ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

FURTHER MATHEMATICS 9649/02

Paper 2 **14 September 2023**

3 hours

Additional Materials: Cover Sheet
Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [50 marks]

1 Using mathematical induction, prove that

$$\sum_{n=1}^{N} \frac{n+3}{2^n (n+1)(n+2)} = \frac{1}{2} - \frac{1}{2^N (N+2)}$$

for all positive integers N. [5]

Deduce the value of

$$\sum_{n=1}^{\infty} \frac{n+3}{2^n (n+1)(n+2)}.$$ [1]

2 **(i)** Let $V = \mathbb{R}^2$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ such that $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Define addition and scalar multiplication as:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 v_1 \\ u_2 v_2 \end{pmatrix} \text{ and } k\mathbf{u} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$$

where k is a real number.

Explain why V is not a vector space when addition and scalar multiplication are defined in this way. [2]

(ii) Let M be the vector space of all real $n \times n$ matrices with the standard addition and scalar multiplication for matrices.

Determine if the following are subspaces of M:

(a) All $n \times n$ matrices $\mathbf{A}$ such that $\mathbf{AB} = \mathbf{BA}$ for a fixed $n \times n$ matrix $\mathbf{B}$.

(b) All $n \times n$ matrices $\mathbf{A}$ such that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$. [5]

3 It is given that $I = \int_{-2}^{2} 3^x \, \mathrm{d}x$.

(i) Use the trapezium rule with five ordinates to find an approximation to *I*. [2]

(ii) Explain whether the approximation to *I* found in **(i)** is an under-estimate or over-estimate of *I*. [2]

(iii) Use Simpson's rule with five ordinates to find an approximation to *I*. [2]

(iv) Find the exact value of *I*. [2]

(v) Explain whether the trapezium rule or Simpson's rule, used with the same number of ordinates, would give a more accurate approximation to *I*. [2]

(vi) Calculate the absolute percentage error in the approximation obtained using Simpson's rule in **(iii)**. [1]

4 The variables *x*, *y* and *z* are related by the simultaneous differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 2z = 8, \qquad (1)$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} - y + 5z = 16x. \qquad (2)$$

When $x = 0$, $y = z = 0$.

(a) By first differentiating (1), show that the simultaneous differential equations can be reduced to the second-order differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9\frac{\mathrm{d}y}{\mathrm{d}x} + 18y = 32x + 40.$$ [2]

(b) Hence solve the differential equations to find *y* and *z* in terms of *x*. [10]

5 **(a)** Find the roots of the equation $2v^4 = 1 + \mathrm{i}\sqrt{3}$, expressing your answers fully in the form $re^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

Let w denote the root with the smallest positive argument p.

(b) Show that $w^n + \frac{1}{w^n} = 2\cos np$ for any positive integer n, and hence show that

$$\cos^4 p = \frac{7 + 4\sqrt{3}}{16}.$$ [4]

(c) On the same Argand diagram, sketch the loci of points representing the complex number z given by the equations

$$|z| = |w| \quad \text{and} \quad \arg(w - z) = \frac{\pi}{3}.$$

Denote the point representing the complex number w by A. [3]

(d) The point of intersection of the two loci is denoted by P. Find the complex number represented by P and determine if it is a root of the equation in **(a)**. [4]

Section B: Statistics [50 marks]

6 A news agency wishes to predict the results of an election. To do so, they interview a random sample of electors a day before the election. 700 electors out of the sample of 900 electors said they would vote for Candidate *T*.

(a) Find a 90% confidence interval for the proportion of the electorate who would vote for Candidate *T*. [3]

(b) In order to obtain a more accurate estimate of the proportion r of the population who would vote for Candidate *T*, the agency proposed taking a larger random sample. Assuming that the sample proportion is unchanged, estimate the required sample size if it is intended that the resulting 90% confidence interval for r should have a width of 0.04. [3]

7 A student carried out an experiment by tossing four similar coins on a table and recording how many coins landed as heads. He repeated the experiment 200 times and the results he obtained were:

Number of heads	0	1	2	3	4
Frequency	5	35	64	66	30

Let p be the probability that a coin will land as heads. The student believes that a binomial distribution with $p = 0.6$ would be a suitable model for the above data. Carry out a chi-squared test of goodness of fit at the 5% level of significance. [4]

If the experiment is repeated 1000 times instead and the proportion for each number of heads is the same as above, comment on whether the result of a chi-squared test of goodness of fit at the 5% level of significance would change. [2]

8 To assess the difference in wear on the two tyres of a motorcycle, 8 motorcycles, each initially with new tyres, were ridden for 1000 km. After this time, the depth of tread on each tyre was measured and recorded. A tyre that has less wear has a deeper depth of tread on it. The results are shown in the table below.

Motorcycle	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Depth of tread on front tyre (mm)	2.4	1.5	2.3	2.4	2.6	2.5	2.1	2.6
Depth of tread on rear tyre (mm)	2.3	1.9	2.1	1.8	1.8	2.8	1.4	2.1

(a) To test the hypothesis that there is no difference in the average wear for the front and rear tyres, explain why a paired-sample *t*-test may not be appropriate. [1]

(b) Test if there is a difference in the average wear for the front and rear tyres at the 10% level of significance, using the Wilcoxon signed-rank test. [4]

(c) It was later discovered that there were two mistakes in the record. The first mistake was that the depths of tread on the front and rear tyres of Motorcycle *F* were swapped. The second mistake was that the depth of tread of the front tyre of Motorcycle *B*, denoted by *b*, was recorded wrongly. Given that there were no changes to the ranks, find the range of values of *b* such that the Wilcoxon signed-rank test, at the 5% level of significance, results in the conclusion that the front tyre has less average wear than the rear tyre. [2]

9 Let X be a continuous random variable with probability density function

$$\mathrm{f}(x)=\begin{cases}4x^3 & 0<x<1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $Y=1-5X^2$.

(a) Find the probability density function of Y. [6]

(b) Find the expectation of Y. [2]

10 A marketing company wants to ensure it provides sufficient bandwidth to handle a certain number of survey responses submitted on its website. The website receives an average of 20 responses per day.

(i) State the conditions under which the number of responses received per day can be well-modelled by a Poisson distribution. [2]

Assume that these conditions in **(i)** hold in a randomly chosen week.

(ii) Find the probability that the marketing company received the first response in the 3rd hour of the first day. [2]

(iii) Find the probability that in one day, the marketing company receives 16 to 18 responses, given that it receives a total of 30 responses over 2 days. [4]

The random variable T denotes the time, in minutes, between receiving two consecutive responses.

(iv) State the probability density function of T. [1]

(v) Find the greatest integer value of n such that there is a probability of at least 0.3 that there are no responses received in n consecutive minutes. [2]

11 There are two weighing machines, *A* and *B*, which are used to measure the masses of packets of rice produced in a factory. An engineer discovers that Machine *A* is faulty and suspects that it is over-stating the mass of the packets. He checks Machine *B* and finds that it is working properly. We assume that the masses of the packets are normally distributed.

To test Machine *A*, the engineer suggests that a certain small number of packets is selected randomly from a large batch produced on a certain day. These packets are weighed using Machine *A*. A hypothesis test is performed to determine if the mean mass of the contents reported by Machine *A* is greater than the mass stated on the packets.

(i) Identify the appropriate hypothesis test for the engineer's proposal. [1]

A sample of eight packets is selected at random from a large batch and measured using Machine *A*. The masses of the packets of rice, *x* g, reported by Machine *A*, are

1998.5, 2000.4, 1999.9, 2005.8, 2011.5, 2007.6, 2001.3, 2002.4.

(ii) A hypothesis test, performed at the α% level, found that there was significant evidence that the mean mass of the packets reported by Machine *A* is greater than 2000 g. Find the minimum value of α. [4]

Another sample of eight packets is selected at random from another large batch and their masses were measured using Machine *B*. The masses, *y* g, are summarised by

$$\sum(y-2000)=33.1, \qquad \sum(y-2000)^2=266.99 .$$

(iii) State the appropriate hypothesis test to determine whether there is significant evidence, at the 5% level, that there is any difference between the mean masses of the packets in the two batches. Carry out the test and state any assumption required for the test. [6]

(iv) Give a possible explanation why the result of the test in **(iii)** could contradict the engineer's finding that Machine *A* is faulty. [1]