RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 11 (Summary and Tutorial)

Topic: Applications of Integration

Summary for Applications of Integration

1 Evaluating Areas Bounded by Curves and Axes

1.1 Area bounded by the curve and the x-axis

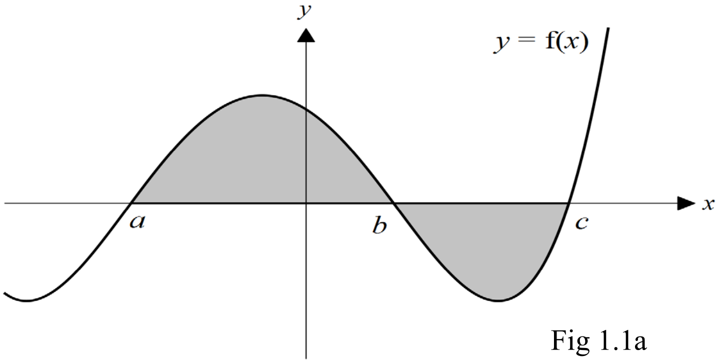


Fig 1.1a

In Fig. 1.1a,

(i) $\int_a^b \mathrm{f}(x)\,\mathrm{d}x > 0$. Thus, area under $y = \mathrm{f}(x)$ from $x = a$ to $x = b$ is given by $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$.

(ii) $\int_b^c \mathrm{f}(x)\,\mathrm{d}x < 0$. Thus, area under $y = \mathrm{f}(x)$ from $x = b$ to $x = c$ is given by $-\int_b^c \mathrm{f}(x)\,\mathrm{d}x$ or $\int_b^c |\mathrm{f}(x)|\,\mathrm{d}x$ (see Fig. 1.1b).

(iii) The area under $y = \mathrm{f}(x)$ from $x = a$ to $x = c$ is given by

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \left(-\int_b^c \mathrm{f}(x)\,\mathrm{d}x\right) \quad \text{or} \quad \int_a^c |\mathrm{f}(x)|\,\mathrm{d}x \text{ (see Fig. 1.1b).}$$

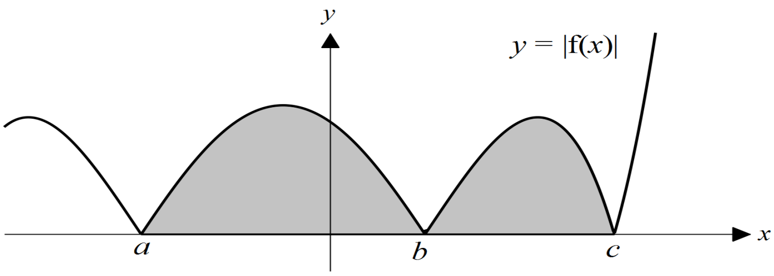


Fig 1.1b

1.2 Area bounded by the curve and the y-axis

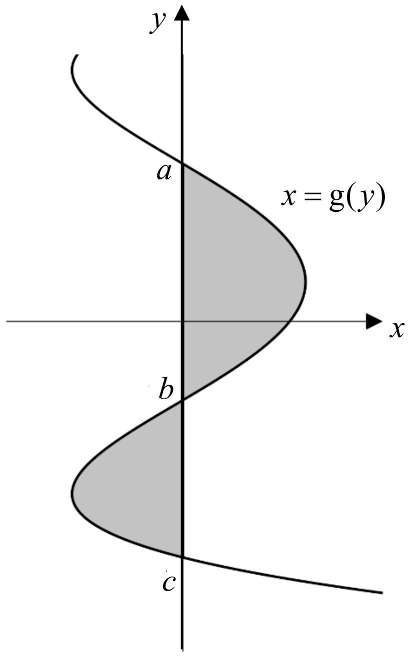


Fig 1.2

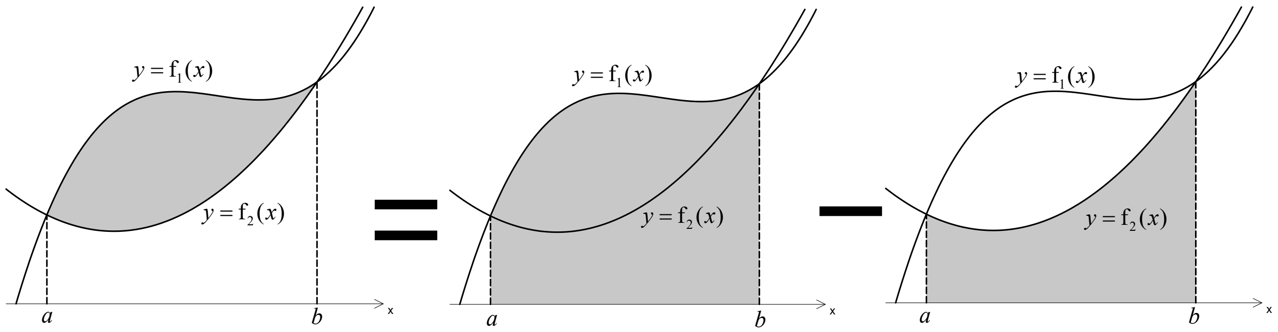
In Fig.1.2,

(i) $\int_b^a \mathrm{g}(y)\ \mathrm{d}y > 0$. Thus, area bounded by $x = \mathrm{g}(y)$ and the y-axis from $y = b$ to $y = a$ is given by $\int_b^a \mathrm{g}(y)\ \mathrm{d}y$.

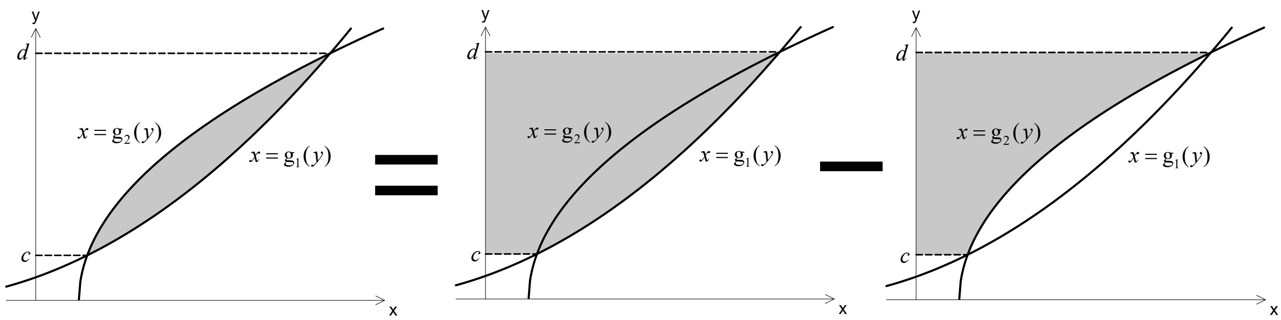
(ii) $\int_c^b \mathrm{g}(y)\ \mathrm{d}y < 0$. Thus, area bounded by $x = \mathrm{g}(y)$ and the y-axis from $y = c$ to $y = b$ is given by $-\int_c^b \mathrm{g}(y)\ \mathrm{d}y$.

(iii) The shaded area bounded by the curve $x = \mathrm{g}(y)$ and the y-axis from $y = c$ to $y = a$ is given by $\int_b^a \mathrm{g}(y)\ \mathrm{d}y + \left(-\int_c^b \mathrm{g}(y)\ \mathrm{d}y\right)$.

1.3 Area bounded between two curves



In general, given 2 curves defined by the equations $y = \mathrm{f}_1(x)$ and $y = \mathrm{f}_2(x)$, and an interval $[a, b]$ where $\mathrm{f}_1(x) \geq \mathrm{f}_2(x)$ within this interval, we can evaluate the area bounded between them from $x = a$ to $x = b$ by evaluating the definite integral $\int_a^b \left(\mathrm{f}_1(x) - \mathrm{f}_2(x)\right) \mathrm{d}x$.



Similarly, given 2 curves defined by the equations $x = \mathrm{g}_1(y)$ and $x = \mathrm{g}_2(y)$, and an interval $[c, d]$ where $\mathrm{g}_1(y) \geq \mathrm{g}_2(y)$ within this interval, we can evaluate the area bounded between them from $y = c$ to $y = d$ by evaluating the definite integral $\int_c^d \left(\mathrm{g}_1(y) - \mathrm{g}_2(y)\right) \mathrm{d}y$.

1.4 Evaluating areas involving curves which are defined parametrically

Essentially, we write the required area as $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ or $\int_c^d \mathrm{g}(y)\,\mathrm{d}y$ like how we did in the earlier sections. However, since both y and x are now expressed in terms of a parameter t, the integration here is done with respect to the parameter t instead of the variable x. To do so, we apply the integration by substitution technique to change the integral to one involving the parameter t. Students are encouraged to **sketch the curve with their GC** in order to identify the required region.

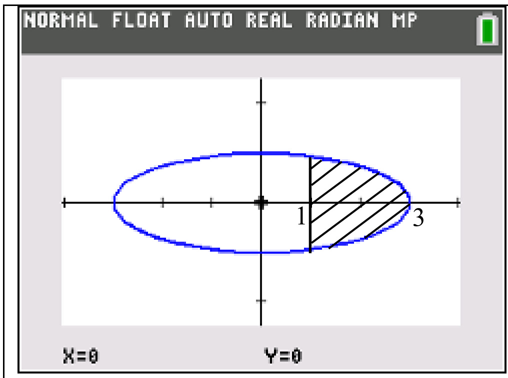
Example

The parametric equations of a curve C is given by

$$x = 3\sin t,\quad y = \cos t \text{, where } 0 \le t \le 2\pi .$$

Find the area of bounded by the curve C, the lines $x = 1$ and $x = 3$.

Solution:



$$\text{Required area } = 2\int_1^3 y\ \mathrm{d}x$$

$$= 2\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}} \left(\cos t\ \frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$

$$= 2\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}} \left(\cos t\ (3\text{cost})\right)\mathrm{d}t$$

and we proceed to solve this integral.

*note the substitution of $y = \cos t,\ \mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$, and also the change of limits.

1.5 Area under a curve as the limiting sum of the areas of the rectangles

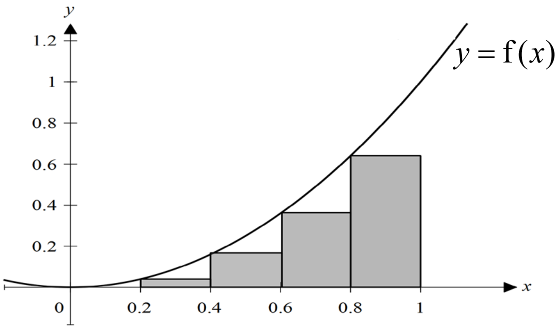


Fig 1.3a

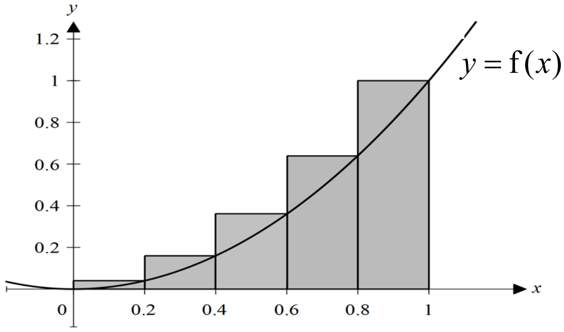


Fig 1.3b

Other than using integration, students are required to be able to estimate the area under a curve using the sum of the areas of the rectangles of equal width.

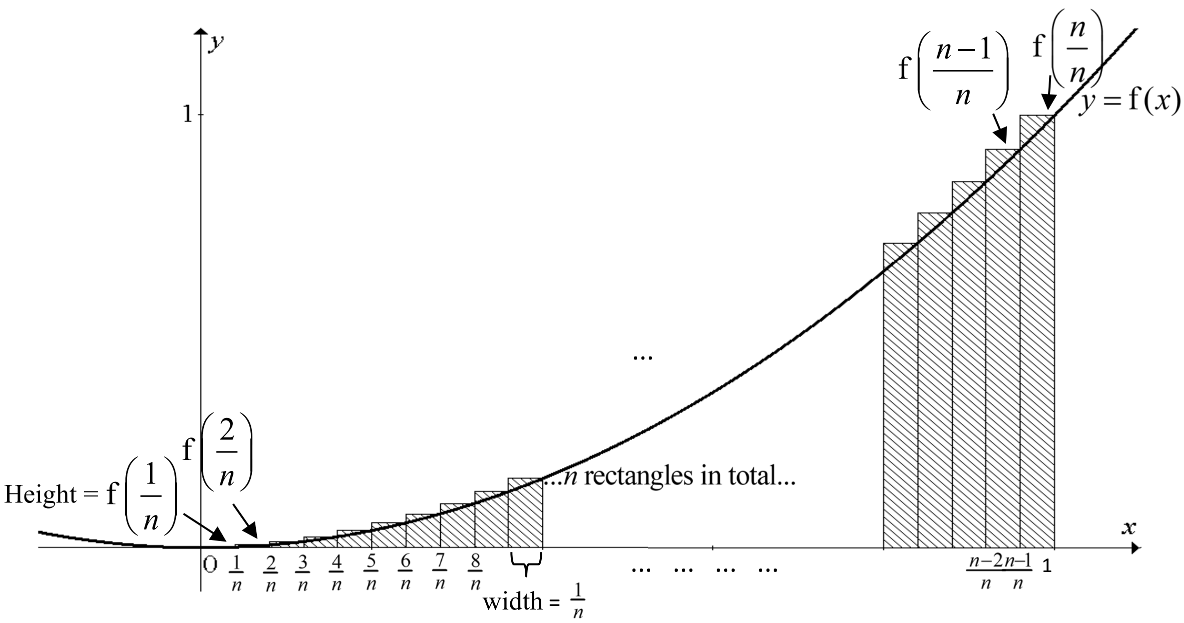
For example, to obtain an estimate for the area under the curve $y = f(x)$ between $x = 0$ and $x = 1$, we can consider finding the sum of the areas of the rectangles as shown in Fig. 1.3a or Fig. 1.3b.

$\int_0^1 f(x)\,dx \approx (0.2)(f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8))$ (in Fig. 1.3a) or

$\int_0^1 f(x)\,dx \approx (0.2)(f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1))$ (in Fig. 1.3b).

From the diagrams, we see that the former gives an approximation which is an **underestimate** while the latter provides an **overestimate** of the actual area under the curve.

To obtain a better estimate, we can consider using more rectangles within the given interval. Let us consider the case in Fig. 1.3b, but instead of using just 5 rectangles, let's use n rectangles.



Thus $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x \approx \dfrac{1}{n}\left(\mathrm{f}\left(\dfrac{1}{n}\right)+\mathrm{f}\left(\dfrac{2}{n}\right)+\mathrm{f}\left(\dfrac{3}{n}\right)+\ldots+\mathrm{f}\left(\dfrac{n}{n}\right)\right)$.

As we increase the number of rectangles n, we find that the sum of the areas of the n rectangles will approach the value of the integral $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$.

Thus, the **limiting sum of the areas of the *n* rectangles** can be used here to evaluate the area under the graph of $y = \mathrm{f}(x)$ from $x = 0$ to $x = 1$.

i.e. $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{1}{n}\mathrm{f}\left(\dfrac{r}{n}\right)$.

Note: This idea can also be similarly applied if we choose to consider the case in Fig. 1.3a.

2 Volume of Solid of Revolution

2.1 Rotation about the x-axis ($y = 0$)

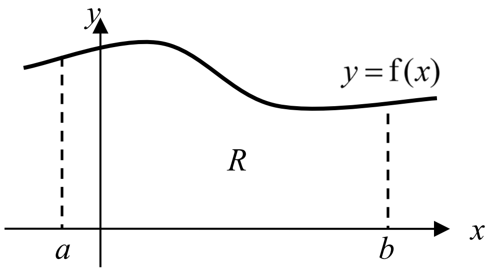


Fig. 2.1a

In general, if the region R bounded by the curve $y = \mathrm{f}(x)$, the x-axis, the lines $x = a$ and $x = b$, as shown in Fig.2.1a is rotated completely about the x-axis, the volume of solid of revolution formed is given by $\pi\int_a^b y^2 \, \mathrm{d}x = \pi\int_a^b \left[\mathrm{f}(x)\right]^2 \, \mathrm{d}x$.

If the region R is bounded by the curves $y = \mathrm{f}_1(x)$, $y = \mathrm{f}_2(x)$, the lines $x = a$ and $x = b$ as shown in Fig. 2.1b,

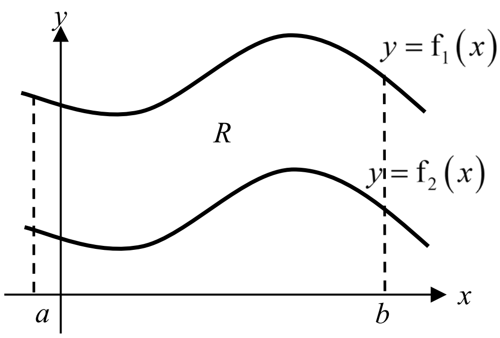


Fig. 2.1b

then the volume of solid of revolution formed when R is rotated completely about the x-axis is given by $\pi\int_a^b \left[\mathrm{f}_1(x)\right]^2 \, \mathrm{d}x - \pi\int_a^b \left[\mathrm{f}_2(x)\right]^2 \, \mathrm{d}x = \pi\int_a^b \left[\mathrm{f}_1(x)\right]^2 - \left[\mathrm{f}_2(x)\right]^2 \, \mathrm{d}x$

2.2 Rotation about the *y*-axis ($x = 0$)

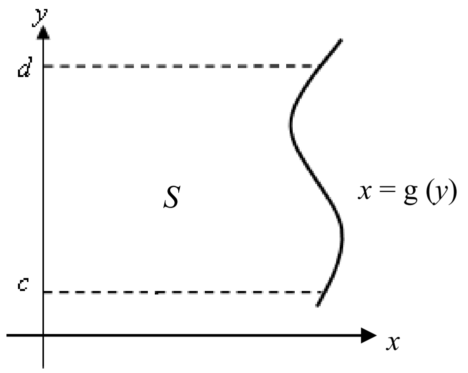


Fig. 2.2a

In general, if the region S bounded by the curve $x = \mathrm{g}(y)$, the y-axis, the lines $y = c$ and $y = d$, as shown in Fig. 2.2a is rotated completely about the y-axis, the volume of solid of revolution formed is given by $\pi\int_c^d x^2 \, \mathrm{d}y = \pi\int_c^d \left[\mathrm{g}(y)\right]^2 \, \mathrm{d}y$

If the region S is bounded by the curves $x = \mathrm{g}_1(y)$, $x = \mathrm{g}_2(y)$, the lines $y = c$ and $y = d$ as shown in Fig. 2.2b,

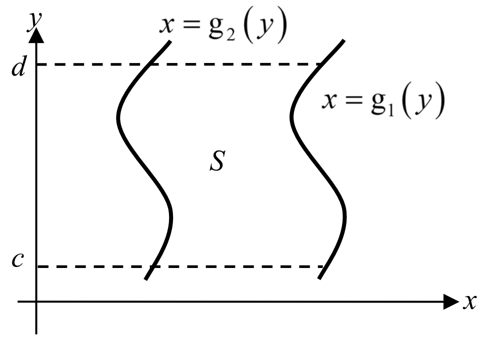


Fig. 2.2b

then the volume of solid of revolution formed when S is rotated completely about the y-axis is given by $\pi\int_c^d \left[\mathrm{g}_1(y)\right]^2 \, \mathrm{d}y - \pi\int_c^d \left[\mathrm{g}_2(y)\right]^2 \, \mathrm{d}y = \pi\int_c^d \left[\mathrm{g}_1(y)\right]^2 - \left[\mathrm{g}_2(y)\right]^2 \, \mathrm{d}y$

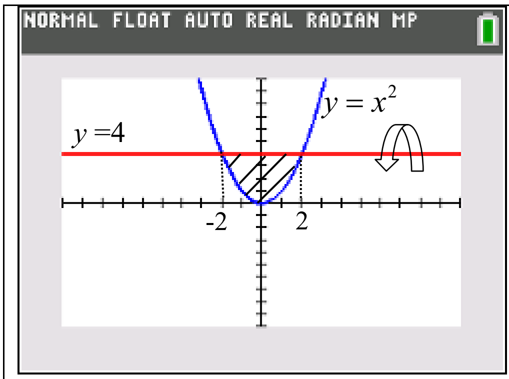
2.3 Rotation about other lines (vertical or horizontal)

When the required region is rotated through 2π about other vertical or horizontal line instead of one of the axes, we will need rewrite the problem (usually via translation of the graph) so that the line of rotation is one of the axis. This idea is illustrated in the following example.

Example

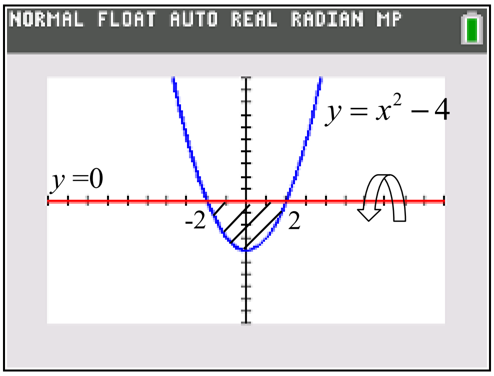
The region R is bounded by the curve $y = x^2$ and the line $y = 4$. Find the volume of the solid formed when R is rotated completely about the line $y = 4$.

Solution:



In order to find the required volume, the region is translated 4 units in the negative y-direction so that the line of rotation is now the x-axis.

Thus, the equation of the curve in consideration is now $y = x^2 - 4$ instead of $y = x^2$.

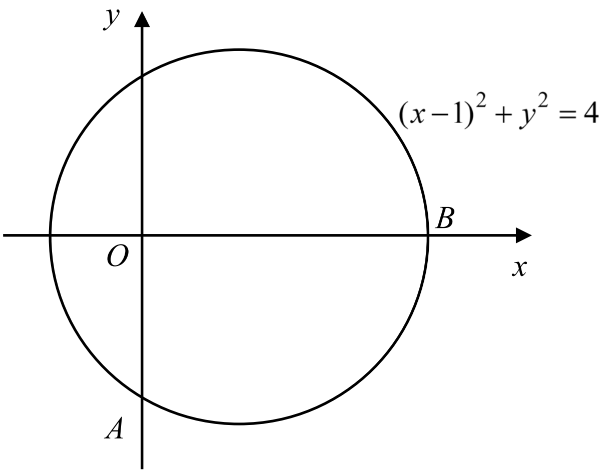


Required volume $= \pi\int_{-2}^{2}(y-4)^2\ \mathrm{d}x = \pi\int_{-2}^{2}(x^2-4)^2\ \mathrm{d}x$

Revision Tutorial Questions

Source of Question: IJC JC2 CT2 9758/2017/P1/Q3

1



The diagram shows a circle with equation $(x-1)^2+y^2=4$.

(i) By using the substitution $x=1+2\sin\theta$, find the exact area bounded by the circle and the y-axis for $x\le 0$. [6]

(ii) The circle cuts the axes at the points A and B as shown in the diagram. The region bounded by the minor arc AB and the line segment AB is rotated through 4 right angles about the x-axis. Find the exact volume of the solid of revolution. [4]

Solution:

(i) [6]

$x=1+2\sin\theta \Rightarrow \mathrm{d}x=2\cos\theta\,\mathrm{d}\theta$

When $y=0$, $(x-1)^2+y^2=4$

$$(x-1)^2=4$$

$$x=-1 \quad \text{or} \quad 3$$

x	-1	0
θ	$-1=1+2\sin\theta$ $\sin\theta=-1$ $\theta=-\frac{\pi}{2}$	$0=1+2\sin\theta$ $\sin\theta=-\frac{1}{2}$ $\theta=-\frac{\pi}{6}$

$$(x-1)^2+y^2=4$$

$$y^2=4-(x-1)^2$$

$$y=\pm\sqrt{4-(x-1)^2}$$

Required Area $=2\int_{-1}^{0}\sqrt{4-(x-1)^2}\ \mathrm{d}x$ ---(1)

	$=2\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}}\sqrt{4-4\sin^2\theta}\ (2\cos\theta)\ \mathrm{d}\theta$ ---(2) $=8\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}}\cos^2\theta\ \ \mathrm{d}\theta$ $=4\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}}\cos 2\theta+1\ \ \mathrm{d}\theta$ $=4\left[\frac{\sin 2\theta}{2}+\theta\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{6}}$ $=4\left[\left(\frac{1}{2}\sin\left(-\frac{\pi}{3}\right)-\frac{\pi}{6}\right)-\left(\frac{1}{2}\sin(-\pi)-\frac{\pi}{2}\right)\right]$ $=4\left[\left(-\frac{\sqrt{3}}{4}-\frac{\pi}{6}\right)-\left(0-\frac{\pi}{2}\right)\right]$ $=\frac{4}{3}\pi-\sqrt{3}\ \ \text{units}^2$
(ii) **[4]**	Volume of revolution $=\pi\int_0^3 4-(x-1)^2\ \mathrm{d}x\ -\ \frac{\pi}{3}\left(\sqrt{3}\right)^2(3)$ $=\pi\int_0^3\left(4-x^2+2x-1\right)\ \mathrm{d}x\ -3\pi$ $=\pi\left[3x-\frac{x^3}{3}+x^2\right]_0^3-3\pi$ $=9\pi-3\pi$ $=6\pi\ \ \text{units}^3$

Source of Question: JJC JC2 CT2 9758/2017/P1/Q6

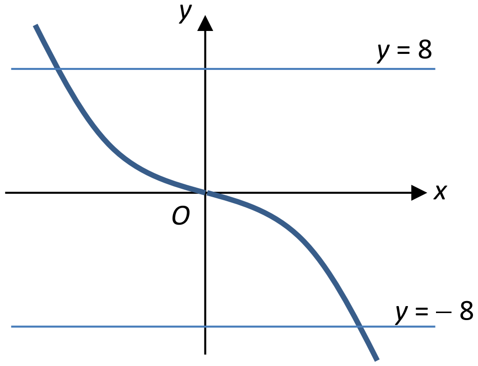
2 **(a)** The region R is bounded by the curve $y=-x^3$, the y-axis and the lines $y=-8$ and $y=8$ Without the use of a calculator, evaluate the area of the region R. [5]

(b) **(i)** The curve $y=-x^3$ is translated by 2 units in the positive x direction and the resulting curve has equation $y=\mathrm{g}(x)$. [1]

(ii) Find The region S is bounded by the curve $y=-x^3$ and the lines $y=-8$, $y=6$ and $x=-2$. The interior of a water container is formed by rotating the region S completely about the line $x=-2$. Using your answer to part **(b)(i)**, calculate the volume of water when the container is filled to the brim. [3]

Solution:

(a) [5]



$y=-x^3 \Rightarrow x=(-y)^{\frac{1}{3}}$

$$\text{Area } = -\int_0^8 (-y)^{\frac{1}{3}}\,\mathrm{d}y+\int_{-8}^0 (-y)^{\frac{1}{3}}\,\mathrm{d}y$$

$$= -2\int_0^8 (-y)^{\frac{1}{3}}\,\mathrm{d}y$$

$$= 2\left[\frac{3y^{\frac{4}{3}}}{4}\right]_0^8$$

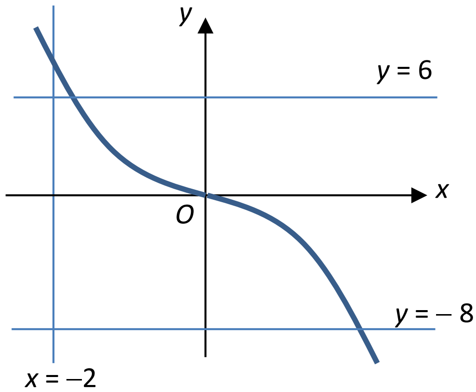
$$= 2\left[\frac{3(8)^{\frac{4}{3}}}{4}-0\right]$$

$= 24$ units^2

(b) (i) [1]

Translating the equation of $y=-x^3$ to right by 2 units $\Rightarrow y=-(x-2)^3$

(b) (ii) [3]



Using $y=-(x-2)^3$, $x=2+(-y)^{\frac{1}{3}}$

$$\text{Volume of container } = \pi\int_{-8}^{6}\left(2+(-y)^{\frac{1}{3}}\right)^2\mathrm{d}y$$

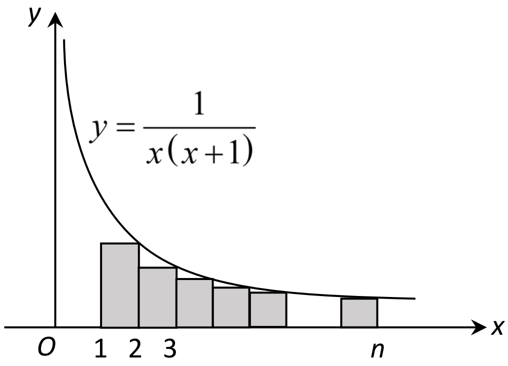
$= 321.63$ units^3

≈ 322 units^3

Source of Question: ACJC Prelim 9758/2018/01/Q7 (modified)

3(i) Express $\dfrac{1}{x(x+1)}$ in partial fractions. [1]

(ii)



The graph of $y=\dfrac{1}{x(x+1)}$, for $0 \le x \le n$, is shown in the diagram. Rectangles, each of width 1 unit, are drawn below the curve from $x = 1$ to $x = n$, where $n \ge 3$.

By considering $\displaystyle\sum_{x=a}^{b} \frac{1}{x(x+1)}$ where a and b are constants to be found, find the total area of the $n-1$ rectangles in terms of n. [3]

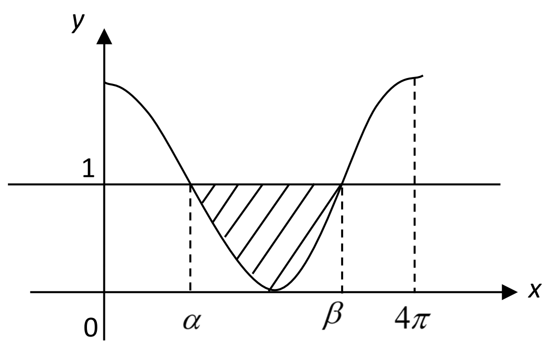
Hence, show that $\dfrac{1}{2} - \ln 2 < \dfrac{1}{n+1} + \ln\left(1 - \dfrac{1}{n+1}\right)$ for all $n \ge 3$. [3]

Solution:

(i) **[1]**	$\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$
(ii) **[3]** **[3]**	Total area of rectangles $= \dfrac{1}{2(2+1)} + \dfrac{1}{3(3+1)} + \ldots + \dfrac{1}{n(n+1)}$ $= \displaystyle\sum_{x=2}^{n} \frac{1}{x(x+1)}$ so $a = 2$, $b = n$ $= \displaystyle\sum_{x=2}^{n} \left(\frac{1}{x} - \frac{1}{x+1}\right)$ $= \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \left(\dfrac{1}{4} - \dfrac{1}{5}\right) + \ldots + \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$ $= \dfrac{1}{2} - \dfrac{1}{n+1}$ Actual area $= \displaystyle\int_1^n \frac{1}{x(x+1)}\,dx = \int_1^n \frac{1}{x} - \frac{1}{(x+1)}\,dx$

	$= \left[\ln x - \ln(x+1)\right]_1^n$ $= \ln n - \ln(n+1) - \ln 1 + \ln 2$ $= \ln n - \ln(n+1) + \ln 2$ Area of rectangles < actual area $\therefore \quad \frac{1}{2} - \frac{1}{n+1} < \ln n - \ln(n+1) + \ln 2$ $\frac{1}{2} - \ln 2 < \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right)$ $\frac{1}{2} - \ln 2 < \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right) \quad (shown)$

Source of Question: PJC JC2 CT1 9758/2017/P1/Q9
4



The diagram shows the curve with equation $y = \cos\left(\frac{1}{2}x\right) + 1$ for $0 \le x \le 4\pi$ and the line with equation $y = 1$. The curve and the line intersect at $x = \alpha$ and $x = \beta$.

(i) State the exact values of α and β. [1]

(ii) Find the area of the region bounded by the curve, the line $y = 1$ and the axes, giving your answer correct to 3 decimal places. [3]

(iii) The shaded region between the curve and the line is rotated completely about the x-axis. Use a non-calculator method to find the volume of revolution. [6]

Solution:

(i) **[1]**	$\alpha=\pi$, $\beta=3\pi$
(ii) **[3]**	Required area = $(\pi)+\int_{\pi}^{2\pi}\cos\left(\frac{1}{2}x\right)+1\ \mathrm{d}x=4.28318\approx 4.283$ Alternatively, required area = $\int_0^1 2\cos^{-1}\left(y-1\right)\ \mathrm{d}y=4.28318\approx 4.283$
(iii) **[6]**	Required volume $=\pi(1)^2(2\pi)-\pi\int_{\pi}^{3\pi}\left(\cos\left(\frac{1}{2}x\right)+1\right)^2\mathrm{d}x$ $=2\pi^2-\pi\int_{\pi}^{3\pi}\cos^2\left(\frac{1}{2}x\right)+2\cos\left(\frac{1}{2}x\right)+1\ \mathrm{d}x$ $=2\pi^2-\pi\int_{\pi}^{3\pi}\left(\frac{1}{2}+\frac{1}{2}\cos x\right)+2\cos\left(\frac{1}{2}x\right)+1\ \mathrm{d}x$ $=2\pi^2-\pi\int_{\pi}^{3\pi}\frac{1}{2}\cos x+2\cos\left(\frac{1}{2}x\right)+\frac{3}{2}\ \mathrm{d}x$ $=2\pi^2-\pi\left[\frac{1}{2}\sin x+4\sin\left(\frac{1}{2}x\right)+\frac{3}{2}x\right]_{\pi}^{3\pi}$ $=2\pi^2-\pi\left[\left(4\sin\left(\frac{3\pi}{2}\right)+\frac{3}{2}(3\pi)\right)-\left(4\sin\left(\frac{\pi}{2}\right)+\frac{3}{2}(\pi)\right)\right]$ $=2\pi^2-\pi\left[\left(-4+\frac{9\pi}{2}\right)-\left(4+\frac{3\pi}{2}\right)\right]$ $=2\pi^2-\pi(3\pi-8)=\pi(8-\pi)$

Source of Question: SAJC JC2 CT2 9758/2017/P1/Q4

5 The region R is bounded by the curve $y=\frac{x^2}{\sqrt{a^2-x^2}}$, the x-axis and the lines $x=\frac{a}{2}$ and $x=\frac{a}{\sqrt{2}}$, where a is a positive constant. Use the substitution $x=a\cos\theta$ to show that the area of R is $a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\cos^2\theta\ \mathrm{d}\theta$, and evaluate the exact value of this integral in terms of a and π

[6]

Solution:

5
[6]

$x = a\cos\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta$

When $x = \dfrac{a}{2}$, $\theta = \dfrac{\pi}{3}$

When $x = \dfrac{a}{\sqrt{2}}$, $\theta = \dfrac{\pi}{4}$

$$\int_{\frac{a}{2}}^{\frac{a}{\sqrt{2}}} \frac{x^2}{\sqrt{a^2 - x^2}}\ dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{a^2\cos^2\theta}{\sqrt{a^2\sin^2\theta}}\left(-a\sin\theta\right)\ d\theta$$

$$= a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\ \ d\theta\ \text{(shown)}$$

$$= \frac{a^2}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\cos 2\theta + 1\right)\ d\theta$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{3}}{4} + \frac{\pi}{3} - \frac{1}{2} - \frac{\pi}{4}\right)$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{3}}{4} + \frac{\pi}{12} - \frac{1}{2}\right)$$

$$= \frac{a^2}{2}\left(\frac{\sqrt{3}-2}{4} + \frac{\pi}{12}\right)\text{units}^2$$

Source of Question: YJC JC2 CT1 9758/2017/P1/Q6

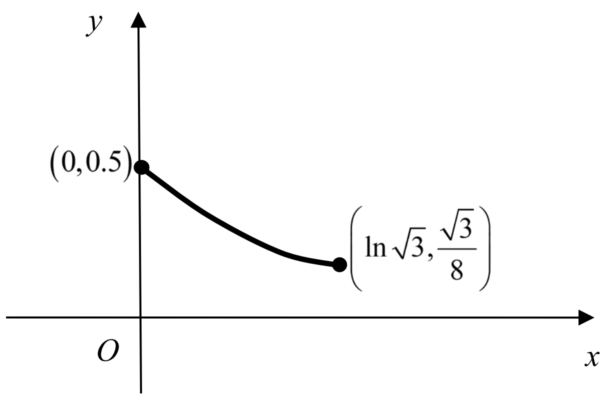
6 A curve C has parametric equations

$$x = \ln \tan t \text{ and } y = 2\sin t \cos^3 t, \quad \text{for } \frac{\pi}{4} \le t \le \frac{\pi}{3}.$$

(i) Sketch the graph of C, indicating clearly the exact coordinates of the end-points. [2]

(ii) The region bounded by C, the x-axis, the y-axis and the line $x = \ln\sqrt{3}$ is denoted by A. Show that the area of A is $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\cos^2 t \, dt$ and evaluate the integral exactly. [5]

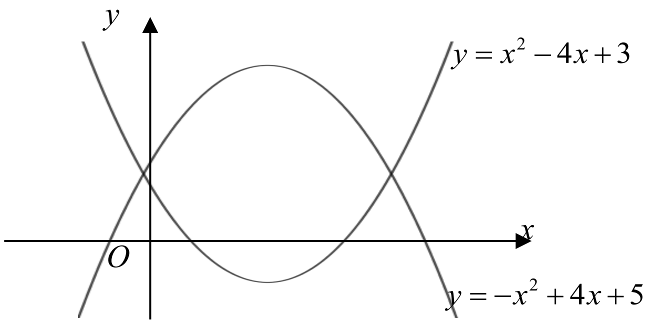
Solution:

(i) **[2]**	
(ii) **[5]**	$x = \ln \tan t \Rightarrow \frac{dx}{dt} = \frac{\sec^2 t}{\tan t}$ when $x = \ln\sqrt{3}$, $\tan t = \sqrt{3}, t = \frac{\pi}{3}$ when $x = 0$, $\tan t = 1, t = \frac{\pi}{4}$ $\int_0^{\ln\sqrt{3}} y \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(2\sin t \cos^3 t\right)\left(\frac{\sec^2 t}{\tan t}\right) dt$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(2\sin t \cos^3 t\right)\left(\frac{\frac{1}{\cos^2 t}}{\frac{\sin t}{\cos t}}\right) dt$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(2\sin t \cos^3 t\right)\left(\frac{1}{\sin t \cos t}\right) dt$

	$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\cos^2 t \text{ d}t$ (shown) $=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1+\cos 2t \text{ d}t$ $=\left[\frac{\sin 2t}{2}+t\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=\left(\frac{\sqrt{3}}{4}+\frac{\pi}{3}\right)-\left(\frac{1}{2}+\frac{\pi}{4}\right)$ $=\frac{\pi}{12}+\frac{\sqrt{3}}{4}-\frac{1}{2}$

Source of Question: VJC JC2 CT1 9758/2018/01/Q10

7 The diagram below shows the graphs of $y=x^2-4x+3$ and $y=-x^2+4x+5$. The region R is the enclosed region between the graphs, and region S is the region bounded by the graph of $y=x^2-4x+3$, the x-axis and the y-axis.



(i) Find the area of region R. [3]

(ii) Find the exact volume of the solid formed when S is rotated completely about the x-axis. [3]

(iii) Show that the volume of the solid formed when S is rotated 2π radians about the y-axis is given by

$\pi\int_a^b \left(5-4\sqrt{y+1}+y\right) \text{d}y$, where a and b are constants to be determined. [3]

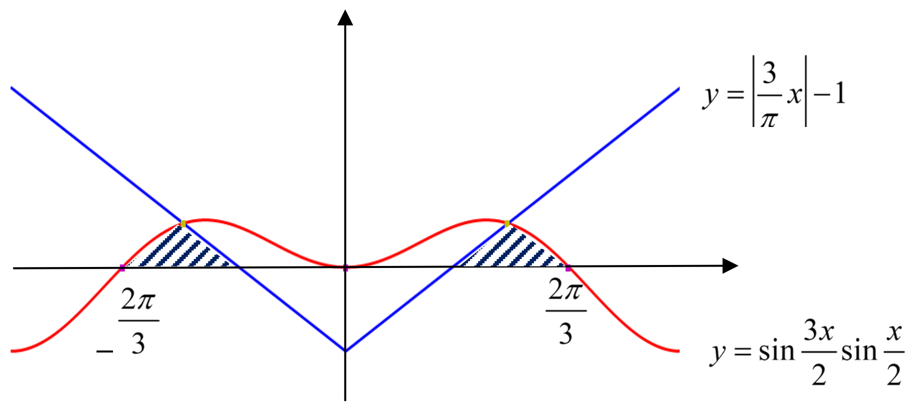
Hence, evaluate this integral. [1]

Solution:

(i) **[3]**	$y=x^2-4x+3$ and $y=-x^2+4x+5$ intersect at $x=-0.23607$ and $x=4.2361$ Area R $=\int_{-0.23607}^{4.2361}\left(-x^2+4x+5-\left(x^2-4x+3\right)\right)\text{d}x$ $=29.8$ (3s.f)

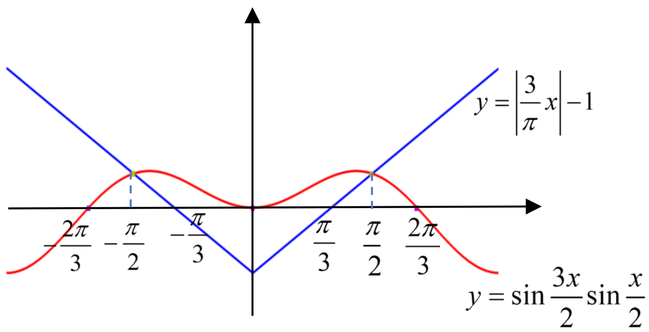
(ii) **[3]**	Volume $=\pi\int_0^1 \left(x^2-4x+3\right)^2 \mathrm{d}x$ $=\pi\int_0^1 x^4-8x^3+22x^2-24x+9 \, \mathrm{d}x$ $=\pi\left[\frac{x^5}{5}-2x^4+\frac{22}{3}x^3-12x^2+9x\right]_0^1$ $=\frac{38\pi}{15}$
(iii) **[3]** **[1]**	$y=x^2-4x+3=\left(x-2\right)^2-1$ $x=2\pm\sqrt{y+1}$ $x=2-\sqrt{y+1},\left(\because x<2\right)$ Volume $=\pi\int_0^3 \left(2-\sqrt{y+1}\right)^2 \mathrm{d}y$ $=\pi\int_0^3 \left(4-4\sqrt{y+1}+y+1\right)\mathrm{d}y$ $=\pi\int_0^3 \left(5-4\sqrt{y+1}+y\right)\mathrm{d}y$ $=2.62\left(3\text{sf}\right)$

Source of Question: AJC JC2 CT2 9758/2017/P1/Q3
8



The diagram shows the graphs of $y=\sin\frac{3x}{2}\sin\frac{x}{2}$ and $y=\left|\frac{3}{\pi}x\right|-1$ in the range $-\pi\le x\le\pi$. Given that $\left(\frac{\pi}{2},\frac{1}{2}\right)$ is a point on both curves, show that the exact area of the shaded region is $\frac{\pi}{12}+\frac{3\sqrt{3}}{4}-1$. [5]

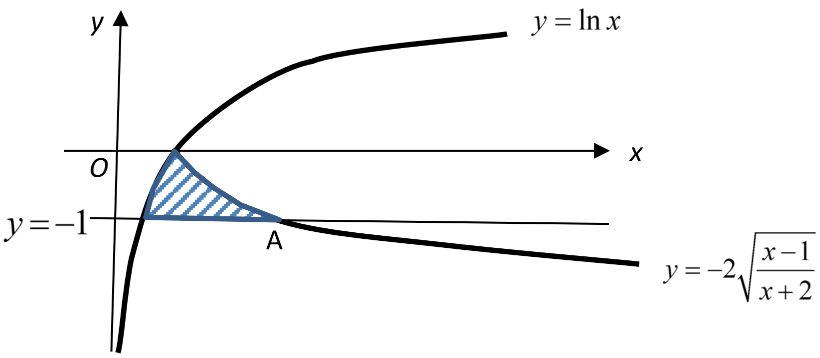
Solution:

8 **[5]**	$y = \left\|\frac{3}{\pi}x\right\| - 1 = 0 \;\Rightarrow\; x = \pm\frac{\pi}{3}$  By symmetry, area of region $= 2\times\left[\frac{1}{2}\times(\frac{\pi}{2}-\frac{\pi}{3})\times\frac{1}{2}\right] + 2\times\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin\frac{3x}{2}\sin\frac{x}{2}\,\mathrm{d}x$ $= \frac{\pi}{12} - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (\cos 2x - \cos x)\,\mathrm{d}x$ $= \frac{\pi}{12} - \left[\frac{\sin 2x}{2} - \sin x\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$ $= \frac{\pi}{12} - \left[(-\frac{\sqrt{3}}{4}) - \frac{\sqrt{3}}{2} - 0 + 1\right] = \frac{\pi}{12} + \frac{3\sqrt{3}}{4} - 1$

Source of Question: AJC JC2 CT1 9758/2017/P1/Q5

9 **(i)** Using integration by parts, show that $\int_{e^{-1}}^{1} (\ln x)^2 \, dx = 2 - \frac{5}{e}$. [4]

(ii) Find $\int \frac{x-1}{x+2} \, dx$. [2]



(iii) The above diagram shows two curves with equations $y = \ln x$ and $y = -2\sqrt{\frac{x-1}{x+2}}$. The line $y = -1$ intersects the curve $y = -2\sqrt{\frac{x-1}{x+2}}$ at the point A where $x = 2$.

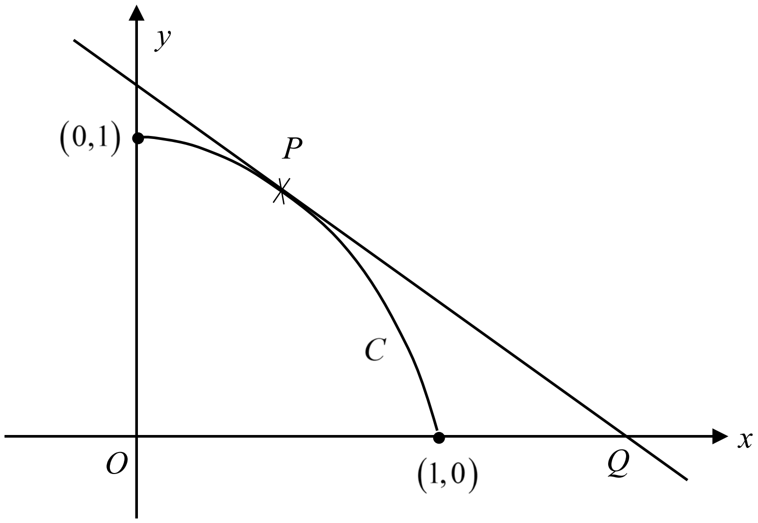
Use a non-calculator method to find the volume of revolution when the shaded region is rotated completely about the x-axis. [4]

Solution:

(i) **[4]**	$\int_{e^{-1}}^{1} (\ln x)^2 dx = \left[x(\ln x)^2\right]_{e^{-1}}^{1} - \int_{e^{-1}}^{1} x\left(2\ln x\left(\frac{1}{x}\right)\right) dx$ $= \left[0 - e^{-1}(-1)^2\right] - 2\int_{e^{-1}}^{1} (\ln x) \, dx$ $= -e^{-1} - 2\left\{\left[x(\ln x)\right]_{e^{-1}}^{1} - \int_{e^{-1}}^{1} x\left(\frac{1}{x}\right) dx\right\}$ $= -e^{-1} - 2\left\{\left[0 - e^{-1}(-1)\right] - \int_{e^{-1}}^{1} 1 \, dx\right\}$ $= -e^{-1} - 2\left\{e^{-1} - [x]_{e^{-1}}^{1}\right\}$ $= -e^{-1} - 2\left\{e^{-1} - \left[1 - e^{-1}\right]\right\}$ $= -e^{-1} - 2e^{-1} + 2 - 2e^{-1}$ $= 2 - 5e^{-1}$

(ii) **[2]**	$\int \frac{x-1}{x+2}\,\mathrm{d}x$ $= \int 1 - \frac{3}{x+2}\,\mathrm{d}x$ $= x - 3\ln\lvert x+2\rvert + c$
(iii) **[4]**	Volume $=\text{cylinder - } \pi \int_{e^{-1}}^{1} (\ln x)^2\,dx - \pi \int_{1}^{2} \left(-2\sqrt{\frac{x-1}{x+2}}\right)^2 dx$ $= \pi(1)^2\left(2-e^{-1}\right) \text{- } \pi \int_{e^{-1}}^{1} (\ln x)^2\,dx - 4\pi \int_{1}^{2} \left(\frac{x-1}{x+2}\right) dx$ $= \pi(1)^2\left(2-e^{-1}\right) \text{- } \pi\left(2-5e^{-1}\right) - 4\pi\left[x - 3\ln\lvert x+2\rvert\right]_1^2$ $= \pi\left(2-e^{-1}-2+5e^{-1}\right) - 4\pi\left[(2-3\ln 4)-(1-3\ln 3)\right]$ $= \pi\left(4e^{-1}\right) - 4\pi\left[1-3\ln 4+3\ln 3\right]$ $= \pi\left(4e^{-1} - 4 + 12\ln\frac{4}{3}\right)$

Source of Question: MJC/Prelim/2016/01/Q11
10



It is given that curve C has parameteric equations

$$x = t^3, \quad y = \sqrt{\left(1-t^2\right)} \quad \text{for } 0 \le t \le 1.$$

The diagram shows the curve C and the tangent to C at P. The tangent at P meets the x-axis at Q.

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is $3p\left(1-p^2\right)-3py\sqrt{\left(1-p^2\right)} = x - p^3$. [3]

(ii) Given further that the line $y = \left(4\sqrt{3}\right)x$ meets the curve at point P, find the exact coordinates of P. [3]

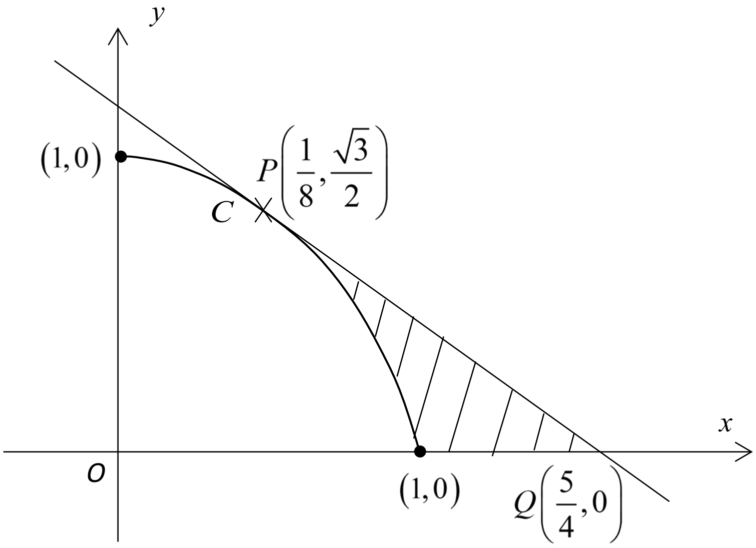
(iii) Hence find the exact coordinates of Q. [2]

(iv) Show that the area of the region bounded by C, the tangent to C at P, and the x-axis is given by $\dfrac{9\sqrt{3}}{32} - \int_{\frac{1}{2}}^{1} 3t^2\sqrt{\left(1-t^2\right)}\,\mathrm{d}t$. [3]

Show that the substitution $t = \sin u$ transforms the above integral to $\dfrac{9\sqrt{3}}{32} - \dfrac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u \,\mathrm{d}u$. Hence, evaluate this area exactly. [6]

Solution:

(i) **[3]**	$\frac{\mathrm{d}x}{\mathrm{d}t}=3t^2, \quad \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{t}{\sqrt{1-t^2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t} \quad =-\frac{t}{\sqrt{1-t^2}}\times\frac{1}{3t^2}$ $=-\frac{1}{3t\sqrt{1-t^2}}$ At point P, $x=p^3$, $y=\sqrt{1-p^3}$. Equation of tangent at P: $y-\sqrt{1-p^2}=-\frac{1}{3p\sqrt{1-p^2}}\left(x-p^3\right)$ $3p\left(1-p^2\right)-3py\sqrt{\left(1-p^2\right)}=x-p^3$ (Shown)
(ii) **[3]**	Substitute $x=p^3$, $y=\sqrt{1-p^2}$ into $y=4\sqrt{3}\,x$ $\sqrt{1-p^2}=4\sqrt{3}\,p^3$ $1-p^2=48p^6$ $48p^6+p^2-1=0$ Using GC, $p=-\frac{1}{2}$ (N.A. since $0\le p\le 1$) or $p=\frac{1}{2}$ $\therefore$ Exact coordinates of P are $\left(\frac{1}{8},\frac{\sqrt{3}}{2}\right)$.
(iii) **[2]**	Equation of tangent to C at P: $3\left(\frac{1}{2}\right)\left[1-\left(\frac{1}{2}\right)^2\right]-\frac{3}{2}y\sqrt{1-\left(\frac{1}{2}\right)^2}=x-\left(\frac{1}{2}\right)^3$ $\frac{9}{8}-\frac{3y}{2}\sqrt{\frac{3}{4}}=x-\frac{1}{8}$ When $y=0$, $\frac{9}{8}=x-\frac{1}{8} \Rightarrow x=\frac{5}{4}$ $\therefore$ Exact coordinates of Q are $\left(\frac{5}{4},0\right)$.

(iv) **[3]** **[6]**	 When $x=\frac{1}{8}$, $t=\frac{1}{2}$. When $x=1$, $t=1$. Area of shaded region = Area of required region $=\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{5}{4}-\frac{1}{8}\right)-\int_{\frac{1}{8}}^{1} y_C \ \mathrm{d}x$ $=\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1} \sqrt{1-t^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$ $=\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1} 3t^2\sqrt{1-t^2}\ \mathrm{d}t$ (shown) Let $t=\sin u$, $\frac{\mathrm{d}t}{\mathrm{d}u}=\cos u$. When $t=\frac{1}{2}$, $u=\frac{\pi}{6}$. When $t=1$, $u=\frac{\pi}{2}$.

	$\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1} 3t^2\sqrt{1-t^2}\ \mathrm{d}t$ $=\frac{9\sqrt{3}}{32}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\sqrt{1-\sin^2 u}\ \cos u\ \mathrm{d}u$ $=\frac{9\sqrt{3}}{32}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\ \cos^2 u\ \mathrm{d}u$ $=\frac{9\sqrt{3}}{32}-\frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sin u\cos u\right)^2\ \mathrm{d}u$ $=\frac{9\sqrt{3}}{32}-\frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 2u\ \mathrm{d}u$ $=\frac{9\sqrt{3}}{32}-\frac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\ \mathrm{d}u$ $\frac{9\sqrt{3}}{32}-\frac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\ \mathrm{d}u$ $=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[u-\frac{1}{4}\sin 4u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\ \mathrm{d}u$ $=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[\left(\frac{\pi}{2}-\frac{1}{4}\sin 2\pi\right)-\left(\frac{\pi}{6}-\frac{1}{4}\sin\frac{2\pi}{3}\right)\right]$ $=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[\frac{\pi}{2}-\frac{\pi}{6}+\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right]$ $=\frac{9\sqrt{3}}{32}-\frac{3}{8}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right]$ $=\frac{9\sqrt{3}}{32}-\frac{3\sqrt{3}}{64}-\frac{\pi}{8}$ $=\frac{15\sqrt{3}}{64}-\frac{\pi}{8}$

❄❄❄❄❄❄❄❄ End of Paper ❄❄❄❄❄❄❄❄