River Valley High School Integrated Programme
2023 JC2 H2 Mathematics (9758)
Lecture Test 2 (Term 1)

Name : ______________________ **Index Number** : ____________

Class : ______________________ **Date** : ____________

Duration : **50 mins** **Max. No. of Marks** : **30**

List of Formulae

Vectors

The point dividing AB in the ratio $\lambda : \mu$ has position vector $\dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$

Vector product:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

[Answer all the questions on writing papers. **Up to 1 mark** will be deducted for poor presentation.]

1. The lines l_1 and l_2 have equations:

$$l_1 : x = \frac{y+2}{2} = z - 1$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R}$$

(i) Verify that the point $A(1, 0, 2)$ lies on l_1. [1]

(ii) Find the position vector of the foot of the perpendicular from A to l_2. Hence find the coordinates of the point of reflection of A about l_2. [5]

(iii) Find the angle between l_1 and l_2. [2]

(iv) Determine whether l_1 and l_2 are intersecting, parallel or skew lines. [3]

2. A baby's toy teaches babies music by having them place lettered balls into boxes. The toy comprises five boxes arranged in a row. Seven balls each with a different letter from {A, B, C, D, E, F, G} written on it, are placed into the boxes. Each box must have exactly one ball in it. There will be two balls not used.

When the "play" button is pushed, the toy will play the notes corresponding to the letters shown on the balls in order.
A baby randomly puts five of the balls into the row of boxes.

Find the probability that

(i) the notes C, E and G are played consecutively, in that order, [3]

(ii) the notes C and G are both played but separated by other notes, [2]

(iii) the notes D, F and A are played given that the five notes are played in alphabetical order. [4]

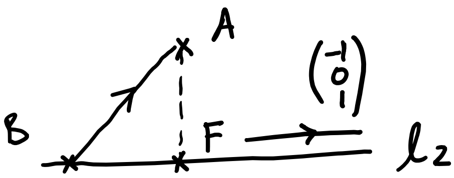
3. The points *A*, *B*, *C*, and *D* are such that *ABCD* is a parallelogram. It is given that $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. The point *P* cuts *BC* such that *BP:BC* is $\lambda : 1$ for some $0 < \lambda < 1$. The point *Q* lies on *AP* produced such that *AP:AQ* is also $\lambda : 1$.

(i) Show that $\overrightarrow{AQ} = \frac{1}{\lambda}\mathbf{a} + \mathbf{b}$. Hence show that *D*, *C* and *Q* are collinear. [3]

(ii) It is given that the area of the parallelogram *ABCD* is equal to the area of the triangle *ADQ*. Find the value of λ. [4]

(iii) Given further that $|\mathbf{a}| = |\mathbf{b}|$, show that $\overrightarrow{AQ}$ is not perpendicular to $\overrightarrow{DQ}$. [3]

~ The End ~

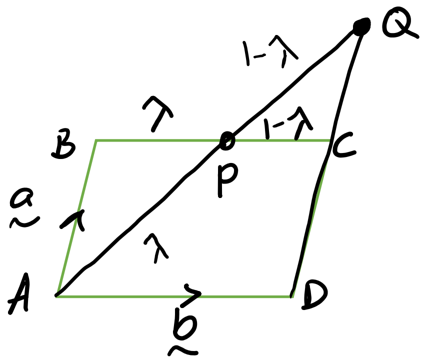
1	Vectors in 3-dimensions [11 marks]	
(i)	Substituting (1, 0, 2) into the Cartesian equation: $(1)=\frac{(0)+2}{2}=(2)-1$ (verified)	
(ii)	 Let B be (5, -5, 0) which lies on l_2 and F be the foot of the perpendicular from A to l_2. Then, $\overrightarrow{BF}=\left(\overrightarrow{BA}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $=\frac{1}{2}\left(\begin{pmatrix}-4\\5\\2\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $=\begin{pmatrix}-3\\0\\3\end{pmatrix}$ $\overrightarrow{OF}=\overrightarrow{OB}+\overrightarrow{BF}$ $=\begin{pmatrix}5\\-5\\0\end{pmatrix}+\begin{pmatrix}-3\\0\\3\end{pmatrix}$ $=\begin{pmatrix}2\\-5\\3\end{pmatrix}$	
	<u>Alternative</u> Let F be the foot of the perpendicular. Since it is a point on l_2, $\overrightarrow{AF}=\overrightarrow{OF}-\overrightarrow{OA}$ $=\begin{pmatrix}5-\beta\\-5\\\beta\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}4-\beta\\-5\\\beta-2\end{pmatrix}$ Since it is perpendicular to l_2, $\begin{pmatrix}4-\beta\\-5\\\beta-2\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}=0$ $\beta-4+\beta-2=0$ $\beta=3$	

	$\overrightarrow{OF}=\overrightarrow{OA}+\overrightarrow{AF}$ $=\begin{pmatrix}1\\0\\2\end{pmatrix}+\begin{pmatrix}1\\-5\\1\end{pmatrix}$ $=\begin{pmatrix}2\\-5\\3\end{pmatrix}$	
	Let A' be the point of reflection of A in l_2 F is the midpoint of AA' $\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OA'}\right)$ $\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$ $\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$ $\overrightarrow{OA'}=2\begin{pmatrix}2\\-5\\3\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}3\\-10\\4\end{pmatrix}$ $A'(3,-10,4)$	
(iii)	Converting l_1 into vector form: $l_1:\mathbf{r}=\begin{pmatrix}0\\-2\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\2\\1\end{pmatrix},\quad \alpha\in\mathbb{R}$ Let θ be the angle between the lines. Then $\cos\theta=\dfrac{\left\|\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}1\\2\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\|}=0$ Therefore, the angle between the lines is 90°.	
(iv)	Since the direction vectors of the lines are not parallel, the lines are either intersecting or skew lines. Solving,	

	$\mathbf{r}=\begin{pmatrix}0\\-2\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}5\\-5\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $(1): \quad \alpha+\beta=5$ $(2): \quad 2\alpha+0\beta=-3$ $(3): \quad \alpha-\beta=-1$ Solving simultaneous, (1) and (3) gives a solution $\alpha=2,\beta=3$ whereas, (2) gives a solution $\alpha=-1.5$. Therefore, the system is inconsistent, and the lines are thus not intersecting. Therefore, they are skew lines.	

2	Probability and Permutations & Combinations [9 marks]	
(i)	Number of ways to fill 5 boxes with 7 balls $=\binom{7}{5}5!$ $\mathrm{P}\left(\text{CEG played in order}\right)=\dfrac{\binom{4}{2}3!}{\binom{7}{5}5!}$ $=\dfrac{(6)(6)}{(21)(120)}=\dfrac{1}{70}$ Alternatively $\mathrm{P}\left(\text{CEG played in order}\right)=\left(\dfrac{1}{7}\right)\left(\dfrac{1}{6}\right)\left(\dfrac{1}{5}\right)(3)$ $=\dfrac{1}{70}$	
(ii)	$\mathrm{P}\left(\text{CG played but separated}\right)=\dfrac{\binom{5}{3}3!\binom{4}{2}2!}{\binom{7}{5}5!}$ $=\dfrac{(10)(6)(6)(2)}{(21)(120)}=\dfrac{2}{7}$ Alternatively $\mathrm{P}\left(\text{CG played but separated}\right)$ $=\dfrac{\begin{bmatrix}\text{Number of ways with C and G in selection}\\ -\text{Number of ways with CG together}\end{bmatrix}}{\binom{7}{5}5!}$ $=\dfrac{\binom{5}{3}(5!)-\binom{5}{3}(4!)(2!)}{\binom{7}{5}5!}$ $=\dfrac{2}{7}$ Alternatively $\mathrm{P}\left(\text{CG played but separated}\right)=\left(\dfrac{2}{7}\right)\left(\dfrac{1}{6}\right)(3)(2!)$ $=\dfrac{2}{7}$	

(iii)	$\mathrm{P}\left(\text{all five alphabetical}\right)=\dfrac{\binom{7}{5}(1)}{\binom{7}{5}(5!)}$ $\mathrm{P}\left(\text{ADF played and all alphabetical}\right)$ $=\dfrac{\binom{4}{2}}{\binom{7}{5}(5!)}$ $\mathrm{P}\left(\text{ADF played} \mid \text{alphabetical}\right)$ $=\dfrac{\mathrm{P}\left(\text{ADF played and all alphabetical}\right)}{\mathrm{P}\left(\text{all alphabetical}\right)}$ $=\dfrac{\binom{4}{2}}{\binom{7}{5}}=\dfrac{2}{7}$	

3	Abstract Vectors (10 marks)	
(i)	 $\overrightarrow{AP} = \lambda\overrightarrow{AC} + (1-\lambda)\overrightarrow{AB}$ $= \lambda(\mathbf{a}+\mathbf{b}) + (1-\lambda)\mathbf{a}$ $= \mathbf{a} + \lambda\mathbf{b}$ $\overrightarrow{AQ} = \dfrac{\lambda+(1-\lambda)}{\lambda}\overrightarrow{AP}$ $= \dfrac{1}{\lambda}\overrightarrow{AP} = \dfrac{1}{\lambda}(\mathbf{a}+\lambda\mathbf{b})$ $= \dfrac{1}{\lambda}\mathbf{a} + \mathbf{b}$ (Shown) $\overrightarrow{DC} = \mathbf{a}$ $\overrightarrow{DQ} = \overrightarrow{DA} + \overrightarrow{AQ}$ $= -\mathbf{b} + \dfrac{1}{\lambda}\mathbf{a} + \mathbf{b}$ $= \dfrac{1}{\lambda}\mathbf{a}$ $= \dfrac{1}{\lambda}\overrightarrow{DC}$ Thus DQ parallel to DC with D as a common point. D, C and Q are collinear.	
(ii)	Area $ABCD =$ Area ADQ $\lvert\mathbf{a}\times\mathbf{b}\rvert = \dfrac{1}{2}\left\lvert\overrightarrow{AD}\times\overrightarrow{AQ}\right\rvert$ $= \dfrac{1}{2}\left\lvert\mathbf{b}\times\left(\dfrac{1}{\lambda}\mathbf{a}+\mathbf{b}\right)\right\rvert$ $= \dfrac{1}{2}\left\lvert\dfrac{1}{\lambda}\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}\right\rvert$ $= \dfrac{1}{2\lambda}\lvert\mathbf{b}\times\mathbf{a}\rvert$ $\lambda = \dfrac{1}{2}$	

(iii)	$\overrightarrow{AQ}\bullet\overrightarrow{DQ}=\left(\frac{1}{\lambda}\mathbf{a}+\mathbf{b}\right)\bullet\left(\frac{1}{\lambda}\mathbf{a}\right)$ $=\frac{1}{\lambda^2}\lvert\mathbf{a}\rvert^2+\frac{1}{\lambda}\mathbf{a}\bullet\mathbf{b}$ $=\frac{1}{\lambda^2}\lvert\mathbf{a}\rvert^2+\frac{1}{\lambda}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ $=\frac{\lvert\mathbf{a}\rvert^2}{\lambda}\left(\frac{1}{\lambda}+\cos\theta\right)$ >0 Since $\frac{1}{\lambda}>1$ and $-1<\cos\theta<1$	
	Alternatively Using the value of $\lambda=\frac{1}{2}$ obtained in part (ii), $\overrightarrow{AQ}\bullet\overrightarrow{DQ}=(2\mathbf{a}+\mathbf{b})\bullet(2\mathbf{a})$ $=4\lvert\mathbf{a}\rvert^2+2\mathbf{a}\bullet\mathbf{b}$ $=4\lvert\mathbf{a}\rvert^2+2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ $=2\lvert\mathbf{a}\rvert^2(2+\cos\theta)$ >0 Since $-1<\cos\theta<1$	