{-1}$ to exist. Often mistakes were made in either finding $\mathrm{D}_{\mathrm{g}^{-1}}$ or keeping $\mathrm{D}_{\mathrm{f}}=\mathbb{R}$ (unrestricted).
(iv) **[1]**	$[1,\infty)\xrightarrow{\mathrm{f}^{-1}}[1,\infty)\xrightarrow{\mathrm{g}^{-1}}[1,2)$ $\mathrm{R}_{\mathrm{g}^{-1}\mathrm{f}^{-1}}=[1,2)$ Note: $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=[1,\infty)$, $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=[1,\infty)$  **OR** Since $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{g}^{-1}}$, $\mathrm{R}_{\mathrm{g}^{-1}\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{g}^{-1}}=\mathrm{D}_{\mathrm{g}}=[1,2)$	Most students who used the arrow diagram successfully found $\mathrm{R}_{\mathrm{g}^{-1}\mathrm{f}^{-1}}$. Students have to note the order of the functions involved and the horizontal asymptote in the graph of g^{-1}. A number of students also managed to recognize the relationship in alternative method, thus were able to state the answer correctly.

2 **(a)** Three consecutive terms of a decreasing geometric progression has a product of 5832. If the first number is reduced by 24, these 3 numbers in the same order will form an arithmetic progression. Find the three terms of the geometric progression. [5]

(b) The fractal called Sierpiński Triangle is depicted below. Fig. 1 shows an equilateral triangle of side 1. In stage 1, the triangle in Fig. 1 is divided into four smaller *identical* equilateral triangles and the middle triangle is removed to give the triangle shown in Fig. 2. In stage 2, the remaining three equilateral triangles in Fig. 2 are each divided into four smaller *identical* equilateral triangles and the middle triangles are removed to give the triangle shown in Fig. 3 and the process continues.

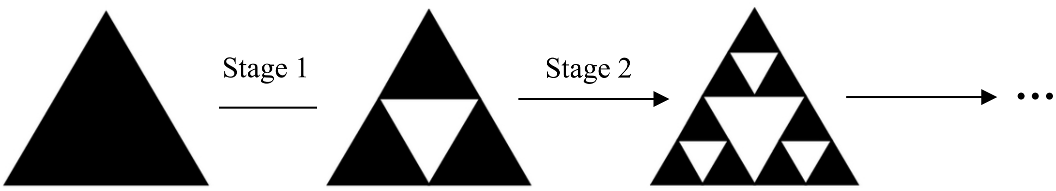


Fig. 1 **Fig. 2** **Fig. 3**

Let T_n be the total area of triangles removed after n stages of the process.

(i) Show that $T_1 = \frac{\sqrt{3}}{16}$. [1]

(ii) Find T_{10}. [3]

(iii) State the exact value of $\lim_{n\to\infty} T_n$. [1]

Solution		Comments
(a) **[5]**	Let the 3 numbers be $\frac{x}{r}$, x and xr, where x is the middle term and r is the common ratio. $\frac{x}{r}(x)(xr) = 5832$ $x^3 = 5832$ $x = 18$ If the first number is reduced by 24, it is now $\frac{x}{r} - 24$. Since the 3 numbers now form an AP, $\left(\frac{x}{r} - 24\right) - x = x - xr$ Substitute $x = 18$ into the above equation,	Most students were able to solve this part successfully, but the efficiency depends on what they wrote as the first 3 terms. Quite many let the 3 numbers be xr^n, xr^{n+1}, xr^{n+2}, and could not solve due to the extra variable n. They were able to get $xr^{n+1} = 18$ but

	$\left(\dfrac{18}{r}-24\right)-18=18-18r$ $3r^2-10r+3=0$ $(3r-1)(r-3)=0$ $\therefore r=\dfrac{1}{3}$ or $r=3$ (rejected $\because$ it is a decreasing GP, i.e. $0<r<1$) Thus, the original 3 numbers are 54, 18, 6.	could not solve it due to failure to recognise this as the second term.
(b) **(i)** **[1]**	Let A be the original area of the triangle in Fig. 1. $A=\dfrac{1}{2}(1)(1)\sin 60°$ $=\dfrac{\sqrt{3}}{4}$ $T_1=\dfrac{1}{4}A=\dfrac{\sqrt{3}}{16}$ (shown)	Students who could not solve this part generally were not able to sieve out the information that there are 4 smaller identical triangles in Figure 2. Many students resorted to finding the height of the triangle in Figure 1 to find the area, when a simple application of the formula suffice.
(ii) **[3]**	Area of triangle removed in stage $1=T_1$ Area of triangles removed in stage $2=\dfrac{3}{4}T_1$ Area of triangles removed in stage $3=\left(\dfrac{3}{4}\right)^2 T_1$ $\vdots$ Area of triangles removed in stage $n=\left(\dfrac{3}{4}\right)^{n-1} T_1$	Students who were able to identify a GP generally managed to get the answer. Many students sort of recognise that there is some addition of the area of triangles, and tried to use the sum of a GP

	Total area of triangles removed after 10 stages, T_{10} $= T_1 + \frac{3}{4}T_1 + \left(\frac{3}{4}\right)^2 T_1 + \cdots + \left(\frac{3}{4}\right)^9 T_1$ $= \dfrac{T_1\left(1-\left(\frac{3}{4}\right)^{10}\right)}{1-\frac{3}{4}}$ $= 0.409$ (3 s.f.)	formula. Not many left the part blank.
(iii) **[1]**	$\lim_{n\to\infty} T_n = \dfrac{T_1}{1-\frac{3}{4}} = \dfrac{\sqrt{3}}{4}$ **OR** $\lim_{n\to\infty} T_n = A = \dfrac{\sqrt{3}}{4}$	It is good to see that many students, while not able to solve the previous part, make an attempt at part (iii).

3 It is given that $\ln y = \sqrt{1+8e^x}$.

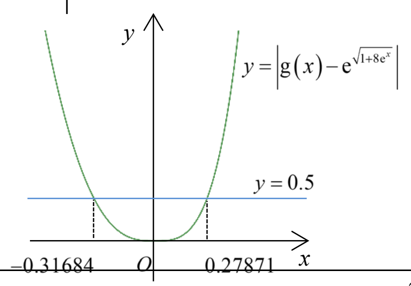
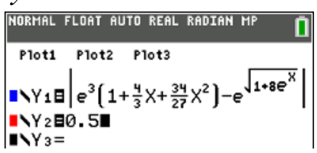
(i) Show that $(\ln y)\frac{dy}{dx} = 4ye^x$. [1]

(ii) Show that the value of $\frac{d^2y}{dx^2}$ when $x = 0$ is $\frac{68}{27}e^3$. [4]

(iii) Hence find the Maclaurin series for $e^{\sqrt{1+8e^x}}$ up to and including the term in x^2. [2]

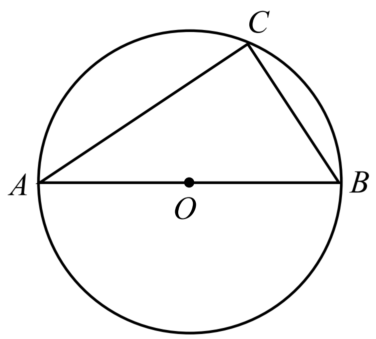
(iv) Denoting the answer found in part **(iii)** as $g(x)$, find the set of values of x for which $g(x)$ is within ± 0.5 of the value of $e^{\sqrt{1+8e^x}}$. [3]

Solution		Comments
(i) **[1]**	$\ln y = \sqrt{1+8e^x}$ Differentiate w.r.t x, $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{1+8e^x}}\left(8e^x\right)$ $\frac{1}{y}\frac{dy}{dx} = \frac{4e^x}{\ln y}$ $(\ln y)\frac{dy}{dx} = 4ye^x$ (shown)	As this is a show question, clear working is vital. Students who did this question in another way should consider these 2 solutions provided as they are the most efficient.

	OR $\ln y=\sqrt{1+8\mathrm{e}^x}$ $(\ln y)^2=1+8\mathrm{e}^x$ Differentiate w.r.t x, $2(\ln y)\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=8\mathrm{e}^x$ $(\ln y)\frac{\mathrm{d}y}{\mathrm{d}x}=4y\mathrm{e}^x$ (shown)	
(ii) **[4]**	$(\ln y)\frac{\mathrm{d}y}{\mathrm{d}x}=4y\mathrm{e}^x$ Differentiate w.r.t x, $(\ln y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y\mathrm{e}^x+4\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}$ When $x=0$, $\mathrm{e}^x=1$. $\ln y=3\Rightarrow y=\mathrm{e}^3$ $3\frac{\mathrm{d}y}{\mathrm{d}x}=4\mathrm{e}^3\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4}{3}\mathrm{e}^3$ $3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{\mathrm{e}^3}\left(\frac{4}{3}\mathrm{e}^3\right)^2=4\mathrm{e}^3+4\left(\frac{4}{3}\mathrm{e}^3\right)$ $3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{16}{9}\mathrm{e}^3=\frac{28}{3}\mathrm{e}^3$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{3}\left(\frac{28}{3}\mathrm{e}^3-\frac{16}{9}\mathrm{e}^3\right)=\frac{68}{27}\mathrm{e}^3$ (shown)	One key to success in Maclaurin Series question is the method used to find the 2nd and 3rd derivatives. So always pay attention to the method used. The method shown here is the most efficient - **differentiate what you have been asked to show**. You are strongly encouraged to use this method and not any others which are inefficient.
(iii) **[2]**	$\ln y=\sqrt{1+8\mathrm{e}^x}\Leftrightarrow y=\mathrm{e}^{\sqrt{1+8\mathrm{e}^x}}$ By Maclaurin Theorem, $\mathrm{e}^{\sqrt{1+8\mathrm{e}^x}}=\mathrm{e}^3+\frac{4}{3}\mathrm{e}^3x+\frac{68}{27}\mathrm{e}^3\frac{x^2}{2!}+\ldots$ $=\mathrm{e}^3\left(1+\frac{4}{3}x+\frac{34}{27}x^2\right)+\ldots$	
(iv) **[3]**	Let $\mathrm{g}(x)=\mathrm{e}^3\left(1+\frac{4}{3}x+\frac{34}{27}x^2\right)$. $\left\|\mathrm{g}(x)-\mathrm{e}^{\sqrt{1+8\mathrm{e}^x}}\right\|<0.5$ 	To solve the inequality, (i) sketch the graphs of $y=\left\|\mathrm{g}(x)-\mathrm{e}^{\sqrt{1+8\mathrm{e}^x}}\right\|$ and $y=0.5$ 

	$x \in (-0.317, 0.279)$ (3 s.f.)	(ii) find the points of intersection (iii) write down the range of values that satisfy the inequality by looking at the graphs

4 **(a)** **(i)**



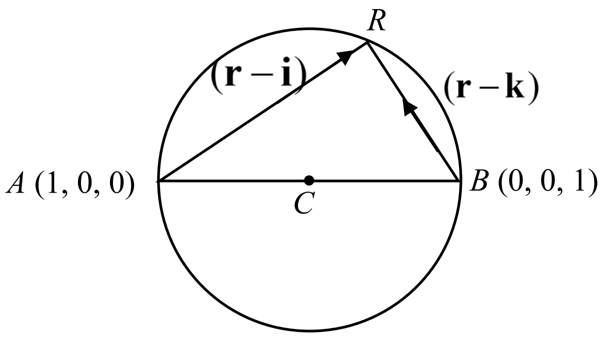
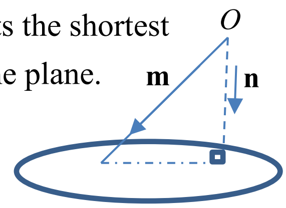
Referred to the origin O, points A, B and C have position vectors **a**, **b** and **c** respectively. The three points lie on a circle with centre O and diameter AB (see diagram).
Using a suitable scalar product, show that the angle ACB is $90°$. [4]

(ii) The variable vector **r** satisfies the equation $(\mathbf{r}-\mathbf{i})\cdot(\mathbf{r}-\mathbf{k})=0$. Describe the set of vectors **r** geometrically. [2]

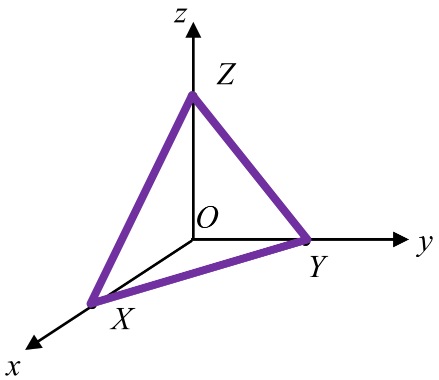
(b) **(i)** The variable vector **r** satisfies the equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}$, where **m** and **n** are constant vectors. Describe the set of vectors **r** geometrically.
Give the geometrical meaning of $|\mathbf{m}\cdot\mathbf{n}|$ if **n** is a unit vector. [2]

(ii) The plane π passes through the points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ where x, y and z are non-zero constants. It is given that d is the perpendicular distance from the origin to π. Show, by finding the normal of π, or otherwise, that $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{1}{d^2}$. [4]

Solution		**Comments**
(a) **(i)** **[4]**	$\overrightarrow{AC}\cdot\overrightarrow{BC}=(\overrightarrow{OC}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OB})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}+\mathbf{a})$ (since $\mathbf{b}=-\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{a}$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ $=\lvert\mathbf{c}\rvert^2-\lvert\mathbf{a}\rvert^2$ (since $\mathbf{c}\cdot\mathbf{c}=\lvert\mathbf{c}\rvert^2, \mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$) $=0$ $\left(\text{since } \lvert\mathbf{c}\rvert=\lvert\mathbf{a}\rvert=\text{radius}\right)$ **OR**	The properties of the circle should be used to show the result. Since A, B and C lies on the circle, we should note that $\lvert\mathbf{a}\rvert=\lvert\mathbf{b}\rvert=\lvert\mathbf{c}\rvert$ is the radius of circle. Also AB is the diameter of the circle, therefore, $\mathbf{b}=-\mathbf{a}$ since they are equal in length but opposite in direction.

	$\overrightarrow{AC}\cdot\overrightarrow{BC}=(\overrightarrow{OC}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OB})$ $=(\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}$ $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot(-\mathbf{a})+\mathbf{a}\cdot(-\mathbf{a})$ (since $\mathbf{b}=-\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ (since $\mathbf{a}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$) $=\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$ $=\lvert\mathbf{c}\rvert^2-\lvert\mathbf{a}\rvert^2$ (since $\mathbf{c}\cdot\mathbf{c}=\lvert\mathbf{c}\rvert^2, \mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$) $=0$ $\left(\text{since } \lvert\mathbf{c}\rvert=\lvert\mathbf{a}\rvert=\text{radius}\right)$ Since $\overrightarrow{AC}\cdot\overrightarrow{BC}=0$, $\therefore \angle ACB=90°$.	
(ii) **[2]**	Let points (1, 0, 0) and (0, 0, 1) be A and B respectively Let R be the point with position vector $\mathbf{r}$ Since $(\mathbf{r}-\mathbf{i})\cdot(\mathbf{r}-\mathbf{k})=0$, ABR is a right-angled triangle. Therefore R lies on a sphere with AB as the diameter of the sphere.  Length of line segment joining $A(1, 0, 0)$ and $B(0, 0, 1)$ is $\sqrt{2}$ Midpoint of (1, 0, 0) and (0, 0, 1) is $\left(\frac{1}{2},0,\frac{1}{2}\right)$ which is C, the centre of the sphere. Set of vectors $\mathbf{r}$ consists of position vectors of points on a sphere with diameter $\sqrt{2}$ (OR radius $\frac{\sqrt{2}}{2}$) and centre $\left(\frac{1}{2},0,\frac{1}{2}\right)$.	The properties of the circle/sphere should be used to describe the set of vectors $\mathbf{r}$ geometrically. You should consider **(a)(i)** is related to **(a)(ii)**. Few students have managed to describe it correctly.
(b) **(i)** **[2]**	Set of vectors $\mathbf{r}$ consists of position vectors of points on a plane that contains the point M with position vector $\mathbf{m}$ and is perpendicular to the vector $\mathbf{n}$. If $\mathbf{n}$ is a unit vector, then $\lvert\mathbf{m}\cdot\mathbf{n}\rvert$ represents the shortest (perpendicular) distance from origin to the plane. 	Note that $\mathbf{m}\cdot\mathbf{n}$ is a fixed scalar value, so $\mathbf{r}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}$ defines a plane. Note that $\lvert\mathbf{m}\cdot\mathbf{n}\rvert$ is the projection of $\mathbf{m}$ onto $\mathbf{n}$ since $\mathbf{n}$ is a unit vector

(ii) **[4]**	**<u>Method 1</u>** Let X, Y and Z be points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ respectively, then $\overrightarrow{OX}=\begin{pmatrix}x\\0\\0\end{pmatrix},\overrightarrow{OY}=\begin{pmatrix}0\\y\\0\end{pmatrix},\overrightarrow{OZ}=\begin{pmatrix}0\\0\\z\end{pmatrix}$ A normal to the plane π $=\overrightarrow{XY}\times\overrightarrow{XZ}=\begin{pmatrix}-x\\y\\0\end{pmatrix}\times\begin{pmatrix}-x\\0\\z\end{pmatrix}=\begin{pmatrix}yz\\xz\\xy\end{pmatrix}$ $\therefore d=\left\|\overrightarrow{OX}\cdot\dfrac{\begin{pmatrix}yz\\xz\\xy\end{pmatrix}}{\sqrt{y^2z^2+x^2z^2+x^2y^2}}\right\|=\left\|\dfrac{xyz}{\sqrt{y^2z^2+x^2z^2+x^2y^2}}\right\|$ $d^2=\dfrac{x^2y^2z^2}{y^2z^2+x^2z^2+x^2y^2}$ $\dfrac{y^2z^2+x^2z^2+x^2y^2}{x^2y^2z^2}=\dfrac{1}{d^2}$ $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{1}{d^2}$ (shown) **<u>Method 2</u>** Let the normal vector to the plane be $\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}$ where $\sqrt{n_x^2+n_y^2+n_z^2}=1$ (unit vector) $d=\left\|\begin{pmatrix}x\\0\\0\end{pmatrix}\cdot\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}\right\|\Rightarrow d=\|xn_X\|\Rightarrow\|n_x\|=\dfrac{d}{\|x\|}$ Similarly, $d=\left\|\begin{pmatrix}0\\y\\0\end{pmatrix}\cdot\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}\right\|\Rightarrow\|n_y\|=\dfrac{d}{\|y\|}$ and, $d=\left\|\begin{pmatrix}0\\0\\z\end{pmatrix}\cdot\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix}\right\|\Rightarrow\|n_z\|=\dfrac{d}{\|z\|}$	**Method 1** first find the normal to the plane and make use length of projection to find d and hence the relationship Some have also tried to find the equation of the plane and use the equation to relate to d which will yield the same result. **Method 2** first make use of the 3 points to find the normal to the plane which is a ***unit vector*** and then make use of the fact that the magnitude of the unit vector is 1 to find the relationship.

	$\sqrt{n_x^2+n_y^2+n_z^2}=1$ Therefore $\Rightarrow \left(\frac{d}{\lvert x\rvert}\right)^2+\left(\frac{d}{\lvert y\rvert}\right)^2+\left(\frac{d}{\lvert z\rvert}\right)^2=1$ $\Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2}$ (Shown) **Method 3 (Geometry)** Let X, Y and Z be points with position vectors $x\mathbf{i}, y\mathbf{j}$ and $z\mathbf{k}$ respectively, then  Using the base OXY, volume of the tetrahedral $OXYZ$ $=\frac{1}{3}(Area \text{ of triangle } OXY)(OZ)$ $=\frac{1}{3}\left(\frac{1}{2}[OX][OY]\right)(OZ)$ $=\frac{1}{6}\lvert xyz\rvert$ Using the base XYZ, volume of the tetrahedral $OXYZ$	**Method 3** makes use of 2 different ways to find the volume of tetrahedral using different base area and then compare these volumes to find the relationship.

	$=\frac{1}{3}\left(Area \text{ of triangle } XYZ\right)(d)$ $=\frac{1}{3}\left(\frac{1}{2}\left\lvert\overrightarrow{XY}\times\overrightarrow{XZ}\right\rvert\right)(d)$ $=\frac{1}{3}\left(\frac{1}{2}\left\lvert\begin{pmatrix}-x\\y\\0\end{pmatrix}\times\begin{pmatrix}-x\\0\\z\end{pmatrix}\right\rvert\right)(d),\quad \overrightarrow{XY}=\begin{pmatrix}-x\\y\\0\end{pmatrix} \text{ and } \overrightarrow{XZ}=\begin{pmatrix}-x\\0\\z\end{pmatrix}$ $=\frac{1}{3}\left(\frac{1}{2}\left\lvert\begin{pmatrix}yz\\xz\\xy\end{pmatrix}\right\rvert\right)(d)$ $=\frac{d}{6}\sqrt{y^2z^2+x^2z^2+x^2y^2}$ Hence $\frac{d}{6}\sqrt{y^2z^2+x^2z^2+x^2y^2}=\frac{1}{6}\lvert xyz\rvert$ $\Rightarrow d^2\left(y^2z^2+x^2z^2+x^2y^2\right)=x^2y^2z^2$ $\Rightarrow \frac{y^2z^2+x^2z^2+x^2y^2}{x^2y^2z^2}=\frac{1}{d^2}$ $\Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{d^2}$	

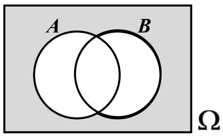
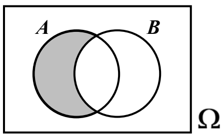
Section B: Probability and Statistics [60 marks]

5 For events A and B it is given that $\mathrm{P}(A)=0.3$, $\mathrm{P}(B|A)=0.4$ and $\mathrm{P}(A'\cap B')=0.15$. Find

(i) $\mathrm{P}(A\cup B)$, [1]

(ii) $\mathrm{P}(B)$, [3]

(iii) $\mathrm{P}(A|B')$. [2]

	Solution	**Comments**
(i) **[1]**	$\mathrm{P}(A\cup B)$ $=1-\mathrm{P}(A'\cap B')$ $=1-0.15$ $=0.85$  $A'\cap B'$	Majority of students did well for this part. The most complicated looking piece of information given in the question is $\mathrm{P}(A'\cap B')$. Either you draw a venn diagram for it and then you can clearly see what you need, or you use the property that $\mathrm{P}(A'\cap B')=\mathrm{P}(A\cup B)'$
(ii) **[3]**	$\mathrm{P}(B\vert A)=\dfrac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)}$ $0.4=\dfrac{\mathrm{P}(B\cap A)}{0.3}$ $\mathrm{P}(B\cap A)=0.12$ $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $0.85=0.3+\mathrm{P}(B)-0.12$ $\mathrm{P}(B)=0.67$	Most student could handle this part. The main idea is to extract out $\mathrm{P}(B)$ using $\mathrm{P}(A\cup B)$ from (i) and $\mathrm{P}(A\cap B)$ which can be found from the given conditional probability. For questions which are more than 1 mark, it is imperative that students show some kind of method rather than just computation of numbers, in case the answer is wrong, they may obtain some method marks.
(iii) **[2]**	$\mathrm{P}(A\vert B')$ $=\dfrac{\mathrm{P}(A\cap B')}{\mathrm{P}(B')}$ $=\dfrac{\mathrm{P}(A)-\mathrm{P}(A\cap B)}{1-\mathrm{P}(B)}$ $=\dfrac{0.3-0.12}{1-0.67}$ $=\dfrac{6}{11}$  $A\cap B'$	This part was generally well done, except for carryover of wrong answer from (ii), or, having obtained 0.67 for (ii) but forgetting to take the complement for the denominator. It is heartening to note most students could recognize the property used in the numerator, from section 2.2 (vi) in the lecture notes. Alternatively, drawing a venn diagram would help.

6 The recruitment manager of the private car hire company, I-ber, claims that the mean weekly earnings of a full-time driver is \$980. The managing director suspects that the mean weekly earnings is less than \$980 and he instructs the recruitment manager to carry out a hypothesis test on a sample of drivers. It is given that the population standard deviation of the weekly earnings is \$88.

(i) State suitable hypotheses for the test, defining any symbols that you use. [2]

The recruitment manager takes a random sample of 10 drivers. He finds that the weekly earnings in dollars, are as follows.

942 950 905 1003 883 987 924 920 913 968

(ii) Find the mean weekly earnings of the sample of these 10 drivers. Carry out the test, at 5% level of significance, for the recruitment manager. Give your conclusion in context and state a necessary assumption for the test to be valid. [5]

(iii) Find the smallest level of significance at which the test would result in rejection of the null hypothesis, giving your answer correct to 1 decimal place. [1]

Solution		**Comments**
(i) **[2]**	Null hypothesis, $H_0: \mu = 980$ Alternative hypothesis, $H_1: \mu < 980$ where μ is the population mean weekly salary.	This is a similar question to CT, the word "population" should be there.
(ii) **[5]**	Let X be the weekly earning of an I-ber driver (in \$). Using GC, $\bar{x} = 939.5$ Perform a 1-tailed test at 5% significance level. Under H_0, $\bar{X} \sim \mathrm{N}\left(980, \frac{88^2}{10}\right)$ p-value $= 0.0728 > 0.05$, hence we do not reject H_0, and conclude that, based on the test carried out by the recruitment manager, there is insufficient evidence for the managing director to conclude at 5% level of significance that the mean weekly earnings of a driver is less than \$980. Assumption: Assume that the weekly earnings of the I-ber drivers are normally distributed.	Defining X should be a default first step, if it is not defined in the question. **Please follow the school prescribed presentations provided in various softcopies (including this one) to avoid losing marks unnecessarily.** The conclusion was sometimes not done properly, for example saying "it is not \$980" instead of "less than \$980"

		There was also some confusion between the claims of the recruitment manager vs the managing director Finally do also take note that we either reject H_0 (and thus accept H_1) or do not reject H_0. We **do not** conclude that we accept H_0
(iii) **[1]**	To reject H_0, smallest level of significance = 7.3% (1 d.p)	Students performed below expectations for this part. It should be noted that the level of significance is usually given in percentage form and thus it should be clear how to proceed.

7 **In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

A company sells hand sanitiser in bottles of two sizes – small and large. The amounts, in ml, of hand sanitiser in the small and large bottles, are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard deviation
Small bottles	108	5
Large bottles	510	σ

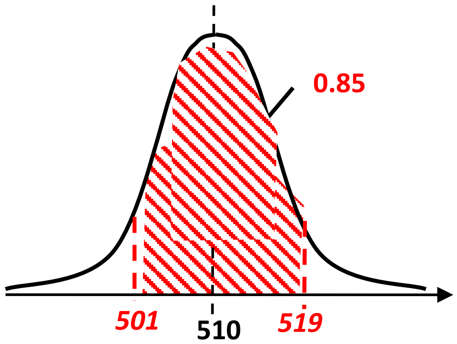
(i) Find the probability that the amount of hand sanitiser in a randomly chosen small bottle is less than 100 ml. [1]

(ii) During a quality control check on a batch of small bottles of hand sanitiser, 100 small bottles are randomly chosen to be inspected by an officer one at a time. Once he finds five bottles, each with amount of hand sanitiser less than 100 ml, that batch will be rejected. Find the probability that he had to check through all 100 bottles to reject that batch. [2]

(iii) Given that the amount of hand sanitiser in 85% of the large bottles lie within 9 ml of the mean, find σ. [3]

(iv) Given instead that $\sigma = 6$, find the probability that the amount of hand sanitiser in a randomly chosen large bottle is less than five times the amount of hand sanitiser in a randomly chosen small bottle. [3]

Solution		**Comments**
(i) **[1]**	Let X and Y be the amount, in ml, of hand sanitiser in a small and large bottle respectively. Then $X \sim \mathrm{N}\left(108, 5^2\right)$ and $Y \sim \mathrm{N}\left(510, \sigma^2\right)$ $\mathrm{P}\left(X < 100\right) = 0.054799 = 0.0548$ (3 s.f.)	Majority did this part well, but some still keyed in wrongly as normalcdf(-E99,100, 108, **25**) instead of normalcdf(-E99,100, 108, **5**)
(ii) **[2]**	Required probability $= {}^{99}C_4 \left(0.054799\right)^4 \left(1-0.054799\right)^{95} \left(0.054799\right)$ $= 0.00880$ (3 s.f.)	There are 4 bottles from the first 99 with the criteria of less than 100ml, and the 5th is the 100th being examined, so some forgot to multiply by 0.054799 for this last bottle. Remember to use 5.s.f and to round your answers properly.

(iii) **[3]**	$\mathrm{P}\left(\lvert Y-510\rvert<9\right)=0.85$ $\mathrm{P}\left(\frac{\lvert Y-510\rvert}{\sigma}<\frac{9}{\sigma}\right)=0.85$ $\mathrm{P}\left(\lvert Z\rvert<\frac{9}{\sigma}\right)=0.85$ From G.C, $\frac{9}{\sigma}=1.4395$ $\sigma=6.25$ (3 s.f.) **OR**  $\mathrm{P}(501<Y<519)=0.85$ $\mathrm{P}\left(\frac{501-510}{\sigma}<\frac{Y-510}{\sigma}<\frac{519-510}{\sigma}\right)=0.85$ $\mathrm{P}\left(-\frac{9}{\sigma}<Z<\frac{9}{\sigma}\right)=0.85$ From G.C, $\frac{9}{\sigma}=1.4395$ $\sigma=6.25$ (3 s.f.)	Majority did this part well, but there were some who mistakenly thought that it is from 505.5 to 514.5 (an interval of 9 ml) which is a misinterpretation of "within 9ml of the mean". Some also thought that the standardization is $\frac{Y-510}{\sigma^2}$, where it should be σ, and not σ^2. Some used invNorm(0.85,0,1, LEFT) instead of invNorm(0.85,0,1, CENTER) which led to a wrong value. Some who did $\mathrm{P}(Y<501)=\frac{1-0.85}{2}=0.075$ were able to successfully get the answer. Those who tried a graphical method by plotting Y_1= normalcdf(501,519,510,X) and Y_2=0.85 were mostly successful. Others who mistakenly thought that σ is an integer, used a tabular method, and were not given full credit as this answer is not to 3.s.f.
(iv) **[3]**	$Y-5X\sim\mathrm{N}\left(510-5(108),6^2+5^2\left(5^2\right)\right)$ i.e $Y-5X\sim\mathrm{N}(-30,661)$ $\mathrm{P}(Y-5X<0)=0.878$ (3 s.f.)	Majority did well on this, but quite some made calculation mistakes for the $\mathrm{Var}(Y-5X)$: swopped the standard deviations for X and Y, forgot the square for the scaling factor 5, used subtraction instead of addition, etc. Some who successfully obtained the correct distribution also made the unfortunate error: used normalcdf(-E99, 0,-30,**661**) instead of $\sqrt{661}$.

8 This question is about arrangements of all eight letters in the word IMMUNITY.

(i) Show that the number of different arrangements of the eight letters that can be made is 10080. [1]

(ii) Find the number of different arrangements that can be made with no two vowels next to each other. [3]

One of the 10080 arrangements in part **(i)** is randomly chosen.
Let A denote the event that the two I's are next to each other and let B denote the event that the two M's are next to each other.

(iii) Determine, with a reason, whether A and B are
(a) mutually exclusive, [1]

(b) independent. [3]

(iv) Find the probability that the chosen arrangement contains no two adjacent letters that are the same. [4]

	Solution	**Comments**
(i) **[1]**	The number of different arrangements $=\dfrac{8!}{2!2!}=10080$ (shown)	This part was expectedly well-done.
(ii) **[3]**	The consonants are M, M, N, T, Y and the vowels are I, I, U. Number of ways to arrange the consonants $=\dfrac{5!}{2!}=60$ $\uparrow\ C_1\ \uparrow\ C_2\ \uparrow\ C_3\ \uparrow\ C_4\ \uparrow\ C_5\ \uparrow$ Number of ways to choose 3 slots to insert the vowels $={}^6C_3=20$ Number of ways to arrange the vowels $=\dfrac{3!}{2!}=3$ $\therefore$ The number of different arrangements $=60\times 20\times 3=3600$	You must know what are vowels (or consonants). Some candidates were confused about these.
	<ins>**Method 2**</ins> (complementary approach) : Case 1 : 3 vowels are together (as a unit) $\boxed{\text{I I U}}$ M M N T Y Number of arrangements = $\dfrac{6!}{2!}\times 3$ = 1080.	We recommend the first approach. Methods 2 and 3 are for your reference.

Case 2 : Group 2 ‘I’s together (as a unit)

M M N T Y $\boxed{\text{I I}}$ U

Number of arrangements = $\dfrac{7!}{2!}$ = 2520.

Case 3 : Group one I and one U together (as a unit)

M M N T Y $\boxed{\text{I U}}$ I

Number of arrangements = $\dfrac{7!}{2!}\times 2 = 5040$.

Hence, the required number of different arrangements
= 10080 – 2520 – 5040 + 1080
= 3600.

A gentle reminder that it is your responsibility to briefly describe the different cases.

For Method 2, do note that the 3 vowels being together also happens in case 2 and case 3.

<u>Method 3</u> (complementary approach) :

Case 1 : 3 vowels are together (as a unit)

$\boxed{\text{I I U}}$ M M N T Y

Number of arrangements = $\dfrac{6!}{2!}\times 3$ = 1080.

Case 2 : Group 2 ‘I’s together (as a unit), and U is another unit not adjacent to I-I

$\boxed{\text{I I}}$ $\boxed{\text{U}}$

M M N T Y
↑ ↑ ↑ ↑ ↑ ↑

Number of ways to arrange the 5 consonants = $\dfrac{5!}{2!}$ = 60.

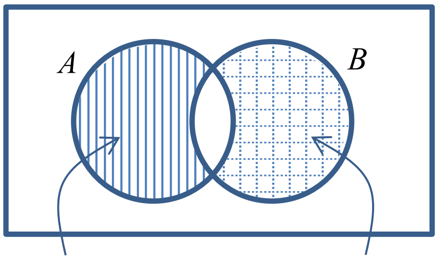
Numbers of ways to insert 2 units = ${}^{6}C_{2}\times 2 = 30$.

Number of different arrangements = 60 × 30 = 1800.

For Method 3, do note that in case 2 and case 3, special care is taken to make sure the 3 vowels don’t come together.

	Case 3 : Group one I and U together (as a unit), and the other I is another unit not adjacent to I-U unit $\boxed{\text{I U}}$ $\boxed{\text{I}}$ M M N T Y ↑ ↑ ↑ ↑ ↑ ↑ Number of ways to arrange the 5 consonants = $\dfrac{5!}{2!}$ = 60. Numbers of ways to insert 2 units = ${}^6C_2 \times 2 \times 2 = 60$. Number of different arrangements = 60 × 60 = 3600. Hence, the required number of different arrangements = 10080 – 1080 – 1800 – 3600 = 3600.	
(iii) (a) [1]	Events A and B are not mutually exclusive. This is because both events can occur at the same time, for example arrangements such as MMUNIITY, IINTUMMY and YUMMIINT.	The most direct approach is to give an explicit example of the existence of such an arrangement. Another viable approach is to find P($A \cap B$) and show that it is non-zero.
(iii) (b) [3]	$\mathrm{P}(A)=\mathrm{P}(B)=\dfrac{7!}{2!}\div 10080=\dfrac{1}{4}$, so $\mathrm{P}(A)\times\mathrm{P}(B)=\dfrac{1}{16}$ $\mathrm{P}(A\cap B)=\dfrac{6!}{10080}=\dfrac{1}{14}$ $\mathrm{P}(A\cap B)\neq \mathrm{P}(A)\times\mathrm{P}(B)$ $\Rightarrow$ Events A and B are not independent.	You **<u>shouldn't</u>** be trying to explain your way through words here. You **<u>ought to be</u>** making attempt to work out actual probabilities to establish <u>one</u> of the following : $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$ $\mathrm{P}(A\mid B)=\mathrm{P}(A)$ $\mathrm{P}(B\mid A)=\mathrm{P}(B)$
(iv) [4]	Required probability $=\mathrm{P}(A'\cap B')$ $=1-\mathrm{P}(A\cup B)$ $=1-\left[\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)\right]$ $=1-\left[\dfrac{1}{4}+\dfrac{1}{4}-\dfrac{1}{14}\right]$ $=\dfrac{4}{7}$	There are plenty instances of candidates getting "carried away" and find <u>number of arrangements</u> instead of <u>probabilities</u>.

Method 2 (Venn Diagram) :



$\mathrm{P}(A)-\mathrm{P}(A\cap B)=\frac{1}{4}-\frac{1}{14}=\frac{5}{28}$ $\qquad$ $\mathrm{P}(B)-\mathrm{P}(A\cap B)=\frac{1}{4}-\frac{1}{14}=\frac{5}{28}$

Required probability $= 1-\frac{5}{28}-\frac{5}{28}-\frac{1}{14}=\frac{4}{7}$.

Method 2. Visualizing with the aid of Venn Diagram can be useful.

Method 3 (complementary approach) :

Case 1 : Group 2 'M's together (as a unit), and the two 'I's are to be separated from each other.

I $\quad$ I

$\uparrow$ $\boxed{\text{MM}}$ $\uparrow$ U $\uparrow$ N $\uparrow$ T $\uparrow$ Y $\uparrow$

Number of ways $= 5! \times {}^{6}C_2 = 1800$.

Case 2 : Group 2 'I's together (as a unit), and the two 'M's are to be separated from each other.

M $\quad$ M

$\uparrow$ $\boxed{\text{II}}$ $\uparrow$ U $\uparrow$ N $\uparrow$ T $\uparrow$ Y $\uparrow$

Number of ways $= 5! \times {}^{6}C_2 = 1800$.

Case 3 : Group 2 'I's together (as a unit), and group 2 'M's together (as a unit)

$\boxed{\text{II}}$ $\boxed{\text{MM}}$ U N T Y

Number of ways $= 6! = 720$.

Hence, the required probability $= 1-\frac{1800}{10080}-\frac{1800}{10080}-\frac{720}{10080}=\frac{4}{7}$.

Methods 3 and 4 are comparable ideas. However, you ought to be careful with the "overlap" in Method 4.

	<u>Method 4</u> (complementary approach) : Case 1 : Group 2 'M's together (as a unit) $\boxed{\text{MM}}$ U N T Y I I Number of ways $= \dfrac{7!}{2!} = 2520.$ Case 2 : Group 2 'I's together (as a unit) $\boxed{\text{I I}}$ U N T Y M M Number of ways $= \dfrac{7!}{2!} = 2520.$ Case 3 : Group 2 'I's together (as a unit), and group 2 'M's together (as a unit) $\boxed{\text{I I}}$ $\boxed{\text{M M}}$ U N T Y Number of ways = 6! = 720. Hence, the required probability $= 1 - \dfrac{2520}{10080} - \dfrac{2520}{10080} + \dfrac{720}{10080} = \dfrac{4}{7}.$	Do note that in case 1 and case 2, there are instances of I-I grouping together, and M-M grouping together.

9 A bag initially contains 3 red balls and 3 black balls. Whenever a red ball is drawn from the bag, it is put back into the bag together with an extra red ball. Whenever a black ball is drawn from the bag, it is not put back into the bag and no extra balls are added.

Isaac draws n balls from the bag, one after another, where $n \in \mathbb{Z}^+$, and R denotes the number of red balls out of the n balls drawn.

(a) Give two reasons why R cannot be modelled using a Binomial distribution. [2]

(b) For $n = 3$, find

(i) $\mathrm{P}(R \geq 1)$, [2]

(ii) the probability that the first ball drawn is black given that at least 1 of the 3 balls drawn is red. [3]

(c) For $n = 31$, show that $\mathrm{P}(R = 31) = \dfrac{1}{714}$. [2]

(d) Isaac wins 100 dollars for each red ball he draws if all the balls he draws from the bag are red, and does not win any money otherwise. What is the maximum amount of money Isaac would win if the probability of all the balls he draws are red exceeds 0.0001? [3]

Solution		**Comments**
(a) **[2]**	1. The probability of drawing red balls from 2nd draw does not remain at a constant. If first ball is red, probability of red for 2nd ball is 4/7 If first ball is black, probability of red for 2nd ball is 3/6. 2. The drawing of balls are not independent of each other as it involves replacement.	Remember to explain in context of the question during A level.
(b) **(i)** **[2]**	$\mathrm{P}(R \geq 1) = \mathrm{P}(\text{at least 1 red}) = 1 - \mathrm{P}(\text{all black})$ $= 1 - \left(\frac{1}{2}\right)\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)$ $= 1 - \frac{1}{20} = \frac{19}{20}$	The most efficient way is to use complement in this case. Those using listing need to evaluate the probability of every term: P(RRR), P(RRB), P(RBR), P(BRR), P(RBB), P(BRB), P(BBR).
(ii) **[3]**	Let A be the event the first ball drawn is black. Then	Some students didn't realise that "at least one of the 3 balls is

	$\mathrm{P}(A \mid R \geq 1)$ $=\dfrac{\mathrm{P}(A \cap \{R \geq 1\})}{\mathrm{P}(R \geq 1)}$ $=\dfrac{\mathrm{P}(\{B,R\}+\{B,B,R\})}{\frac{19}{20}}$ $=\dfrac{\frac{1}{2}\left(\frac{3}{5}\right)+\frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{3}{4}\right)}{\frac{19}{20}}$ $=\dfrac{9}{19}$ Alternatively, $\mathrm{P}(A \mid R \geq 1)$ $=\dfrac{\mathrm{P}(A \cap \{R \geq 1\})}{\mathrm{P}(R \geq 1)}$ $=\dfrac{\mathrm{P}(B)-P(BBB)}{\frac{19}{20}}$ $=\dfrac{\frac{1}{2}-\frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)}{\frac{19}{20}}$ $=\dfrac{9}{19}$	red" refers to "$R \geq 1$", and calculate P(at least 1 of the 3 balls is red). For numerator, P(BR) = P(BRR) + P(BRB) Note that P(BBR)≠P(BRB)
(c) **[2]**	P(all 31 balls red) $=\left(\frac{3}{6}\right)\left(\frac{4}{7}\right)\left(\frac{5}{8}\right)\left(\frac{6}{9}\right)\left(\frac{7}{10}\right)\cdots\left(\frac{30}{33}\right)\left(\frac{31}{34}\right)\left(\frac{32}{35}\right)\left(\frac{33}{36}\right)$ $=\dfrac{3(4)(5)}{34(35)(36)}$ $=\dfrac{1}{34(7)(3)}$ $=\dfrac{1}{714}$	The idea here is to list down sufficient terms to observe that (6)(7)(8)…(32)(33) appears in both the numerator and denominator.

(d)
[3]

P(all red balls in first n draws) $= \mathrm{P}(R = n)$

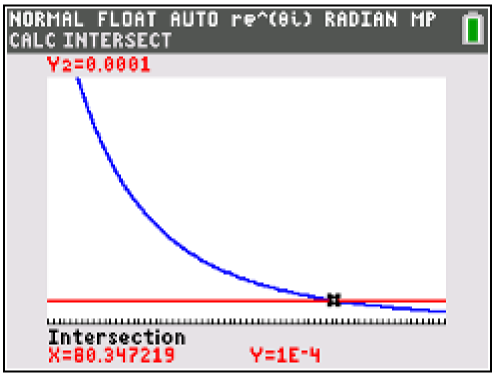
$$= \left(\frac{3}{6}\right)\left(\frac{4}{7}\right)\left(\frac{5}{8}\right)\left(\frac{6}{9}\right)\cdots\left(\frac{n}{n+3}\right)\left(\frac{n+1}{n+4}\right)\left(\frac{n+2}{n+5}\right)$$

$$= \frac{60}{(n+3)(n+4)(n+5)}$$

From GC,

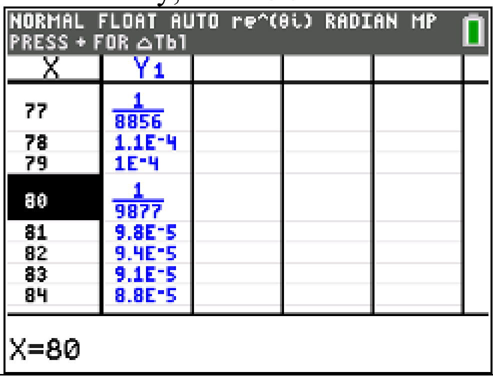
$$\frac{60}{(n+3)(n+4)(n+5)} > 0.0001$$

$$\Rightarrow 0 < n < 80.347$$



Maximum amount Isaac would win with probability exceeding 0.0001 is $8000.

Alternatively, from GC table of values:



NORMAL FLOAT AUTO re^(θi) RADIAN MP
PRESS + FOR ΔTbl

X	Y1
77	1/8856
78	1.1E-4
79	1E-4
80	1/9877
81	9.8E-5
82	9.4E-5
83	9.1E-5
84	8.8E-5

X=80

The idea is to see that $\mathrm{P}(R = n) > 0.0001$ where part (c) provides the hint on getting the general form for $\mathrm{P}(R = n)$.

GC does the rest of the calculations. Some students managed to get the general form but gotten different answers from their GCs, most likely due to carelessness.

10 A bag contains four balls numbered 1, 2, 3 and 4. In a game, a ball is drawn at random from the bag and then a fair coin is tossed a number of times that is equal to the number shown on the ball drawn. The random variable X is the number of heads recorded.

(i) Show that $\text{P}(X=0)=\dfrac{15}{64}$. Find $\text{P}(X=x)$ for all other possible values of x. [5]

(ii) Denoting the expectation and variance of X by μ and σ^2 respectively, find $\text{P}(X>\mu)$ and show that $\sigma^2=\dfrac{15}{16}$. [3]

Adam plays this game 10 times.

(iii) Find the probability that there are at least two games with at least 2 heads recorded. [2]

Bill plays this game 50 times.

(iv) Using a suitable approximation, estimate the probability that the average number of heads recorded is less than 1. [3]

Solution		**Comments**
(i) **[5]**	The following table * shows the probabilities of obtaining the number of heads corresponding to the number of throws.	

		Number of heads recorded, X				
		0	1	2	3	4
No. of tosses (No. shown on the ball drawn)	1	T: $\frac{1}{4}\times\binom{1}{0}\left(\frac{1}{2}\right)$	H: $\frac{1}{4}\times\binom{1}{1}\left(\frac{1}{2}\right)$	-	-	-
	2	TT: $\frac{1}{4}\times\binom{2}{0}\left(\frac{1}{2}\right)^2$	HT or TH: $\frac{1}{4}\times\binom{2}{1}\left(\frac{1}{2}\right)^2$	HH: $\frac{1}{4}\times\binom{2}{2}\left(\frac{1}{2}\right)^2$	-	-
	3	TTT: $\frac{1}{4}\times\binom{3}{0}\left(\frac{1}{2}\right)^3$	HTT or THT or HTT: $\frac{1}{4}\times\binom{3}{1}\left(\frac{1}{2}\right)^3$	HHT or HTH or THH: $\frac{1}{4}\times\binom{3}{2}\left(\frac{1}{2}\right)^3$	HHH: $\frac{1}{4}\times\binom{3}{3}\left(\frac{1}{2}\right)^3$	-
	4	TTTT: $\frac{1}{4}\times\binom{4}{0}\left(\frac{1}{2}\right)^4$	HTTT, THTT, TTHT or TTTH: $\frac{1}{4}\times\binom{4}{1}\left(\frac{1}{2}\right)^4$	HHTT, HTHT, HTTH, THHT, THTH or TTHH: $\frac{1}{4}\times\binom{4}{2}\left(\frac{1}{2}\right)^4$	HHHT, HHTH, HTHH or THHH: $\frac{1}{4}\times\binom{4}{3}\left(\frac{1}{2}\right)^4$	HHHH: $\frac{1}{4}\times\binom{4}{4}\left(\frac{1}{2}\right)^4$

	$\text{P}(X=0)=\text{P}(\text{T, TT, TTT or TTTT})=\frac{1}{4}\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right]=\frac{15}{64}$ (shown)	
	<table><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>$\text{P}(X=x)$</td><td>$\frac{15}{64}$</td><td>$\frac{13}{32}$</td><td>$\frac{1}{4}$</td><td>$\frac{3}{32}$</td><td>$\frac{1}{64}$</td></tr></table>	$\text{P}(X=x)$ can be obtained by adding the respective probabilities in the corresponding column of table *. Do check that $\sum_{x=0}^{4}\text{P}(X=x)=1$.
(ii) **[3]**	$\mu=\text{E}(X)$ $=\left(0\times\frac{15}{64}\right)+\left(1\times\frac{13}{32}\right)+\left(2\times\frac{1}{4}\right)+\left(3\times\frac{3}{32}\right)+\left(4\times\frac{1}{64}\right)$ $=\frac{5}{4}=1.25$ $\text{P}(X>\mu)=\text{P}\left(X>\frac{5}{4}\right)$ $=\text{P}(X=2)+\text{P}(X=3)+\text{P}(X=4)$ $=\frac{1}{4}+\frac{3}{32}+\frac{1}{64}$ $=\frac{23}{64}$ or 0.359375 $\text{E}(X^2)=\left(0^2\times\frac{15}{64}\right)+\left(1^2\times\frac{13}{32}\right)+\left(2^2\times\frac{1}{4}\right)+\left(3^2\times\frac{3}{32}\right)+\left(4^2\times\frac{1}{64}\right)=\frac{5}{2}$ $\sigma^2=\text{Var}(X)$ $=\text{E}(X^2)-[\text{E}(X)]^2$ $=\frac{5}{2}-\left(\frac{5}{4}\right)^2$ $=\frac{15}{16}$ (shown)	X is a discrete random variable. Many mistook X to be normally distributed, and ***wrongly concluded*** that $\text{P}(X>\mu)=\frac{1}{2}$.
(iii) **[2]**	Let Y be the number of games, out of ten, with at least two heads. $Y\sim\text{B}\left(10,\frac{23}{64}\right)$	$\text{P}(X\geq 2)=\text{P}(X>\mu)$, whose value has been found in (ii)

	$P(Y \geq 2)$ $= 1 - P(Y \leq 1)$ $= 1 - 0.076953$ $= 0.923$ (3 s.f.)	
(iv) **[3]**	Let $\overline{X}$ be the average number of heads recorded in 50 games. Since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(1.25, \dfrac{15}{16(50)}\right)$ or $\overline{X} \sim \mathrm{N}\left(1.25, \dfrac{0.9375}{50}\right)$ approximately. $P\left(\overline{X} < 1\right) = 0.0339$ (3 s.f.) Alternatively, Since $n = 50$ is large, by Central Limit Theorem, $X_1 + X_2 + \cdots X_{50} \sim \mathrm{N}\left(50 \times 1.25, 50 \times \dfrac{15}{16}\right)$ approximately. $P\left(\overline{X} < 1\right) = P\left(X_1 + X_2 + \cdots X_{50} < 50\right) = 0.0339$ (3 s.f.)	It is ***wrong*** to state that $X \sim \mathrm{N}\left(1.25, \dfrac{15}{16}\right)$ due to Central Limit Theorem. Note that $\overline{X} = \dfrac{X_1 + X_2 + \cdots X_{50}}{50}$