VICTORIA JUNIOR COLLEGE

JC 1 WEIGHTED ASSESSMENT 1 2023

H2 Further Mathematics 9649/01

Paper 1 3 hours

Additional Materials: Answer Paper
Graph Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

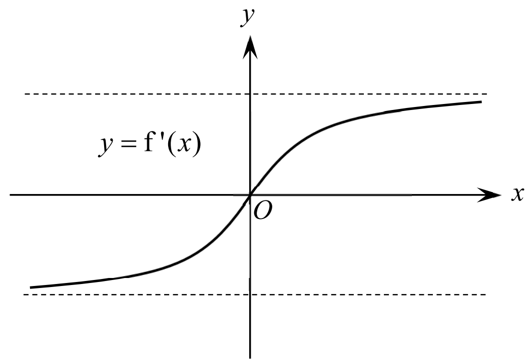
Write your name and CT group on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages

1 The graph of the first derivative of a function f is shown in the diagram below. It is symmetrical about the origin O and approaches the lines $y = 0.5$ and $y = -0.5$ for large values of x. Sketch the graph of $y = \mathrm{f}(x)$ given that it has a pair of asymptotes that intersect at the origin. [3]



2 The terms in the sequence $u_0, u_1, u_2, \ldots$ satisfy the recurrence relation

$$u_{n+2} - u_{n+1} = r(u_{n+1} - u_n),$$

where r is a non-zero constant.

(i) Find the general solution of this recurrence relation. [2]

(ii) Given that $u_0 = 0$ and the sequence converges to a finite value L, find an expression for u_n in terms of L, n and r. State a necessary condition on r. [3]

3 A curve is defined parametrically by $x = \dfrac{t^2}{1+t^2}$, $y = t^3 - \lambda t$, where λ is a positive constant.

(i) Sketch the curve, stating the equation of its asymptote. [2]

(ii) Find in terms of λ, the x-coordinate of the point P where the curve intersects itself. [1]

(iii) Show that the area of the region bounded by the curve between P and the origin is given by an integral of the form

$$4\int_0^{\mathrm{f}(\lambda)} \mathrm{g}(t^2)\ \mathrm{d}t,$$

where $\mathrm{f}(\lambda)$ is a function of λ and $\mathrm{g}(t^2)$ is a function of t^2 to be determined. [5]

4 It is given that the equation $1 + \cos(\pi x) - 2\sqrt{x} = 0$ has a root α in the interval [0, 1].

Use linear interpolation once on the interval [0, 1] to obtain an approximation x_1 to α. [2]

Using x_1 as an initial estimate, apply the Newton-Raphson method to find α, correct to 2 decimal places. [4]

With the help of an appropriate graph, explain how Newton-Raphson method using another initial estimate x_1^* in the interval [0, 1] fails to give an approximation to α. [2]

5 **(a)** For a positive constant a, there is an angle ϕ such that $\sin\phi = a$ and $\frac{\pi}{2} < \phi < \pi$.

Evaluate $\displaystyle\int_{-1}^{0} \frac{1}{\sqrt{1-a^2x^2}}\,\mathrm{d}x$, leaving your answer in terms of a, ϕ and π. [2]

(b) Using the substitution $t = \tan\frac{x}{2}$, show that

$$\int \frac{\cos x}{1+\cos x - \sin x}\,\mathrm{d}x = \int \frac{1+t}{1+t^2}\,\mathrm{d}t.$$

Hence determine $\displaystyle\int \frac{\cos x}{1+\cos x - \sin x}\,\mathrm{d}x$. [6]

6 The curve G has equation $y = \dfrac{x^2 - 2kx + k}{x-k}$, where k is a non-zero constant and $k \neq 1$.

(i) State, in terms of k, the equations of the asymptotes of G. [2]

(ii) Determine the set of values of k for which G has two stationary points. [3]

(iii) Give a sketch of G for $k > 1$, stating in terms of k, the coordinates of the point of intersection of its asymptotes. [1]

(iv) With the help of your sketch in part **(iii)**, determine, in exact form, the value of m $(m < 0)$ such that the line $y = m(x-k)$ is a line of symmetry of G. [3]

~~**7** **(a)** Show that $\mathrm{e}^{\mathrm{i}n\theta} - \mathrm{e}^{-\mathrm{i}n\theta} = 2\mathrm{i}\sin n\theta$, where n is a positive integer. [1]~~

~~**(b)** Show that $\sin^5\theta$ can be expressed in the form~~

$$\cancel{a\sin\theta + b\sin 3\theta + c\sin 5\theta},$$

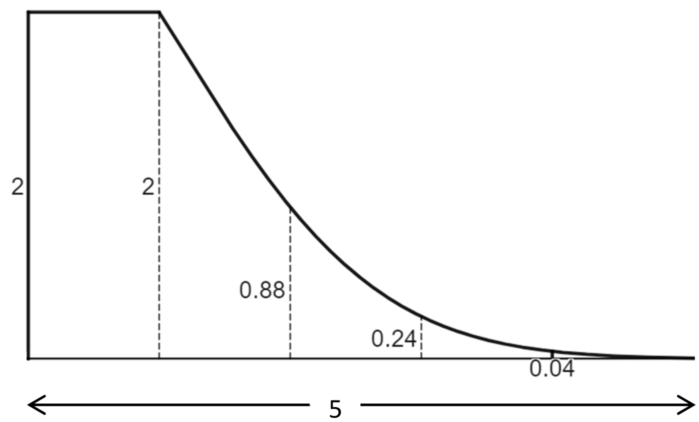
~~where a, b and c are constants to be determined.~~

~~Use this result to deduce a similar expression for $\cos^5\theta$. [6]~~

~~**(c)** By considering $\sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}nx}$, show that~~

$$\cancel{\sin x + \sin 2x + \sin 3x + \cdots + \sin Nx = \operatorname{cosec}\left(\frac{1}{2}x\right)\sin\left(\frac{N}{2}x\right)\sin\left(\frac{N+1}{2}x\right).} \quad \cancel{[4]}$$

~~**8** The diagram (not drawn to scale) shows the cross-section of a skateboard ramp. The ramp is 5 metres long and its height at equal intervals of 1 metre are as shown.~~



~~**(i)** Using all the information in the sketch and trapezium rule, find an estimate for the cross-sectional area of the ramp.~~ [3]

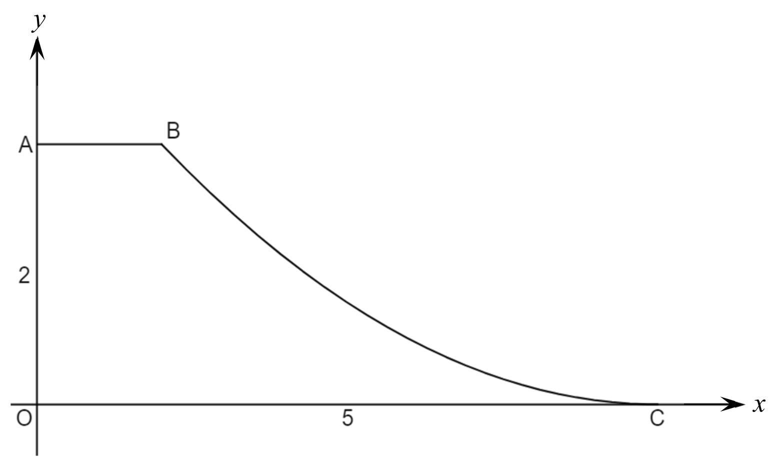
~~**(ii)** The ramp is to be made of concrete and a builder makes the amount of concrete based on the estimate found in part **(i)** multiplied by the width of the ramp. State, with a reason, whether he makes enough concrete to construct the ramp.~~ [1]

~~**(iii)** Repeat part **(i)** using Simpson's rule instead.~~ [3]

~~The function $y(x)$ where~~

$$\cancel{y(x)} = \begin{cases} 2, & 0 \le x \le 1 \\ \mathrm{p}(x), & 1 \le x \le 5 \end{cases}$$

~~is used to model another ramp and its graph is shown in the diagram below.~~



~~**(iv)** Given that the x-axis is tangential to the curve BC at C, write down the quadratic function $\mathrm{p}(x)$.~~ [1]

~~A ramp is obtained by rotating the region bounded by the line AB, curve BC and the axes completely about the y-axis.~~

~~**(v)** Using $\mathrm{p}(x)$ found in part **(iv)**, determine to the nearest integer, the least amount of cement that the builder needs to make to construct the ramp.~~ [3]

9 It is given that $I_n = \int_0^{\pi} \cos^n(2\theta)\,d\theta$, where n is a positive integer.

(i) Without using the calculator, evaluate I_2. [2]

(ii) For $n \geq 3$, show that $I_n = \dfrac{n-1}{n} I_{n-2}$. [5]

(iii) Deduce that for all odd values of n, I_n is independent of n. [1]

(iv) For even values of n, show that

$$I_n = \frac{(n!)\pi}{2^n\left[\left(\frac{n}{2}\right)!\right]^2}.$$ [3]

10 In a membership drive, a fitness club is trying to recruit new members. The sales manager models the number of members that the club has at the end of each month assuming that a certain portion p $(0 < p < 1)$ of its members in the previous month will be lost to competitors, and that it will recruit a constant number, k, of new members in each month.

Let M_n $(n \geq 1)$ be the number of members that the club has n months after the start of the membership drive.

(i) Write down an expression for M_{n+1} in terms of M_n. [1]

(ii) Given that the club has 500 members at end of the first month, determine M_n in terms of n, p and k. [5]

The sales manager sets a target for the club membership to reach 750 at the end of 6 months.

(iii) Given that $k = 80$, show that to meet its target, the club needs to retain approximately 95% of its members, month-by-month. [3]

(iv) Given that the club can only retain 90% of its members, month-by-month, find the least number of members it must recruit each month to meet or exceed its target. [3]

~~**11** **(a)** On the same Argand diagram, sketch the loci of points given by each of the following equations.~~

~~**(i)**~~ $\cancel{|z+3-3i| = 3\sqrt{2},}$

~~**(ii)**~~ $\cancel{\arg\left(z-3\sqrt{2}+3-3i\right) = \frac{5\pi}{6}.}$

~~Find, in the form $x+iy$, the exact complex number represented by the point of intersection of the loci in parts **(i)** and **(ii)**. [6]~~

~~**(b)** Shade on another Argand diagram, the set S, of complex numbers w for which~~

$$\cancel{-\frac{3\pi}{4} \leq \arg\left(\frac{w-3}{2i}\right) \leq 0 \quad \text{and} \quad |w-3| \leq 5.}$$ ~~[5]~~

~~Hence find the greatest and least possible exact values of $|w-3i|$, where $w \in S$. [3]~~