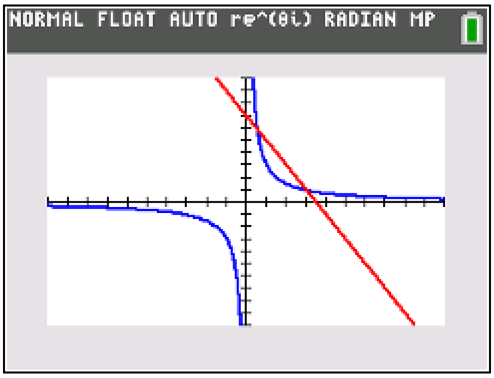
Raffles Institution
H2 Mathematics (9758)
Solution for 2018 A-Level Paper 1

Question 1

No.	Suggested Solution	Remarks for Student
(i)	$y=\dfrac{\ln x}{x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1-\ln x}{x^2}$	
(ii)	$\displaystyle\int_1^{\mathrm{e}} \frac{\ln x}{x^2}\,\mathrm{d}x = \int_1^{\mathrm{e}} \frac{\ln x-1+1}{x^2}\,\mathrm{d}x$ $\displaystyle = \int_1^{\mathrm{e}} \frac{1}{x^2}\,\mathrm{d}x-\int_1^{\mathrm{e}} \frac{1-\ln x}{x^2}\,\mathrm{d}x$ $\displaystyle = \left[-\frac{1}{x}\right]_1^{\mathrm{e}}-\left[\frac{\ln x}{x}\right]_1^{\mathrm{e}}$ $\displaystyle =1-\frac{1}{\mathrm{e}}-\frac{1}{\mathrm{e}}$ $\displaystyle =1-\frac{2}{\mathrm{e}}$	

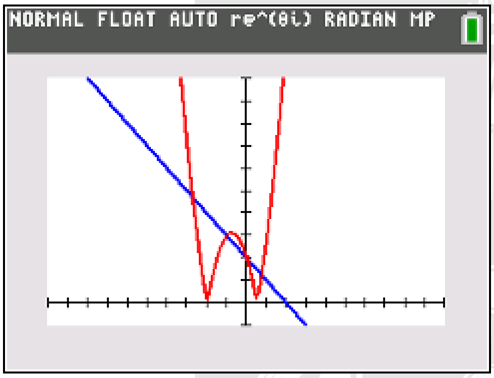
Question 2

No.	Suggested Solution	Remarks for Student
(i)	 Sub $y=\frac{3}{x}$ into $y+2x=7$ $\Rightarrow \frac{3}{x}+2x=7$ $\Rightarrow 2x^2-7x+3=0$ $(2x-1)(x-3)=$ $x=\frac{1}{2},\quad x=3$ $y=6,\quad y=1$	
(ii)	Required volume $=\pi\int_{\frac{1}{2}}^{3}(7-2x)^2-\frac{9}{x^2}\,\mathrm{d}x$ $=\pi\left[-\frac{1}{6}(7-2x)^3\right]_{\frac{1}{2}}^{3}+\pi\left[\frac{9}{x}\right]_{\frac{1}{2}}^{3}$ $=\pi\left[-\frac{1}{6}+36+3-18\right]$ $=\frac{125\pi}{6}$	Note that question said exact and did not mention answer in terms of π

Question 3

No.	Suggested Solution	Remarks for Student
(i)	$x\dfrac{dy}{dx}=2y-6 \quad ...(1)$ $y=ux^2 \quad ...(2)$ $\dfrac{dy}{dx}=x^2\dfrac{du}{dx}+2xu \quad ...(3)$ Sub (2) and (3) into (1): $x\left(x^2\dfrac{du}{dx}+2xu\right)=2ux^2-6$ $x^3\dfrac{du}{dx}=-6$ $\dfrac{du}{dx}=-\dfrac{6}{x^3}$	
(ii)	$u=\dfrac{3}{x^2}+C$ $\dfrac{y}{x^2}=\dfrac{3}{x^2}+C \Rightarrow y=3+Cx^2$ $y=2$ when $x=1$: $2=3+C \Rightarrow C=-1$ $\therefore y=3-x^2$	

Question 4

No.	Suggested Solution	Remarks for Student
(i)	$\lvert 2x^2+3x-2\rvert = 2-x$ $\left(2x^2+3x-2\right)^2 = \left(2-x\right)^2$ $\left(2x^2+3x-2\right)^2 - \left(2-x\right)^2 = 0$ $\left(2x^2+3x-2+2-x\right)\left(2x^2+3x-2-2+x\right)=0$ $\left(2x^2+2x\right)\left(2x^2+4x-4\right)=0$ $x\left(x+1\right)=0$ or $x^2+2x-2=0$ $x=0, x=-1, x=\dfrac{-2\pm\sqrt{12}}{2}$ $=-1\pm\sqrt{3}$	
(ii)	 $-1-\sqrt{3}<x<-1$ or $0<x<-1+\sqrt{3}$	

Question 5

No.	Suggested Solution	Remarks for Student
	$\mathrm{f}: x \mapsto \dfrac{x+a}{x+b}$ for $x \in \mathbb{R}, x \neq -b, a \neq -1$ $\mathrm{g}: x \mapsto x$ for $x \in \mathbb{R}$ Given ff = g, $\dfrac{\dfrac{x+a}{x+b}+a}{\dfrac{x+a}{x+b}+b} = x$ $\dfrac{x+a+ax+ab}{x+a+bx+b^2} = x$ $\dfrac{x(1+a)+a(1+b)}{x(1+b)+a+b^2} = x$ $x(1+a)+a(1+b) = x^2(1+b)+x(a+b^2)$ Comparing coefficients of x^2: $b=-1$ $\mathrm{ff}(x) = x$ $\mathrm{f}^{-1}(x) = \mathrm{f}(x) = \dfrac{x+a}{x-1}$	ff = g, we assume question is just referring to the rule.

Question 6

No.	Suggested Solution	Remarks for Student
	Given $\underset{\sim}{a} \times 3\underset{\sim}{b} = 2\underset{\sim}{a} \times \underset{\sim}{c}$	
(i)	$\underset{\sim}{a} \times (3\underset{\sim}{b} - 2\underset{\sim}{c}) = \underset{\sim}{a} \times 3\underset{\sim}{b} - \underset{\sim}{a} \times 2\underset{\sim}{c}$ $= 2\underset{\sim}{a} \times \underset{\sim}{c} - \underset{\sim}{a} \times 2\underset{\sim}{c}$ $= \underset{\sim}{0}$ $\therefore \underset{\sim}{a} // (3\underset{\sim}{b} - 2\underset{\sim}{c})$ Thus, $3\underset{\sim}{b} - 2\underset{\sim}{c} = \lambda \underset{\sim}{a}$, where λ is a constant.	
(ii)	$\underset{\sim}{b}.\underset{\sim}{c} = \lvert\underset{\sim}{b}\rvert \lvert\underset{\sim}{c}\rvert \cos 60^\circ = 2$ $(3\underset{\sim}{b} - 2\underset{\sim}{c}).(3\underset{\sim}{b} - 2\underset{\sim}{c}) = \lambda\underset{\sim}{a}.\lambda\underset{\sim}{a}$ $9\lvert\underset{\sim}{b}\rvert^2 - 12\underset{\sim}{b}.\underset{\sim}{c} + 4\lvert\underset{\sim}{c}\rvert^2 = \lambda^2 \lvert\underset{\sim}{a}\rvert^2$ $144 - 24 + 4 = \lambda^2 \Rightarrow \lambda = \pm\sqrt{124} = \pm 2\sqrt{31}$	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	$\dfrac{x^2-4y^2}{x^2+xy^2}=\dfrac{1}{2}$ $2x^2-8y^2=x^2+xy^2$ Differentiate with respect to x: $4x-16y\dfrac{dy}{dx}=2x+2xy\dfrac{dy}{dx}+y^2$ $\dfrac{dy}{dx}(2xy+16y)=2x-y^2$ $\dfrac{dy}{dx}=\dfrac{2x-y^2}{2xy+16y}$	
(ii)	When $x=1$, $\dfrac{1-4y^2}{1+y^2}=\dfrac{1}{2}$ $2-8y^2=1+y^2$ $y=\pm\dfrac{1}{3}$ Let P and Q be $\left(1,\dfrac{1}{3}\right)$ and $\left(1,-\dfrac{1}{3}\right)$ respectively, At $P\left(1,\dfrac{1}{3}\right)$: $\dfrac{dy}{dx}=\dfrac{2-\frac{1}{9}}{\frac{2}{3}+\frac{16}{3}}=\dfrac{17}{54}$ Tangent: $y-\dfrac{1}{3}=\dfrac{17}{54}(x-1)$ $\dfrac{17}{54}x-y=-\dfrac{1}{54}$ At $Q\left(1,-\dfrac{1}{3}\right)$: $\dfrac{dy}{dx}=\dfrac{2-\frac{1}{9}}{-\frac{2}{3}-\frac{16}{3}}=-\dfrac{17}{54}$ Tangent: $y+\dfrac{1}{3}=-\dfrac{17}{54}(x-1)$ $\dfrac{17}{54}x+y=-\dfrac{1}{54}$ Solving coordinates of N is $\left(-\dfrac{1}{17},0\right)$	

Question 8

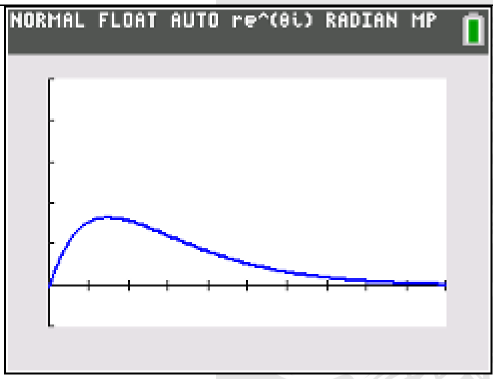
No.	Suggested Solution	Remarks for Student
(i)	$u_1 = 5$ and $u_2 = 15 \Rightarrow 15 = 2(5) + A$ $A = 5$ $u_3 = 2u_2 + 2A = 40$	
(ii)	$u_n = a(2^n) + bn + c$ $u_1 = 5: \quad 2a + b + c = 5 \quad ...(1)$ $u_2 = 15: \quad 4a + 2b + c = 15 \quad ...(2)$ $u_3 = 40: \quad 8a + 3b + c = 40 \quad ...(3)$ Using GC: $a = \frac{15}{2}, b = -5, c = -5$	
(iii)	$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \left(\frac{15}{2}(2^r) - 5r - 5 \right)$ $= \frac{15}{2}\left(\frac{2(2^n - 1)}{2 - 1} \right) - 5\left(\frac{1}{2} n(n+1) \right) - 5n$ $= 15(2^n - 1) - \frac{5}{2} n(n+1) - 5n$	

Question 9

No.	Suggested Solution	Remarks for Student
	$x = 2\theta - \sin 2\theta,\ y = 2\sin^2\theta$ for $0 \le \theta \le \pi$	
(i)	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2 - 2\cos 2\theta,\ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 4\sin\theta\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sin\theta\cos\theta}{2-2\cos 2\theta}$ $= \frac{4\sin\theta\cos\theta}{2-2\left(1-2\sin^2\theta\right)}$ $= \frac{4\sin\theta\cos\theta}{4\sin^2\theta}$ $= \frac{\cos\theta}{\sin\theta}$ $= \cot\theta$	Ok to have 0 and π included though cot is not defined.
(ii)	Point where $\theta = \alpha$ is $\left(2\alpha - \sin 2\alpha,\ 2\sin^2\alpha\right)$ Equation of normal: $y - 2\sin^2\alpha = -\tan\alpha\left(x - 2\alpha + \sin 2\alpha\right)$ At point A, $y = 0$, $-2\sin^2\alpha = -\frac{\sin\alpha}{\cos\alpha}\left(x - 2\alpha + \sin 2\alpha\right)$ $2\sin\alpha\cos\alpha = x - 2\alpha + \sin 2\alpha$ $\therefore x = 2\alpha$, that is, $k = 2$	
(iii)	Total length of $C = \int_0^\pi \sqrt{\left(2-2\cos 2\theta\right)^2 + 4\sin^2 2\theta}\ \mathrm{d}\theta$ $= \int_0^\pi \sqrt{\left(4 - 8\cos 2\theta + 4\cos^2 2\theta\right) + 4\sin^2 2\theta}\ \mathrm{d}\theta$ $= \int_0^\pi \sqrt{\left(8 - 8\cos 2\theta\right)}\ \mathrm{d}\theta$ $= \int_0^\pi \sqrt{8 - 8\left(1 - 2\sin^2\theta\right)}\ \mathrm{d}\theta$ $= \int_0^\pi 4\sin\theta\ \mathrm{d}\theta$ $= -4\left[\cos\theta\right]_0^\pi$ $= -4\left(-1-1\right)$ $= 8$	

Question 10

No.	Suggested Solution	Remarks for Student
	$L\frac{\mathrm{d}I}{\mathrm{d}t}+RI+\frac{q}{C}=V$, where $I=\frac{\mathrm{d}q}{\mathrm{d}t}$	
(i)	Differentiate with respect to t, $L\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{1}{C}\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}t}$ $L\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{I}{C}=\frac{\mathrm{d}V}{\mathrm{d}t}$, since $I=\frac{\mathrm{d}q}{\mathrm{d}t}$ Therefore, $L\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{I}{C}=0$, when $\frac{\mathrm{d}V}{\mathrm{d}t}=0$ that is, V is a constant.	
(ii)	$I=Ate^{-\frac{Rt}{2L}}$ $\frac{\mathrm{d}I}{\mathrm{d}t}=Ae^{-\frac{Rt}{2L}}+Ate^{-\frac{Rt}{2L}}\left(-\frac{R}{2L}\right)=Ae^{-\frac{Rt}{2L}}-\frac{AR}{2L}te^{-\frac{Rt}{2L}}$ $\frac{\mathrm{d}^2I}{\mathrm{d}t^2}=-\frac{AR}{2L}e^{-\frac{Rt}{2L}}-\frac{AR}{2L}e^{-\frac{Rt}{2L}}+\frac{AR^2}{4L^2}te^{-\frac{Rt}{2L}}=-\frac{AR}{L}e^{-\frac{Rt}{2L}}+\frac{AR^2}{4L^2}te^{-\frac{Rt}{2L}}$ Sub into $L\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{I}{C}=0$, $-ARe^{-\frac{Rt}{2L}}+\frac{AR^2}{4L}te^{-\frac{Rt}{2L}}+ARe^{-\frac{Rt}{2L}}-\frac{AR^2}{2L}te^{-\frac{Rt}{2L}}+\frac{A}{C}te^{-\frac{Rt}{2L}}=0$ $-ARe^{-\frac{Rt}{2L}}+ARe^{-\frac{Rt}{2L}}+\frac{AR^2}{4L}te^{-\frac{Rt}{2L}}-\frac{AR^2}{2L}te^{-\frac{Rt}{2L}}+\frac{A}{C}te^{-\frac{Rt}{2L}}=0$ $-\frac{R^2}{4L}+\frac{1}{C}=0\Rightarrow C=\frac{4L}{R^2}$	
(iii)	Note that the given values satisfy (ii), when $R=4$ and $L=3$, from (ii), $C=\frac{4L}{R^2}=\frac{3}{4}$	By right, we should sub R = 4, L = 3 and C = 0.75 into I and check that it still a solution of the DE.

	$I = Ate^{-\frac{2}{3}t}$ $\frac{\mathrm{d}I}{\mathrm{d}t} = 0 \Rightarrow Ae^{-\frac{2}{3}t} - \frac{2A}{3}te^{-\frac{2}{3}t} = 0$ $\Rightarrow t = \frac{3}{2}$ $I = \frac{3A}{2\mathrm{e}}$ Sub into $L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = 0,$ $3\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + 4(0) + \frac{1}{0.75}\frac{3A}{2\mathrm{e}} = 0$ $\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = -\frac{A}{1.5\mathrm{e}} < 0$	
(iv)	 Remember to label the max point $\left(\frac{3}{2}, \frac{3A}{2\mathrm{e}}\right)$	

Question 11

No.	Suggested Solution	Remarks for Student
(i) **(a)**	$100\left(1+\frac{0.2}{100}\right)^{12}=102.4266\approx 102.43$	Given answer in exam report is 102.42
(b)	$100(1.002)+100(1.002)^2+...+100(1.002)^{12}$ $=100\left(\frac{1.002\left(1.002^{12}-1\right)}{1.002-1}\right)$ $=1215.714974$ ≈ 1215.71	
(c)	$S_n=100\left(\frac{1.002\left(1.002^{n}-1\right)}{1.002-1}\right)>3000$ $S_n=50100\left(1.002^n-1\right)>3000$ Using GC, $n=29,\ S_n=2988.6$ $n=30,\ S_n=3094.8$ note that $2988.6+100=3088.6$ So, total will first exceed \$3000 on the first day of June of 2018.	
(ii) **(a)**	$100+12b$	
(b)	$(100+b)+(100+2b)+...+(100+24b)=2800$ $2400+b(1+2+3+...+24)=2800$ $b\left[\frac{24}{2}(1+24)\right]=400$ $b=\frac{4}{3}\approx 1.33$	
(iii)	$100\left(\frac{1.01\left(1.01^{60}-1\right)}{1.01-1}\right)=6000+b\left[\frac{60}{2}(1+60)\right]$ $10100\left(1.01^{60}-1\right)=6000+1830b$ $b=1.2287632\approx 1.23$	