Hypothesis Testing

1

X = mass of a packet of cat food $\sim \mathrm{N}\left(\mu,\ 2.7^2\right)$

Test $\mathrm{H_0}: \mu = 375$

Against $\mathrm{H_1}: \mu \neq 375$

Two-tail at 5% significance level

Test statistic:

Under $\mathrm{H_0}$, $\bar{X} \sim \mathrm{N}\left(375,\ \frac{2.7^2}{100}\right) \Rightarrow Z \sim \mathrm{N}(0,1)$

Reject $\mathrm{H_0}$ if p-value < 0.05.

Observed sample mean $\bar{x} = 375.31$

p-value = $2 \times \mathrm{P}\left(\bar{X} > 375.31\right) = 0.251 \not< 0.05$, $\therefore \mathrm{H_0}$ is not rejected.

There is insufficient evidence at 5% significance level that the **mean** mass of the packets of cat food differs from 375 grams.

Alternatively,

At 5% significance level, the rejection region is $\bar{X} < 374.471$ or $\bar{X} > 375.529$. Since the observed sample mean $\bar{x} = 375.31$ does not fall in either rejection region, $\mathrm{H_0}$ is not rejected.

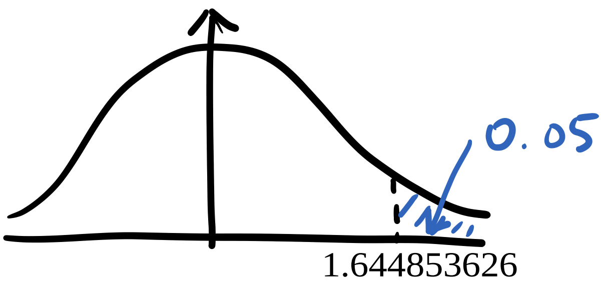
Let Y = mass of a packet of cat food packed by the new machinery $\sim \mathrm{N}\left(\mu,\ 2.7^2\right)$

Test $\mathrm{H_0}: \mu = 375$

Against $\mathrm{H_1}: \mu > 375$

One-tail test at 5% significance level

Under $\mathrm{H_0}$, $\bar{Y} \sim \mathrm{N}\left(375,\ \frac{2.7^2}{100}\right) \Rightarrow Z \sim \mathrm{N}(0,1)$

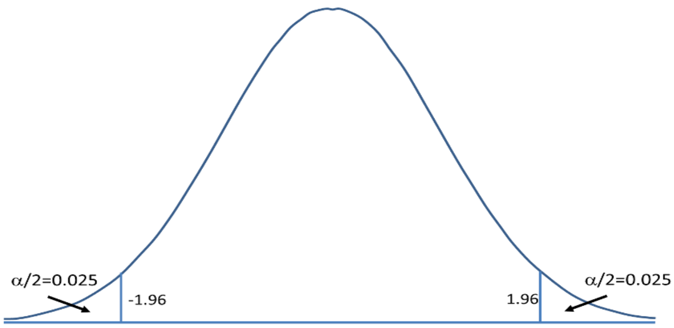


To reject $\mathrm{H_0}$, $z_{calculate} > z_{critical}$

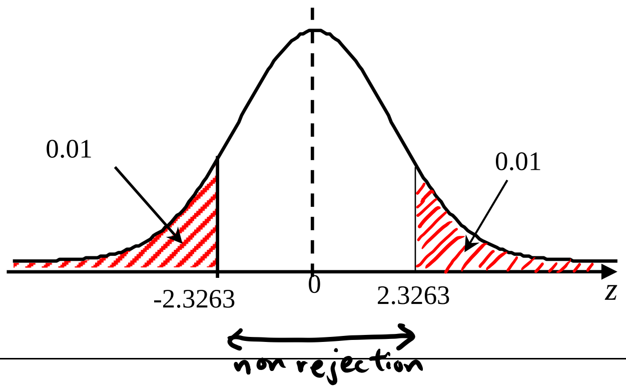
$$\Rightarrow \frac{\bar{y} - 375}{\frac{2.7}{\sqrt{100}}} > 1.644853626$$

$$\Rightarrow \bar{y} > 375.4441105$$

Therefore, $\bar{y} \geq 375.5$ (1 d.p.)

2a(i)	Unbiased estimate of population mean = $\bar{y} = \frac{3072}{600} = 5.12$ Unbiased estimate of population variance $= s^2 = \frac{1}{599}\left(16688 - \frac{3072^2}{600}\right) = 1.601602671 = 1.60 \text{ (3 sf)}$
(ii)	Let μ denote the population mean weight of the bags of multigrain rice. To test $H_0: \mu = 5$ Against $H_1: \mu > 5$ at 2% significance level Under H_0, $Z = \frac{\bar{y} - 5}{\sqrt{\frac{s^2}{600}}} \sim N(0,1)$ where $s^2 = 1.601602671$ *p*-value = 0.0100995741 = 0.0101 (3 sf) < 2%. Reject H_0. There is sufficient evidence at 2% level of significance to conclude that the machine dispenses more than 5 kg of rice.
(iii)	"Machine dispenses more than 5 kg of rice", i.e. reject H_0 $\therefore$ *p*-value < $\frac{\alpha}{100}$ i.e. $\alpha\,\% > 0.0100995741$ The smallest level of significance to reject H_0 is 1.01 %.
b(i)	Let μ denote the population mean weight of the bags of rice. To test $H_0: \mu = 5$ Against $H_1: \mu \neq 5$ at 5% sig level Under H_0, $Z = \frac{\bar{X} - 5}{\sqrt{\frac{0.153^2}{55}}} \sim N(0,1)$ by Central Limit Theorem since *n* is large. Do <u>not</u> reject H_0 so  $-1.95996 < \frac{\bar{x} - 5}{0.153/\sqrt{55}} < 1.95996$ giving $4.96 < \bar{x} < 5.04$ (3 sf)
(ii)	There is NO need for any assumption about the population distribution because Central Limit Theorem is applicable due to the large sample size, to have $\bar{X}$ follow normal distribution.

3(i)	Unbiased estimate of the population mean $= \dfrac{\sum(x-4.5)}{65} + 4.5$ $= \dfrac{21}{65} + 4.5$ $= 4.823076923$ ≈ 4.82 (3 s.f.) Unbiased estimate of the population variance $= \dfrac{1}{65-1}(85 - \dfrac{21^2}{65})$ = 1.222115385 ≈ 1.22 (3 s.f)
(ii)	Test $H_0 : \mu = 4.5$ Against $H_1 : \mu > 4.5$ Level of significance: 1% = 0.01 Test statistic: $Z = \dfrac{\overline{X} - \mu_0}{\dfrac{s}{\sqrt{n}}} \sim \text{N}(0,1)$, where $\overline{x} = 4.8231$, $s = \sqrt{1.22212}$ and n = 65 Using G.C., p-value = 0.00923 < 0.01 Therefore we reject H_0, and conclude that there is sufficient evidence at 1% level of significance that the students from the school spend more time reading magazines as compared to the national average. No assumption needed. This is because the sample size is large and thus by Central Limit Theorem, $\overline{X}$ follows a normal distribution.
(iii)	Test $H_0 : \mu = \mu_0$ Against $H_1 : \mu < \mu_0$ The null hypothesis is rejected at 1% level of significance $\Rightarrow \dfrac{4.8231 - \mu_0}{\sqrt{\dfrac{1.2221}{65}}} < -2.32634$ $\mu_0 > 5.1421$ $\mu_0 > 5.14$ (to 3 s.f.)

4(i)	Unbiased estimate of population mean, $\bar{x} = \frac{10012}{60} = 166.87 = 167$ (3 s.f.) Unbiased estimate of population variance, $s^2 = \frac{6259}{60-1} = 106.08 = 106$ (3 s.f.)
(ii)	Let X denote the breaking strength of the rope (in kN) To test $H_0 : \mu = 169.7$ Against $H_1 : \mu < 169.7$ Conduct 1-tail test at 2% significance level. Under H_0, since $n = 60$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(169.7, \frac{106.08}{60}\right)$ approximately. Using a Z-test, $p\text{-value} = 0.0167$ (3 s.f.) Since $p\text{-value} < 0.02$, we reject H_0 and conclude that there is sufficient evidence at 2% significance level <u>that the mean breaking strength is less than 169.7 kN</u> / <u>that the manufacturer's claim is invalid</u> / <u>to reject the manufacturer's claim</u>
(iii)	It is not necessary to have any assumptions about the population. Since the sample size n is large, the sample mean $\bar{X}$ can be approximated to a normal distribution by Central Limit Theorem.
(iv)	Let Y denote the breaking strength of the rope using the new weaving process (in kN) To test $H_0 : \mu = \mu_0$ Against $H_1 : \mu \neq \mu_0$ Conduct 2-tail test at 2% significance level. Under H_0, since $n = 50$ is large, by Central Limit Theorem, $\bar{Y} \sim N\left(\mu_0, \frac{90.25}{50}\right)$ approximately 

	Given $\bar{y}=171$, and in order not to reject H_0, $-2.3263<\dfrac{171-\mu_0}{\sqrt{\dfrac{90.25}{50}}}<2.3263$ $-3.1254<171-\mu_0<3.1254$ $\Rightarrow 167.87<\mu_0<174.13$ $\therefore$ set of values of μ_0 is: $\{\mu_0\in\mathbb{R}:168<\mu_0<174\}$

5(i)	$\sum(x-150)=-35$ $\left(\sum x\right)-50(150)=-35$ $\sum x=7465$ $\bar{x}=\dfrac{7465}{50}=\dfrac{1493}{10}=149.3$ $s^2=\dfrac{1}{49}(167)=\dfrac{167}{49}=3.40816\approx 3.41(3\text{ sf})$
(ii)	$H_0:\mu=150$ against $H_1:\mu<150$ Test at 5% significance level, $Z=-2.68$ $p=0.00367<0.05$ Reject H_0 and conclude that there is sufficient evidence at 5% significance level, the content of the drinks in the bottle is less than what the packaging claimed to be / the complaints are valid.
(iii)	5% significance level means there is a 5% chance that we say that the complaints are valid when the contents are actually not less than 150ml.
(iv)(a)	If the sample was not randomly chosen, the conclusion made can be unreliable. For example, the sample may have been the first 50 bottles that are produced in the same batch and the mean quantity could have changed as production continued.
(iv)(b)	The conclusion is unaffected whether the distribution is normal or not as this is a large sample. By Central Limit Theorem, the mean quantity in the bottled drink is approximately normally distributed.
(v)	$H_0:\mu=150$ against $H_1:\mu\neq 150$ $p=(0.00367)(2)=0.00734<0.05$ Reject H_0 at 5% significance level that there is sufficient evidence that, on average, the bottled soft drink does not contain 150ml of drinks

6(ai)	Unbiased estimate of population mean,

	$\bar{x} = \frac{\sum x}{n} = \frac{970.02}{90} = 10.778$ Unbiased estimate of population variance, $s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$ $= \frac{1}{89}\left(11326 - \frac{(970.02)^2}{90}\right)$ $= 9.78792 = 9.79\ (3\text{ s.f.})$
(aii)	Let X months be the r.v. denoting the walking age of a toddler and μ months be the population mean walking age. To test $\mathrm{H}_0 : \mu = 11.2$ against $\mathrm{H}_1 : \mu \neq 11.2$ at 5% level of significance Since n is large, under H_0, approximately by Central Limit Theorem, Test statistic: $Z = \frac{\bar{X} - 11.2}{\sqrt{\frac{9.7879}{90}}} \sim N(0,1)$ Using G.C, *p*-value = 0.20067= 0.201 (3s.f) Since *p*-value = 0.201 > 0.05, do not reject H_0 and conclude that there is insufficient evidence to conclude at 5% level of significance that the toddlers' walking ability is not equal to 11.2 months.
(bi)	To test $\mathrm{H}_0 : \mu = 11.2$ against $\mathrm{H}_1 : \mu < 11.2$
(bii)	At 5% level of significance Under H_0, $Z = \frac{\bar{X} - 12}{\frac{10.3}{\sqrt{20}}} \sim N(0,1)$ We do not reject H_0 if z > -1.64485

	$\frac{k-11.2}{\frac{10.3}{\sqrt{20}}} > -1.64485$ $k-11.2 > -1.64485\left(\frac{10.3}{\sqrt{20}}\right)$ $k > 7.4117$ Set of values of k = $\{k \in \mathbb{R} : k > 7.41\}$ (3 s.f.)

7	Let X = weight of students and μ = mean weight $H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$ Under H_0, $\overline{X} \sim N\left(\mu_0, \frac{7^2}{250}\right)$ Test statistics $Z = \frac{\overline{X}-\mu_0}{\sqrt{\frac{49}{250}}} \sim N(0,1)$ $\bar{x} = 70 + \frac{900}{250} = 73.6$ (exact) At 5%, we reject H_0 if $z \leq -1.9600$ or $z \geq 1.9600$. Since we do not reject H_0 $-1.9600 < \frac{73.6-\mu_0}{\sqrt{\frac{49}{250}}} < 1.9600$ $72.7 < \mu_0 < 74.5$ (3 s.f.)

8(i)	Unbiased estimate for population mean is $\bar{x} = \frac{475}{100} = 4.75$ Unbiased estimate for population variance is $s^2 = \frac{1}{99}\left(2439 - \frac{475^2}{100}\right)$ $= 1.8459$ ≈ 1.85
(ii)	H_0: $\mu = 4.4$ H_1: $\mu > 4.4$ Level of significance : 5% Since n = 100 is large, by Central Limit Theorem, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ approximately.

	Test statistic : $Z=\dfrac{\overline{X}-4.4}{s/\sqrt{n}}\sim \mathrm{N}(0,1)$ $\bar{x}=4.75,\ s=\sqrt{1.8459},\ n=100$ From the GC, p-value = 0.0049960. Since p- value <0.05, reject H_o. Sufficient evidence at 5% level of significance to conclude that the mean number of hours a student spends on the research database daily has been understated.
(iii)	As n = 100 is large, Central Limit Theorem applies. There is no need to assume that the population is normally distributed.
	$s^2=\dfrac{100}{99}(3.6875)=3.7247$ H_0: $\mu=4$ H_1: $\mu<4$ Level of significance : 5% Under H_o, $Z=\dfrac{\overline{X}-4}{s/\sqrt{n}}\sim \mathrm{N}(0,1)$ To reject H_o, $z\le -1.6448$. Hence, $\dfrac{\bar{x}-4}{\sqrt{\dfrac{3.7247}{100}}}\le -1.6448$ $\bar{x}\le 3.6825$ Answer: $\bar{x}\le 3.68$ **<u>Alternative</u>** H_0: $\mu=4$ H_1: $\mu<4$ Level of significance : 5% Under H_o, $Z=\dfrac{\overline{X}-4}{s/\sqrt{n}}\sim \mathrm{N}(0,1)$ To reject H_o, $z\le -1.6448$. Hence, $\dfrac{\bar{x}-4}{\sqrt{\dfrac{3.6875}{100}}}\le -1.6448$. $\bar{x}\le 3.6842$ Answer: $\bar{x}\le 3.68$

9(i)	Let $m=x+y$

	$\sum m = 212.8 + 102.6 = 315.4 \qquad \sum m^2 = 490.65 + 230.125 = 720.775$ Unbiased estimate of population mean using combined sample, $\overline{m} = \frac{315.4}{150} = \frac{1577}{750} = 2.1027 = 2.10$ (3.s.f) Unbiased estimate of population variance using combined sample, $s_m{}^2 = \frac{1}{149}\left(720.775 - \frac{315.4^2}{150}\right) = 0.38654 = 0.387$ (3 s.f.)
(ii)	Let M be the mass of a randomly chosen tin of milk powder. Let μ denote the population mean mass of a tin of milk powder. H_0: $\mu = 2$ H_1: $\mu > 2$ Since $n = 150$ is large, by Central Limit Theorem, $\therefore \overline{M} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ approximately. Test statistic: $Z = \frac{\overline{M} - \mu}{S/\sqrt{n}}$ Level of Significance: 5% Reject H_0 if p-value < 0.05 Under H_0, using G.C., p-value = 0.021565 = 0.0216 (3.s.f) Since p-value = 0.0216 < 0.05, we reject H_0 and conclude that at 5 % level of significance, there is sufficient evidence that the mean mass of a tin of milk powder is more than 2 kg. 0.0216 is the probability of obtaining a sample mean greater than or equal to 2.1027kg, assuming that the population mean mass of a tin of milk powder is 2kg.
(iii)	The null hypothesis is still rejected in favour of the alternative hypothesis, as p-value = 2(0.021565) < 0.05. Therefore there is sufficient evidence that the mean mass of a tin of milk powder is not 2 kg.
(iv)	Level of Significance: 5% Reject H_0 if z-value > 1.6449

	$\dfrac{2.1027-\mu_0}{\sqrt{0.38654}/\sqrt{150}} > 1.6449$ $\mu_0 < 2.1027-(1.6449)\left(\sqrt{0.38654}/\sqrt{150}\right)$ $0 < \mu_0 < 2.02$

10(i)	Unbiased estimates of the population mean, $\bar{x} = 100+\dfrac{1}{80}\Sigma(x-100)$ $= 100+\dfrac{1}{80}(-19.2)$ $= 99.76$ Unbiased estimates of the population variance, $s^2 = \dfrac{1}{79}\left(\sum(x-100)^2 - \dfrac{\left(\sum(x-100)\right)^2}{80}\right)$ $= \dfrac{1}{79}\left(129.3-\dfrac{(-19.2)^2}{80}\right)$ $= 1.5784 \approx 1.58$
(ii)	Let X be the random variable denoting the mass of a cupcake. To test $H_0: \mu = 100$ $H_1: \mu < 100$ Significance level: 5% Test Statistics: Under H_0, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(100, \dfrac{1.5784}{80}\right)$ approximately. Rejection criteria: Reject H_0 if p-value < 0.05 $\bar{x}$ = 99.76, n = 80, $s = \sqrt{1.5784}$ Using GC, p-value = 0.043760 Since p-value < 0.05, there is sufficient evidence to reject H_0 at 5% level of significance, i.e. there is sufficient evidence to conclude that the shop has overestimated the mean mass of a cupcake, ie. complaint is valid.
(iii)	Let X be the random variable denoting the mass of a cupcake. To test $H_0: \mu = 100$ $H_1: \mu < 100$ Significance level: 5%

	Test Statistics: Under H_0, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(100, \frac{1.5784}{n}\right)$ approximately. Rejection criteria: Reject H_0 if p-value < 0.05 For the customer's complaint to be not valid, H_0 not to be rejected. p-value $\geq$ 0.05 $\mathrm{P}\left(\bar{X} \leq 99.76\right) \geq 0.05$ $\mathrm{P}\left(\frac{\bar{X}-100}{\left(\sqrt{\frac{1.5784}{n}}\right)} \leq \frac{99.76-100}{\left(\sqrt{\frac{1.5784}{n}}\right)}\right) \geq 0.05$ $\mathrm{P}\left(Z \leq \frac{-0.24}{\left(\sqrt{\frac{1.5784}{n}}\right)}\right) \geq 0.05$ $\frac{-0.24}{\left(\sqrt{\frac{1.5784}{n}}\right)} \geq -1.64485$ $\frac{0.24}{\left(\sqrt{\frac{1.5784}{n}}\right)} \leq 1.64485$ $\sqrt{n} \leq \left(\frac{1.64485}{0.24}\right)\sqrt{1.5784}$ $n \leq 74.139$ Largest possible n is 74.

11(i)	To test H_0: $\mu = 3.6$ against H_1: $\mu < 3.6$ at 1 % level of significance
(ii)	Under H_0, $\bar{X} \sim \mathrm{N}(3.6, \frac{1.2^2}{n})$ Test statistics: $Z = \frac{\bar{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{3.5-3.6}{\frac{1.2}{\sqrt{n}}} = \frac{-0.1}{\frac{1.2}{\sqrt{n}}}$ p-value = P($\bar{X}$ < 3.5) = P(Z < $\frac{-0.1}{\frac{1.2}{\sqrt{n}}}$) < 0.01= P(Z < -2.32635) ,

	Reject H_0, hence $\dfrac{-0.1}{\frac{1.2}{\sqrt{n}}} < -2.32635$. $n > (2.32635 \times 12)^2 = 779.314$ Hence minimum $n = 780$

12(i)	Unbiased estimates for population mean and population variance, $\bar{x} = \dfrac{511.5}{75} = 6.82$ $s^2 = \dfrac{1}{n-1}\left[\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}\right] = \dfrac{1}{74}\left[4027.89 - \dfrac{511.5^2}{75}\right] = 7.29$
12(ii)	Given μ denote mean blood glucose level, $H_0 : \mu = 6.0$ To test : $H_1 : \mu > 6.0$ at 5% level of significance Under H_0, since sample size of 75 is large, using Central Limit Theorem, $\bar{X} \sim N\left(6, \dfrac{7.29}{75}\right)$ approximately, and test statistic, $Z = \dfrac{\bar{X} - 6}{\sqrt{7.29/75}} \sim N(0,1)$ Critical Region: Reject H_0 if $p\text{-value} \le 0.05$ Calculations : Using GC, $z_{cal} = 2.630$ and p-value = 0.00427 Conclusion: Since p-value < 0.05, we reject H_0. There is sufficient evidence, at 5% level of significance, that Natalie's average blood glucose level is higher than 6.0.
12(iii)	Readings at weekend may be biased by different life style, so results may not be valid.
12(iv)	$H_0 : \mu = 6.0$ To test $H_1 : \mu < 6.0$ at 10% level of significance Under H_0, since n = 75 is large by Central Limit Theorem, $\bar{X} \sim N\left(6, \dfrac{s^2}{75}\right)$ $Z = \dfrac{\bar{X} - 6}{\sqrt{S^2/75}} \sim N(0,1)$ approximately, and test statistic, Reject H_0 if $z_{cal} \le -1.55477$ Calculations : $\bar{x} = \dfrac{420}{75} = 5.6$ and using $z_{cal} = \dfrac{5.6 - 6}{s / \sqrt{75}}$

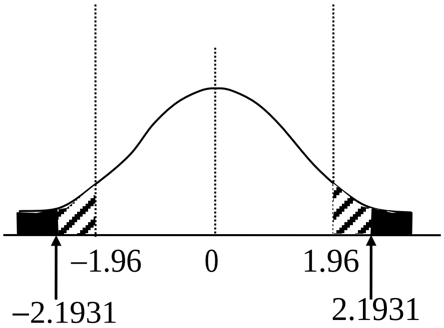
	Since H_0 is rejected, $z_{\text{cal}} = \dfrac{5.6-6}{s/\sqrt{75}} \le -1.55477 \quad \Rightarrow \quad s^2 \le 4.96$ Required range is $s^2 \le 4.96$
12(v)	Since sample sizes are large enough, Natalie may use Central Limit Theorem to approximate the distribution of the sample mean to be normal. Therefore, no need to know anything about the population distribution of the glucose blood levels.

13(i)	Unbiased estimate of the population mean = $\dfrac{438}{50} = 8.76$ Unbiased estimate of the population variance, $s^2 = \dfrac{1}{49}\left(\sum(x-9)^2 - \dfrac{\left(\sum(x-9)\right)^2}{50}\right)$ $= \dfrac{1}{49}\left(45 - \dfrac{\left(\sum x - 50\times 9\right)^2}{50}\right)$ $= \dfrac{1}{49}\left(45 - \dfrac{\left(438-450\right)^2}{50}\right)$ = 0.8595918367 = 0.860 (to 3 s.f.)
(ii)	$H_0 : \mu = 9$ $H_1 : \mu < 9$ Under H_0, the test statistic is $Z = \dfrac{\bar{X} - \mu}{s/\sqrt{n}} \sim N(0,1)$ approximately by CLT, where $\mu = 9$, $s = \sqrt{0.8595918367}$, $\bar{x} = 8.76$ and $n = 50$. Level of significance : 5% = 0.05. From Graphing Calculator, we have $z = -1.8304173$ p-value = 0.03359378 Since the p-value 0. 03359378 < 0.05, we reject H_0. Hence, at 5% level of significance, there is sufficient evidence to conclude that the student council has overstated the average number of hours spent on studying per day by a student.

	Probability of concluding that the mean studying time is less than 9 hours when it is actually 9 hours is 0.05.
	$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$ Under H_0, the test statistic is $Z = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} \sim N(0,1)$ approximately by CLT, where $\mu = \mu_0$, $s = \sqrt{0.8595918367}$, $\bar{x} = 8.76$ and $n = 50$. Level of significance : 5% = 0.05. Critical region : $\|z\| > 1.96$. To reject H_0, test-value $\dfrac{\bar{x} - \mu}{s/\sqrt{n}} < -1.96$ or $\dfrac{\bar{x} - \mu}{s/\sqrt{n}} > 1.96$. $\Rightarrow \mu_0 < 8.50$ or $\mu_0 > 9.02$

14	10 - Let X – breaking strength of a cord. $H_0: \mu = 9$ vs $H_1: \mu > 9$. Under H_0, $\overline{X} \sim N(9, \frac{2.8}{n})$ by CLT. Since σ^2 is known and n is large, we carry out a z-test. z-value $= \dfrac{9.5-9}{\sqrt{\frac{2.8}{n}}} = \dfrac{0.5\sqrt{n}}{\sqrt{2.8}}$ Since at 5% sig level, the batch is accepted, then we must reject H_0. ie $\dfrac{0.5\sqrt{n}}{\sqrt{2.8}} > 1.645 \Rightarrow n > 30.3 \Rightarrow n = 31$

15(i)	Unbiased estimate of μ, $\bar{x} = \dfrac{\sum x}{n} = \dfrac{38100}{40} = 952.50$ Unbiased estimate of $\sigma^2,\ s^2 = \dfrac{1}{39}(731800)$ $= 18764.10$ (2 d.p.)
(ii)	H_0: $\mu = 1000$ H_1: $\mu \neq 1000$ Level of Significance: 5%

	Under H_0, $Z = \dfrac{\overline{X} - 1000}{S/\sqrt{40}} \sim N(0,1)$ approximately by Central Limit Theorem <u>Method 1: *Compare critical region and observed test statistic*</u> Critical region: $\lvert z \rvert > 1.960$ $z = \dfrac{952.5 - 1000}{s/\sqrt{40}} \approx -2.1931$ $\left(\text{where } s = \sqrt{\dfrac{731800}{39}}\right)$ Since $\lvert z \rvert = 2.1931 > 1.960$, we reject H_0.  <u>Method 2: *Using p-value*</u> *p*-value = 0.028299 Since *p*-value = 0.0283 < 0.05, we reject H_0. We conclude that there is sufficient evidence at 5% level of significance that the mean amount of loans borrowed by its clients differs from $1000.
(iii)	The meaning of 'at the 5% significance level' is that there is a probability of 0.05 of rejecting the claim that the <u>**mean**</u> amount of loans borrowed by its clients is $1000 given that it is true. **OR** The meaning of 'at the 5% significance level' is that there is a probability of 0.05 that it was **wrongly concluded** that the <u>**mean**</u> amount of loans borrowed by its clients differs from $1000.
(iv)	Test Statistic: $z = \dfrac{k - 1000}{250/\sqrt{40}}$

	Do not reject $H_0 \Rightarrow -1.96 < z < 1.96$ $-1.96 < \dfrac{k-1000}{250/\sqrt{40}} < 1.96$ $-77.4758 < k - 1000 < 77.4758$ $922.524 < k < 1077.4758$ $922.53 \le k \le 1077.47$

16(i)	$\bar{x} = \dfrac{\sum(x-25)}{80} + 25 = 25.3$ $s^2 = \dfrac{1}{80-1}\left[230 - \dfrac{(24)^2}{80}\right] = 2.8203 \approx 2.82\,(3\text{sf})$
(ii)	An estimate of a population parameter is an unbiased estimate if its expected value is equal to the true value of the population parameter.
(iii)	Test $H_0: \mu = 25$ vs $H_1: \mu > 25$ Under H_0, since n = 80 is large, by Central Limit Theorem, $\bar{X} \sim N\left(25, \dfrac{2.8203}{80}\right)$ approx. Level of significance: 5%. Critical region: $z \ge 1.6449$ Standardized test statistic: $z = \dfrac{\bar{x} - \text{"claimed value"}}{s/\sqrt{n}} = \dfrac{25.3-25}{\sqrt{\dfrac{2.8203}{80}}} = 1.5978$ From GC, p-value = $0.055045 > 0.05$ Since the p-value is more than the level of significance, we do not reject H_o. There is sufficient evidence at 5% level of significance to doubt the owner's claim.

(iv)	Test $H_0 : \mu = 25$ vs $H_1 : \mu > 25$ Standardised Test statistic, $z = \dfrac{\bar{x}-\mu}{\sqrt{\dfrac{\sigma^2}{n}}} = \dfrac{k-25}{\sqrt{\dfrac{2}{100}}}$ Critical region: $z \geq 1.2816$ For fish farm's claim to be valid, H_0 rejected. $\dfrac{k-25}{\sqrt{\dfrac{2}{100}}} \geq 1.2816$ $k \geq 25.18$ (2dp)

17(i)	$s^2 = \dfrac{1}{49}(3969) = 81$
(ii)	The mean of the estimator is equal to the parameter it is estimating, i.e. $E(S^2) = \sigma^2$
(iii)	Null hypothesis, $H_0 : \mu = 250$ Alternative hypothesis, $H_1 : \mu \neq 250$ Under H_0, by Central Limit Theorem, $\bar{X} \sim N\left(250, \dfrac{81}{50}\right)$ approximately For $\alpha = 10$, null hypothesis is rejected $\Rightarrow P(\bar{X} \leq \bar{x}) \leq 0.05$ or $P(\bar{X} \geq \bar{x}) \leq 0.05$ $\Rightarrow \bar{x} \leq 247.906$ or $\bar{x} \geq 252.093$ Hence $\{\bar{x} \in \mathbb{R} : 0 < \bar{x} \leq 247.9$ or $\bar{x} \geq 252.1\}$ (corrected to 1dp)
(iv)	$\sum x = 12641 \Rightarrow \bar{x} = \dfrac{12641}{50} = 252.82$ From GC, p-value = 0.026719 Null hypothesis is not rejected $\Rightarrow p$-value > α % Hence $\{\alpha \in \mathbb{R} : 0 < \alpha \leq 2.67\}$

18	To test $H_0: \mu = 10$ vs $H_1: \mu < 10$ at 5% level of significance Under H_0, using Central Limit Theorem, $Z = \dfrac{\bar{X} - 10}{s/\sqrt{n}} \sim N(0,1)$ approx Reject H_0 if p-value < 0.05 Since $\bar{x} = 10 - \dfrac{25.6}{64} = 9.6$, $s^2 = \dfrac{1}{63}\left[151.99 - \dfrac{(614.4 - 10\times 64)^2}{64}\right] = 2.25$ Thus, p-value = 0.0164 Since p-value < 0.05, there is sufficient evidence at 5% level of significance level to reject H_0, and we conclude that the factory's claim may not be correct. Since sample mean remains unchanged and s^2 decreases, z will decrease. Thus we will still reject H_0.

19(i)	$\bar{x} = 8.31$, $s^2 = \dfrac{1}{79}\left[232.2 - \dfrac{(-15.2)^2}{80}\right] = 2.90$ (3 s.f.)
(ii)	By CLT, $\bar{X} \sim N(8.31, \dfrac{2.9027}{n})$ approximately P($\bar{X} < 8.2$) < 0.3 $\dfrac{8.2 - 8.31}{\sqrt{\dfrac{2.9027}{n}}} < -0.5244$ $-0.11\sqrt{n} < -0.8934$ $\sqrt{n} > 8.1218$ $n > 65.96$ Least value of n is 66

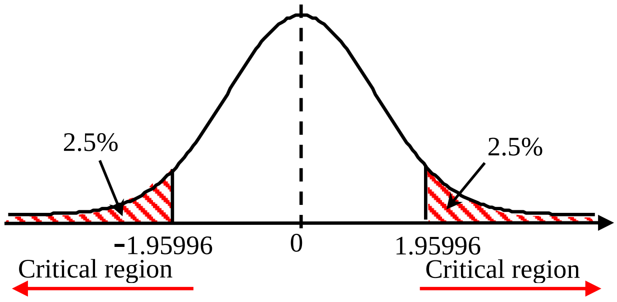
20	Test $H_o: \mu = \mu_0$ (manager's claim) vs $H_1: \mu < \mu_0$ (manager overstated the average time) Test Statistic: Since X is normally distributed, and population variance is known, we use a Z Test. $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ Use a one-tailed test at 6%, and reject H_o if $p < 0.06$. For manager to justify that he has not overstated the average time, i.e. do not reject H_0 at 6% (z calc does not lie in critical region) $z_{calc} = \dfrac{4.2375 - \mu_0}{0.466/\sqrt{8}} > -1.5547736$ $4.2375 - \mu_0 > -1.5547736\left(\dfrac{0.466}{\sqrt{8}}\right)$

	$\mu_0 < 4.493658$ hrs $\mu_0 <$ 4 hrs 29.62 mins Largest possible value of μ_0 is 4 hours 29 mins

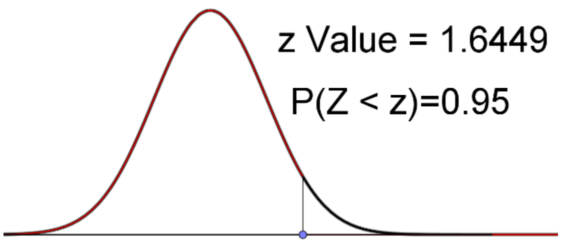
21(i)	$\bar{x} = \frac{980}{100} = 9.8$ $s^2 = \frac{1}{99}\left[9700 - \frac{980^2}{100}\right] = \frac{32}{33}$
(ii)	Let μ be the population mean of the mass of one packet of salt. $H_o : \mu = 10$ $H_1 : \mu < 10$ Since n = 100 is large, Under H_o, $Z = \frac{\bar{x} - 10}{s/10} \sim N(0,1)$ Using GC, the p-value = 0.0211<0.05 Reject the null hypothesis and there is sufficient evidence at 5% significance level to reject the company's claim that the mass is at least 10g.
(iii)	For a 2-tailed test, the p-value = 2(0.0211) = 0.0422 < 0.05. Thus reject the null hypothesis and there is sufficient evidence at 5% significance level that the mass of the packets of salt it packs differs from 10 grams.
(iv)	Let Y be the r.v. of the mass of each packet of salt in the new packaging system. $H_o : \mu = 10$ $H_1 : \mu < 10$ Since n = 60 is large, Under H_o, $Z = \frac{\bar{Y} - 10}{\frac{0.9}{\sqrt{60}}} \sim N(0,1)$ Since company's claim is valid, H_o is not rejected. $\frac{m - 10}{\frac{0.9}{\sqrt{60}}} \geq -2.3263$ $m \geq 9.7297 \approx 9.73$ Least possible value of m is 9.73.

22(i)	Unbiased estimate of population mean

	$\bar{x}=\frac{\sum(x-30)}{50}+30=\frac{-27}{50}+30=29.46$ Unbiased estimate of population variance $s^2=\frac{1}{n-1}\left[\sum(x-30)^2-\frac{\left(\sum(x-30)\right)^2}{n}\right]$ $=\frac{1}{49}\left[167-\frac{(-27)^2}{50}\right]$ $=3.110612245$ $\approx 3.11 \quad \text{or} \quad \frac{7621}{2450}$
(ii)	Let X be the random variable the running time of a random advertisement. Test $\quad H_0 : \mu = 30$ seconds Against $H_1 : \mu \neq 30$ seconds Two-tailed test at 5% level of significance. Under H_0, $\overline{X} \sim N\left(30, \frac{3.1106}{50}\right)$ approximately by Central Limit Theorem. By GC: $\quad p$-value = 0.030388 < 0.05 Since p-value < 0.05, we reject H_0 and conclude that at 5% level there is sufficient evidence that the mean running time differs from 30 seconds.
(iii)	There is a probability of 0.05 that we concluded that the advertisement mean running time differs from 30 seconds when it is actually 30 seconds.
(iv)	Not necessary to assume normal distribution for the running times since n = 50 is large enough for Central Limit Theorem to be used so that sample mean is normally distributed.
(v)	Test $\quad H_0 : \mu = 30$ seconds Against $H_1 : \mu \neq 30$ seconds Two-tailed test at 5% level of significance. Under H_0, $\overline{X} \sim N\left(30, \frac{3.1106}{50}\right)$ approximately by Central Limit Theorem. Test statistic is $Z=\frac{\overline{X}-\mu}{s/\sqrt{n}} \sim N(0,1)$ Value of test statistic is $z=\frac{\bar{x}-30}{\sqrt{3.1106/50}}$

	 If we do not reject H_0, $-1.95996 < z_{test} < 1.95996$ i.e. $-1.95996 < \dfrac{\bar{x}-30}{\sqrt{3.1106/50}} < 1.95996$ i.e. $29.5111 < \bar{x} < 30.4889$ Hence the range of values of sample mean is $29.5 < \bar{x} < 30.5$

23(i)	Let X be the random variable the fuel consumption of a car in litres/100 km. (i) Using GC $\bar{x} = 9.69$, s^2 =0.27651 ≈0.277
(ii)	Test $H_0 : \mu = 9.5$ against $H_1 : \mu > 9.5$ at 5% level of significance. Under H_0, $\bar{X} \sim N(9.5, \dfrac{0.27651}{60})$ approximately by Central Limit theorem since n is large. From GC p- value is 0.00256 < 0.05. H_0 is rejected. Hence there is sufficient evidence to conclude that the manufacturer's claim that the mean fuel consumption is at most 9.5 litres per 100 km is rejected at 5% level of significance.
(iii)	It is not necessary to assume that the distribution is normal as we can apply Central Limit theorem since n is large.
(iv)	The p-value is the probability that the mean fuel consumption is either equal to 9.69 km or more for 100 litres assuming that H_0 is true. OR p- value is the lowest significance level at which the evidence is sufficient to conclude that the mean fuel consumption is more than 9.5 litres/100 km

	Test Statistic $= \dfrac{\bar{x}-9.5}{\sqrt{\dfrac{2.5}{50}}} \geq 1.6449$ $\bar{x} \geq 9.8678 \approx 9.87$ 

24(i)	Unbiased estimate of population mean, $\bar{x} = \dfrac{\sum x}{80} = \dfrac{268}{80} = 3.35$. Unbiased estimate of population variance, $s^2 = \dfrac{1}{79}\left[195\right]$ $= 2.4684$ $= 2.47$ (to 3 s.f.)
(ii)	Let X be the random variable that represents the time, in hours, spent per week outdoors by children in Singapore. To test $\mathrm{H}_0 : \mu = 3$ against $\mathrm{H}_1 : \mu \neq 3$ at 5% level of significance. Since $n = 80$ is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(3, \dfrac{2.4684}{80}\right)$ approximately under H_0 . Test statistic: $Z = \dfrac{\bar{X}-3}{\sqrt{\dfrac{2.4684}{80}}} \sim \mathrm{N}(0,1)$. $z = \dfrac{3.35-3}{\sqrt{\dfrac{2.4684}{80}}} = 1.99$ Using GC, $\mu_0 = 3, s = \sqrt{2.4684}, \bar{x} = 3.35, n = 80, z = 1.99$ p–value $= 0.0463$ (3 s.f.) . Since p–value $= 0.0463 < 0.05$, we reject H_0 and conclude that there is sufficient evidence at 5% significance level that the researcher's claim is not valid.
(iii)	Let Y be the random variable that represents the time, in hours, spent per week outdoors by children in School A.

	To test $\mathrm{H_0}: \mu = 3$ against $\mathrm{H_1}: \mu > 3$ at 5% level of significance. Since $n = 100$ is large, by Central Limit Theorem, $\bar{Y} \sim \mathrm{N}\left(3, \dfrac{1.25^2}{100}\right)$ approximately under $\mathrm{H_0}$ For teacher's claim to be valid, we reject $\mathrm{H_0}$. $z_{test} = \dfrac{m-3}{\sqrt{\dfrac{1.25^2}{100}}} > 1.6449$ $m > 3.2056$ Least value of m = 3.21 (to 2 d.p)

25(i)	$\bar{x} = \dfrac{\sum x}{n} = \dfrac{1590}{100} = 15.9$ $s^2 = \dfrac{1}{n-1}\left(\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}\right) = \dfrac{1}{99}\left(25381.4 - \dfrac{(1590)^2}{100}\right) = 1.01414 \approx 1.01$
(ii)	$\mathrm{H_0}: \mu = 16.0$ $\mathrm{H_1}: \mu < 16.0$ Level of significance: 4% Since n (=100) is large, by **Central Limit Theorem**, $\dfrac{\bar{X} - \mu}{\sqrt{1.01414/100}} \sim N(0,1)$ approximately. If $\mathrm{H_0}$ is true, $\mu = 16$. p-value = 0.160 = 16% > 4% Since p-value > level of significance, $\mathrm{H_0}$ is not rejected. Hence there is insufficient evidence at 4% level of significance to justify the restaurant manager's suspicion.
(iii)	$\mathrm{H_0}: \mu = \mu_0$ $\mathrm{H_1}: \mu \neq \mu_0$ P(Z < – 2.5758) = 0.005 To reject $\mathrm{H_0}$ at 1% level of significance,

	$\Rightarrow z < -2.5758$ or $z > 2.5758$ $\Rightarrow \dfrac{15.9-\mu_0}{\sqrt{\dfrac{1.014}{100}}} < -2.5758$ or $\dfrac{15.9-\mu_0}{\sqrt{\dfrac{1.014}{100}}} > 2.5758$ $\Rightarrow \mu_0 > 16.2$ or $\mu_0 < 15.6$

26(i)	Unbiased estimate of the population mean, $\bar{x} = \dfrac{\sum(x-50)}{100} + 50 = \dfrac{-96}{100} + 50 = 49.04$ **[A1]** Unbiased estimate of the population variance, $s^2 = \dfrac{1}{99}\left[1879 - \dfrac{(-96)^2}{100}\right] = 18.048 = 18.0$ (3 s.f.)
(ii)	The expected value of the 'unbiased estimate' is the population parameter.
(iii)	$H_0 : \mu = k$ $H_1 : \mu > k$ Since σ is unknown, but sample size 100 is large, under H_0, test statistic $Z = \dfrac{\overline{X}-\mu}{s/\sqrt{n}} \sim N(0,1)$ From GC, $z_{critical} = 1.6449$ Since H_0 was rejected, $z_{calc} > z_{critical} = 1.6449$ $\dfrac{49.04-k}{\sqrt{18.048}/10} > 1.6449$ $k < 48.3$
(iv)	No assumptions are necessary since sample size 100 is large, by CLT, mean height of minions in a randomly chosen sample is normally distributed.

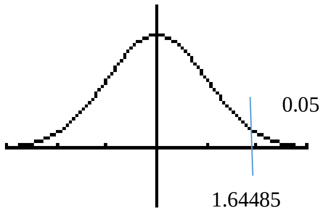
27	Let $w = x-3$, then , $\sum w = 45$, $\sum w^2 = 425$ and Unbiased estimate of μ is $\bar{x} = 3 + \bar{w} = 3 + \dfrac{45}{60} = 3.75$ Unbiased estimate of σ^2 is $s_x^2 = s_w^2 = \dfrac{1}{n-1}\left[\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}\right]$ $= \dfrac{1}{59}\left[425 - \dfrac{(45)^2}{60}\right] = 6.631356 \approx 6.63$
	$\mathrm{H}_0 : \mu = 3$

<table>
<tr><td></td><td>$H_1 : \mu > 3$
Level of significance:5%
Test Statistic: Since n = 60 is sufficiently large, so s_x^2 is a good estimate of σ^2.
$\therefore \bar{X} \sim N\left(3, \frac{s_x^2}{n}\right)$ when H_0 is true. $\therefore Z = \frac{\bar{X}-3}{s_x/\sqrt{n}} \sim N(0,1)$.
Rejection region: $z \geq 1.6449$
Computation: $\bar{x} = 3.75$, $n = 60$, $s_x = \sqrt{6.631356}$
$\therefore z = 2.25598 \approx 2.26$
$p-\text{value} = 0.0120358 \approx 0.0120$
Conclusion: Since p – value = 0.0120 < 0.05, $\therefore H_0$ is rejected at 5% significance level. Hence there is sufficient evidence to conclude that the machine is not working correctly at the 5% significance level.

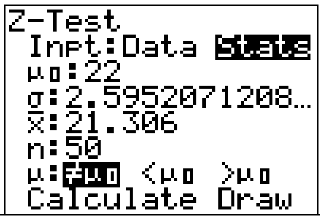
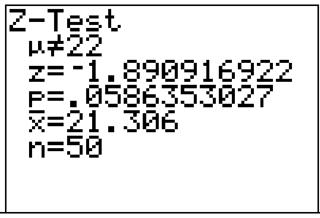
Yes. The test is valid since $n = 60$ is sufficiently large, by Central Limit Theorem, the sample mean length of a nail ($\bar{X}$) is approximately normally distributed.</td></tr>
<tr><td></td><td>If $\sigma = 0.1$, then when H_0 is true, $\bar{X} \sim N\left(3, \frac{\sigma^2}{n}\right)$
P (presuming machine has gone wrong when in fact it is working correctly) = 0.01

$\therefore P\left(\bar{X} > a \text{ when } H_0 \text{ is true}\right) = 0.01$
$\Rightarrow P\left(Z > \frac{a-3}{0.1/\sqrt{n}}\right) = 0.01$
From GC: $P(Z > 2.32635) = 0.01$
$\therefore \frac{a-3}{0.1/\sqrt{n}} = 2.32635$
$\Rightarrow a = 3 + 2.32635\left(\frac{0.1}{\sqrt{n}}\right)$
$\approx 3 + \frac{0.233}{\sqrt{n}}$</td></tr>
</table>

<table>
<tr><td>28(i)</td><td>Unbiased estimate for □□= $\frac{75}{60} + 50$ =51.25
Unbiased estimate for □^{2} = $\frac{1}{59}\left(2016 - \frac{75^2}{60}\right)$ =32.5805
= 32.6</td></tr>
<tr><td>(ii)</td><td>H_0: □□□□□□
H_1: □□□□□□</td></tr>
</table>

	Under H_0, $Z=\dfrac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}} \sim N(0,1)$ approx. by CLT where $\mu=50,\ \bar{x}=51.25,\ s=\sqrt{32.5805},\ n=60$ Level of significance: 5% From GC, p-value = 0.0449. Since p-value < 5%, we reject H_0 and conclude that at the 5% level, there is significant evidence to support the resident's claim.
(iii)	"5% level of significance" means that there is a probability of 0.05 to support the resident's claim when the population mean speed is actually 50 km/h.
(iv)	H_0: □□□□□□□ H_1: □□□□□□□ Under H_0, $Z=\dfrac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ approx. by CLT where $\mu=50,\ \sigma=14,\ n=100$ Level of significance: 5% Reject $H_0 \Rightarrow z > 1.64485$  $\dfrac{c-50}{\left(\frac{14}{\sqrt{100}}\right)} > 1.64485$ $c > 50+1.64485\left(\frac{14}{\sqrt{100}}\right)$ $c > 52.30279$ $c > 52.3$

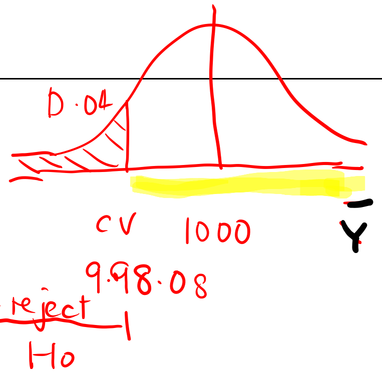
29(i)	An unbiased estimate is an estimate in which the <u>expectation</u> of the estimator is equal to the <u>population parameter</u>.
(ii)	Unbiased estimate for population mean $=\dfrac{\sum(x-18)}{50}+18$ $=21.306$ Unbiased estimate of population variance σ^2 $=s^2=\dfrac{1}{49}\left[\sum(x-18)^2-\dfrac{\left(\sum(x-18)\right)^2}{50}\right]$ $=6.7351$ $=6.74$ (3 s.f.)
(iii)	Let X be the mass of each bag of beans produced in Factory A.

	To test $H_0 : \mu = 22$ Against $H_1 : \mu \neq 22$ Conduct a Two-tailed test at 5% level of significance: Under H_0, since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim N\left(22, \frac{6.7351}{50}\right)$ approximately Using a Z-test, *p*-value = 0.0586 > 0.05 Since *p*-value > 0.05, we do not reject H_0 and conclude that there is insufficient evidence at the 5% level of significance that the claim is not valid / that the mean mass of each bag of beans is not 22 kg. 
(iv)	*p*-value is the lowest level of significance for which the null hypothesis of the mean mass of the bag of beans of 22 kg will be rejected.

30(i)	Let X be the volume of detergent in a randomly selected bottle, and μ the population mean. Ho: $\mu = 1000$ [claim $\mu \geq 1000$] H_1: $\mu < 1000$ Under Ho, and since n=100 >50 is large, by **CLT** $\overline{X} \sim N\left(1000, \frac{120.1}{100}\right)$ Perform one-tail test at 4% significance level. Reject Ho if $p \leq 0.04$. $p = P(\overline{X} < 998) = 0.0340$. Since p< 0.05, we reject Ho. There is **sufficient evidence** at 5% significance level to conclude that the manager's claim is **not valid.**
(ii)	Let Y be the volume of detergent in a randomly selected bottle from this batch, and μ the population mean. Ho: $\mu = 1000$ H_1: $\mu < 1000$ [manager's suspicion] at 4% Under Ho, by CLT, $\overline{Y} \sim N\left(1000, \frac{120.1}{100}\right)$

First find critical value C: $P(\overline{Y} < C) = 0.04$

$C = 998.081$



Suspicion not justified (do not reject Ho)

$\overline{y} = \frac{k}{100} > 998.081$.

$k > 99808.1$

Smallest integer value of k is 99809

(iii) Let W be the volume of a randomly selected bottle of deluxe range of detergent, and μ the population mean.

Ho: $\mu = \mu_0$ [manager's claim]

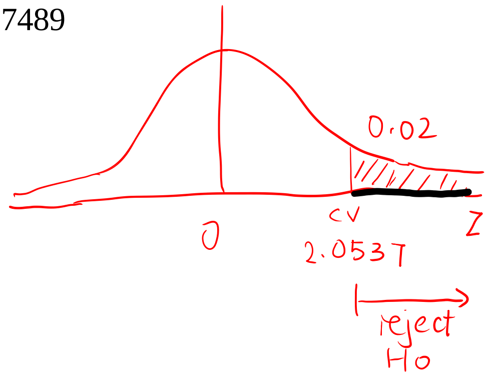
H1: $\mu > \mu_0$ [understating] at 2%

Under Ho, $\overline{W} \sim N\left(\mu_0, \frac{10.2^2}{60}\right)$ and $Z = \frac{\overline{W} - \mu_0}{\sqrt{\frac{10.2^2}{60}}} \sim N(1,0)$

First find critical value: $P(Z > C_1) = 0.02$

$P(Z < C_1) = 0.98$

$C_1 = 2.0537489$



Manager is understating (Reject Ho)

$z \geq 2.0537489$

$$\frac{690 - \mu_0}{\sqrt{\frac{10.2^2}{60}}} \geq 2.0537489$$

$$690 - \mu_0 \geq 2.0537489\sqrt{\frac{10.2^2}{60}}$$

$$-\mu_0 \geq 2.0537489\sqrt{\frac{10.2^2}{60}} - 690$$

$$\mu_0 \leq -2.0537489\sqrt{\frac{10.2^2}{60}} + 690$$

$$\mu_0 \leq -2.704 + 690$$

$$\mu_0 \leq 687.3$$

Maximum $\mu_0 = 687$

31(ai) unbiased estimates of the population mean $= \frac{\sum x}{n} = \frac{7916}{80} = 98.95$

unbiased estimates of the population variance

$$= \frac{1}{n-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right] = \frac{1}{79}\left[784976 - \frac{(7916)^2}{80}\right] = 21.365 = 21.4$$

(aii)	Let X be the random variable denoting the mass of a randomly chosen mini-pack of chocolates in grams. $H_0 : \mu = 100$ $H_1 : \mu < 100$ at 5% significance level Under H_0, since n = 80 is large, by CLT, $Z = \frac{\overline{X} - 100}{\sqrt{\frac{21.365}{80}}} \sim N(0,1)$ approx. Test statistic, $z = \frac{98.95 - 100}{\sqrt{\frac{21.365}{80}}} = -2.0318$ Use GC, p-value = 0.021086 < 0.05 Conclusion: There is sufficient evidence at 5% sig level that mean mass of mini-packs of chocolate produced by company A is less than 100 grams. No need Because by CLT, the sample mean is normally distributed already.
(b)	Let Y be the random variable denoting the mass of a randomly chosen mini-pack of chocolates in grams produced by company B $H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$ at 5% significance level Under H_0, $\overline{Y} \sim N\left(\mu_0, \frac{7^2}{30}\right)$ **assuming it to be normally distributed,** Test statistic, $Z = \frac{105 - \mu_0}{\sqrt{\frac{49}{30}}}$ For H_0 not rejected, $-1.95996 < \frac{105 - \mu_0}{7/\sqrt{30}} < 1.95996$ $\rightarrow 102 < \mu_0 < 108$ or $103 \leq \mu_0 \leq 107$ Assumption: mass of a randomly chosen packet of chocolates from factory B is normally distributed.

32

Let μ be the population mean length of metal pieces.

$\bar{x} = \frac{390}{48} = 8.125$ and $s^2 = \frac{1}{47}\left(3181 - \frac{390^2}{48}\right) = 0.26064$ (5sf)

H_0: $m = 8$
H_1: $m \neq 8$

Level of significance: 4%

Under H_0, since the sample size n = 48 is large, by Central Limit Theorem, the test statistic: $Z = \dfrac{\overline{X} - 8}{\left(\dfrac{\sqrt{0.26064}}{\sqrt{48}}\right)} \sim N(0,1)$ approximately.

From GC, $z_{\text{cal}} = 1.696$

p-value = 0.0898 > 0.04

Since the p-value is more than the level of significance, we do not reject H_0.
Hence there is **insufficient** evidence at the 4% level of significance to conclude that the mean length of the square metal pieces produced by the machine is **not** 8 cm.

"At the 4% level of significance" means there is a probability of 0.04 that we wrongly conclude that the mean length of metal pieces produced by the machine is not 8 cm when it is indeed 8 cm.

Expected number of inspections that draw wrong conclusion = $12 \times 0.04 = 0.48$

H_0: $m = 8$
H_1: $m > 8$

Given level of significance = 3%, $s = 0.4$ and $n = 80$

Under H_0, since the sample size $n = 80$ is large, by Central Limit Theorem,

the test statistic: $Z = \dfrac{\overline{X} - 8}{\left(\dfrac{0.4}{\sqrt{80}}\right)} \sim N(0,1)$ approximately.

At 3% level of significance, the critical region is $\{z: z \geq 1.8808\}$
Since the machine will not be replaced, H_0 is not rejected.

$$\frac{\bar{x} - 8}{\left(\dfrac{0.4}{\sqrt{80}}\right)} < 1.8808$$

$$\bar{x} < 8 + 1.8808\left(\frac{0.4}{\sqrt{80}}\right)$$

Hence $0 < \bar{x} < 8.08$

33	(i) 'Keep the recorded values small since they are around 49' or 'give an indication of the variations around the hypothesized mean of 49. (ii) Unbiased estimate of μ, $\bar{x} = \dfrac{\sum(x-49)}{100} + 49$ $= \dfrac{313}{100} + 49$ $= 52.13$

Unbiased estimate of σ^2, s^2

$$= \frac{1}{100-1}\left(\sum (x-49)^2 - \frac{\left[\sum (x-49)\right]^2}{100} \right)$$

$$= \frac{1}{99}\left(23280 - \frac{(313)^2}{100} \right)$$

$$= 225.26$$

$$\approx 225$$

(iii) $H_0 : \mu = 49$

$H_1 : \mu \neq 49$

Under H_0, the test statistic is $Z = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} \sim \mathrm{N}(0,1)$ approximately (by CLT)

where μ = 49, $\overline{x} = 52.13$, n = 100, $s^2 = 225.26$

p- value = 0.037027 ≈ 0.0370

Since the p-value < 0.05 (the significance level), we reject H_0 and conclude that at the 5% level, there is significant evidence that the Exam Board's claim is not valid.

(iv) No assumptions needed since n = 100 is large, by Central Limit Theorem, the distribution of the sample mean marks is approximately normal.

(v) $H_0 : \mu = 49$

$H_1 : \mu > 49$

Under H_0, the test statistic is $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$

where μ = 59.2, n = 50, $\sigma = 11.0$

Critical region: $z > 1.7507$

Since the mean mark of candidates has not increased, we do not reject $H_0 \Rightarrow$

$$\frac{k-\mu}{\sigma/\sqrt{n}} < 1.7507$$

$$\frac{k-49}{11.0/\sqrt{50}} < 1.7507$$

$$k < 51.723$$

$$\{k \in \mathbb{R} : k \leq 51.7\}$$

Qn	Suggested Solution
34(i)	Unbiased estimate of the population mean, $\bar{x}=\dfrac{216}{60}=3.6$ Unbiased estimate of the population variance, $s^2=\dfrac{\sum(x-\bar{x})^2}{60-1}=\dfrac{6.4}{59}=\dfrac{32}{295}$
(ii)	$H_0: \mu = 3.5$ $H_1: \mu > 3.5$ where μ is the population mean time taken Perform 1-tail test at 1% significance level. Under H_0, $\overline{X} \sim \mathrm{N}\left(3.5, \dfrac{8}{4425}\right)$ approximately since sample size of 60 is large. From sample, $\bar{x}=3.6$, $s=\sqrt{(32/295)}$ and sample size $n = 60$. From GC, p-value = 0.00934 (3 s.f.) Since p-value $\le$ 0.01, we reject H_0 and conclude that there is sufficient evidence at 1% significance level that the manager has understated the average time taken.
(iii)	There is a probability of 0.01 we may wrongly conclude that the manager has understated the average time taken, when in fact he has not.
(iv)	$H_0: \mu = 3.5$ $H_1: \mu \neq 3.5$ Perform 2-tail test at 10% significance level. Under H_0, $\overline{X} \sim \mathrm{N}\left(3.5, \dfrac{32}{295(n+60)}\right)$ approximately since sample size = $n+60$ is large. From sample, $\bar{x}=3.55$, $s=\sqrt{(32/295)}$ and sample size = $n+60$. For the manager's claim not to be rejected, $-1.6449<\dfrac{3.55-3.5}{\sqrt{\dfrac{32}{295(n+60)}}}<1.6449$

	Since n is positive, $\sqrt{\frac{295(n+60)}{32}} < 32.898$ Hence $n < 57.4 \Rightarrow$ the largest $n = 57$ **Alternative** $-1.6449 < \frac{3.55-3.5}{\sqrt{\frac{32}{295(n+60)}}} < 1.6449$ [table below] $\Rightarrow$ the largest $n = 57$

n	$\frac{3.55-3.5}{\sqrt{\frac{32}{295(n+60)}}}$
56	1.6351 < 1.6449
57	1.6421 < 1.6449
58	1.6491 > 1.6449

35. Solution:

A sample is random if every fitness tracker battery from the thousands produced on a day has an equal chance of being selected and that the selections are independent of each other.
Let X hour be the lifetime of a fitness tracker battery and μ_x hour be the population mean. $H_0 : \mu_x = 2000$ $H_1 : \mu_x < 2000$ Level of significance: 1% Test statistic: When H_0 is true, $Z = \frac{\bar{X} - 2000}{S/\sqrt{200}} \sim N(0,1)$ approximately Computation: $n = 200$, $\bar{x} = 1995$, sample standard deviation $= 25.5$ $s^2 = \frac{n}{n-1} \times \text{sample variance}$ $= \frac{200}{199} \times 25.5^2$ $= \frac{130050}{199} \approx 653.52$ p-value = 0.0028373 Conclusion: Since $p-\text{value} = 0.00284 < 0.01$, H_0 is rejected at 1% significance level. Hence there is sufficient evidence to conclude that the population mean lifetime is less than 2000 hours, that is, there is sufficient evidence to doubt the company's claim.
Approximation (1)

Since $n = 200$ is large, sample mean lifetime of a fitness tracker battery follows a normal distribution approximately. Approximation (2) s^2 is used as an unbiased estimate of population variance, σ^2
Let Y hours be the lifetime of a second type battery and μ_y hours be the population mean. $\bar{y} = 2012.625$ $H_0 : \mu_y = 2000$ $H_1 : \mu_y > 2000$ Level of significance: 5% Test statistic: $Z = \dfrac{\bar{Y} - 2000}{\frac{\sigma}{\sqrt{8}}} \sim N(0,1)$ For H_0 to be rejected, $z \geq 1.6449$. $\dfrac{2012.625 - 2000}{\frac{\sigma}{\sqrt{8}}} \geq 1.6449$ $\sigma \leq 21.709$ $\sigma^2 \leq 471.28$ $\therefore$ when $\sigma^2 \leq 471$, H_0 is rejected at the 5% level of significance. There is sufficient evidence that the mean lifetime of second type of battery is more than 2000 hours.

Qn	Suggested Solution
36(a)	Let X be the mass of a randomly chosen mooncake. $H_0 : \mu = 150$ $H_1 : \mu < 150$ where μ is the population mean mass of mooncakes. Since sample size of 9 is small, assume X follows a normal distribution. Under H_0, $\bar{X} \sim N\left(150, \dfrac{6.73^2}{9}\right)$ From GC, p-value = 0.186322 = 0.186 (3 s.f.) Since the p-value > 0.1, we do not reject H_0 and conclude that there is insufficient evidence at the 10% significance level that the mean mass of the mooncake is less than 150 g, i.e. insufficient evidence to reject owner's claim.
(b)	Let Y be the working hours of a randomly chosen teacher in the school. $s^2 = \dfrac{n}{n-1}(\text{sample variance}) = \dfrac{50k^2}{49}$ hours^2 $H_0 : \mu = 60$ $H_1 : \mu \neq 60$ Under H_0, $\bar{Y} \sim N\left(60, \dfrac{k^2}{49}\right)$ approximately by Central Limit Theorem since sample size of 50 is large.

	In order to reject H_0, p-value = $2\,P(\overline{Y} \geq 62) \leq 0.05$ From GC (graph), $0 < k \leq 7.14299$ Set of values of k is $\{k \in \mathbb{R} : 0 < k \leq 7.14\}$. **Alternative** In order to reject H_0, $\overline{y}$ must lie within the critical region. i.e, $\overline{y} \geq \overline{y}_{\text{critical}}$ $\therefore \overline{y}_{\text{critical}} \leq 62$ From GC (graph), $0 < k \leq 7.14$ (to 3sf) Set of values of k is $\{k \in \mathbb{R} : 0 < k \leq 7.14\}$.

37 (i)	The branch manager obtains a sampling frame consisting of all the customers of the branch, **numbering all the customers with a distinct number from 1 to *N*.** **Randomly select 80** of these customers by **generating 80 distinct random numbers (using a random number generator)** and **select the corresponding** customers.
(ii)	Let T be the random variable denoting the waiting time of a customer at the branch in minutes and μ be the population mean waiting time. Unbiased estimate of population mean, $\bar{t}$ $= \dfrac{\sum(t-15)}{50} + 15$ $= \dfrac{-60}{50} + 15 = 13.8$ (Exact) Unbiased estimate of population variance, s^2 $= \dfrac{1}{n-1}\left[\sum(t-15)^2 - \dfrac{\left(\sum(t-15)\right)^2}{n}\right]$ $= \dfrac{1}{50-1}\left(1168 - \dfrac{(-60)^2}{50}\right)$ $= 22.367$ $= 22.4$ (to 3 sf)
(iii)	Test $H_0 : \mu = 15$ against $H_1 : \mu < 15$ at 5% significance level.

	Under H_0, since $n = 50 > 30$ is large, $\bar{T} \sim N\left(15, \frac{22.367}{50}\right)$ approximately by Central Limit Theorem. Using a 1-tailed z-test, The test statistic value $\bar{t} = 13.8$ gives z_{calc}= -1.7942 and p-value = 0.036393=0.0364 ≤ 0.05 Since p-value = $0.0364 \le 0.05$, we reject H_0 and conclude that at the 5% level of significance, there is sufficient evidence to conclude that the mean waiting time is less than 15 minutes.
(iv)	The p-value of **0.0364** is the **probability that sample mean waiting time of a customer** in the branch is **at most 13.8 minutes** when the **(population) mean waiting time of a customer in the branch is actually 15 minutes**.
(v)	Test $H_0 : \mu = k$ against $H_1 : \mu \ne k$ at 2% significance level. Under H_0, since $n = 50 > 30$ is large, $\bar{T} \sim N\left(k, \frac{22.367}{50}\right)$ approximately by Central Limit Theorem. Using a 2-tailed z-test, the test statistic $Z = \dfrac{\bar{T} - k}{\sqrt{\frac{22.367}{50}}} \sim N(0,1)$ Given that there is insufficient evidence to reject H_0, $-2.3263 < \dfrac{13.8 - k}{\sqrt{\frac{22.367}{50}}} < 2.3263$ $-1.5559 < 13.8 - k < 1.5559$ $-15.3559 < -k < -12.2441$ $12.2441 < k < 15.3559$ $12.2 < k < 15.4$