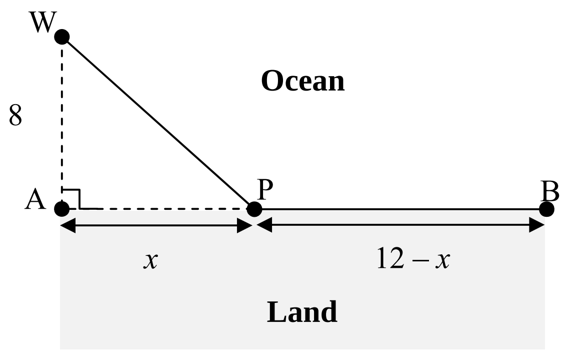
## Section A: Pure Mathematics [40 marks]

**1**




Let $x$ be the distance between $A$ and $P$.

$PW = \sqrt{8^2 + x^2} = \sqrt{64 + x^2}$

Cost of laying the pipe, $C = 60000\sqrt{64 + x^2} + 45000(12 - x)$

$$\frac{\mathrm{d}C}{\mathrm{d}x} = 0$$

$$\frac{60000x}{\sqrt{64 + x^2}} - 45000 = 0$$

$$60000x = 45000\sqrt{64 + x^2}$$

$$4x = 3\sqrt{64 + x^2}$$

$$16x^2 = 9\left(64 + x^2\right)$$

$$x^2 = \frac{576}{7}$$

$$x = \sqrt{\frac{576}{7}} \quad \text{(reject } -\sqrt{\frac{576}{7}} \text{ as } x > 0\text{)}$$

To show that $C$ is minimum when $x = \sqrt{\frac{576}{7}}$ :

**<u>Method 1: First Derivative Test</u>**

| $x$ | $x = \left(\sqrt{\frac{576}{7}}\right)^-$ | $x = \sqrt{\frac{576}{7}}$ | $x = \left(\sqrt{\frac{576}{7}}\right)^+$ |
|---|---|---|---|
| $\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{60000x}{\sqrt{64 + x^2}} - 45000$ | $< 0$ | $0$ | $> 0$ |

$\therefore C$ is a minimum when $x = \sqrt{\frac{576}{7}}$ .

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| | **Method 2: Second Derivative Test**<br>$\dfrac{\mathrm{d}^2C}{\mathrm{d}x^2} = \dfrac{60000\sqrt{64+x^2} - \dfrac{60000x^2}{\sqrt{64+x^2}}}{64+x^2}$<br>$= \dfrac{60000\left(64+x^2\right)-60000x^2}{\left(64+x^2\right)^{\frac{3}{2}}} = \dfrac{3840000}{\left(64+x^2\right)^{\frac{3}{2}}} > 0$ for all $x$<br>$\therefore C$ is a minimum when $x=\sqrt{\dfrac{576}{7}}$. |

| | | |
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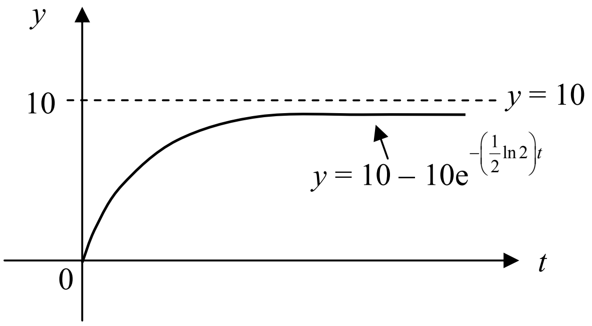
| **2** | **(i)** | $\dfrac{\mathrm{d}y}{\mathrm{d}t} = k(10-y)$<br>$\int \dfrac{1}{10-y}\ \mathrm{d}y = \int k\ \mathrm{d}t$ |
| | | $-\ln\lvert 10-y\rvert = kt + c$ |
| | | $\lvert 10-y\rvert = \mathrm{e}^{-kt-c}$<br>$10-y = \pm\mathrm{e}^{-kt-c}$<br>$10-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-kt} = A\mathrm{e}^{-kt}$, where $A = \pm\mathrm{e}^{-c}$<br>$y = 10 - A\mathrm{e}^{-kt}$ |
| | | When $t = 0$, $y = 0 \Rightarrow A = 10$ |
| | | When $t = 2$, $y = 5 \Rightarrow 5 = 10 - 10\mathrm{e}^{-2k} \Rightarrow 10\mathrm{e}^{-2k} = 5$<br>$k = \dfrac{1}{2}\ln 2$ |
| | | $\therefore\ y = 10 - 10\mathrm{e}^{-\left(\frac{1}{2}\ln 2\right)t}$ |
| | **(ii)** |  |
| | | The amount of material memorised tends to 10 units. |
| | **(iii)** | $\dfrac{\mathrm{d}y}{\mathrm{d}t} = k(10-y) - \alpha y,\ \alpha > 0$ |

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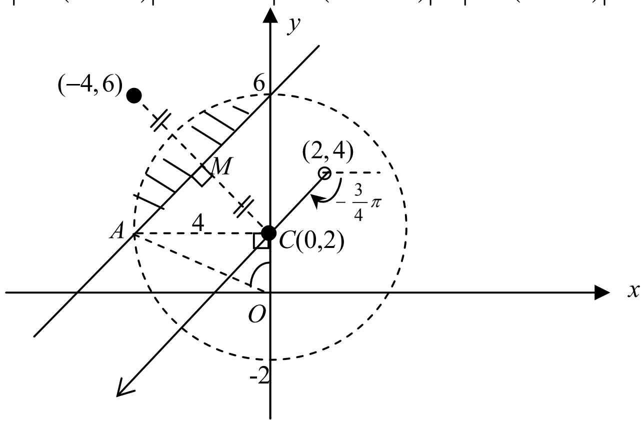
| **3 (i)** | $\lvert z-(0+2\mathrm{i})\rvert<4$ and $\lvert z-(-4+6\mathrm{i})\rvert \le \lvert z-(0+2\mathrm{i})\rvert$   |
| **(ii)** | $\angle AOC=\tan^{-1}\left(\frac{4}{2}\right)=1.1071$ rad |
| | Least possible arg $z=\frac{\pi}{2}$<br>Largest possible arg $z=\frac{\pi}{2}+1.1071=2.68$<br>$\therefore \frac{\pi}{2}<\arg z<2.68$ |
| **(iii)** | $\arg\left(w-(2+4\mathrm{i})\right)=\arg(-1-\mathrm{i})=-\frac{3}{4}\pi$ |
| | $M\left(\frac{0-4}{2},\frac{2+6}{2}\right)\Rightarrow M(-2,4)$ |
| | Least value of $\lvert z-w\rvert$ = $CM=\sqrt{(-2-0)^2+(4-2)^2}$ |
| | $=2\sqrt{2}$ |

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| **4 (i)** | $n_2=\begin{pmatrix}-1\\1\\1\end{pmatrix}\times\begin{pmatrix}4\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\4\\-5\end{pmatrix}$ |
| | Acute angle between $\Pi_1$ and $\Pi_2=\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}1\\-2\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\4\\-5\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}-1\\4\\-5\end{pmatrix}\right\rvert}$ |
| | $=\cos^{-1}\left(\frac{14}{\sqrt{6}\sqrt{42}}\right)$<br>$=28.1^\circ$ (to 1 dp) |

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| **(ii)** | $\begin{pmatrix}3\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\4\\-5\end{pmatrix}=-12$<br><br>$\therefore \Pi_2:\ \mathbf{r}\bullet\begin{pmatrix}-1\\4\\-5\end{pmatrix}=-12 \Rightarrow -x+4y-5z=-12$ |
| | $\Pi_1 :\ x-2y+z=4$<br>$\Pi_2 :\ -x+4y-5z=-12$<br>Using GC, eqn of $l_1$ is $\mathbf{r}=\begin{pmatrix}-4\\-4\\0\end{pmatrix}+\alpha\begin{pmatrix}3\\2\\1\end{pmatrix}$, where $\alpha\in\mathbb{R}$. |
| **(iii)** | Given $\overrightarrow{OA}=\begin{pmatrix}6\\3\\-5\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\overrightarrow{AB}=\begin{pmatrix}2\\3\\1\end{pmatrix}-\begin{pmatrix}6\\3\\-5\end{pmatrix}=\begin{pmatrix}-4\\0\\6\end{pmatrix}$<br>Length of projection of $\overrightarrow{AB}$ onto the line $l_1$<br>$=\dfrac{\left\lvert\begin{pmatrix}-4\\0\\6\end{pmatrix}\bullet\begin{pmatrix}3\\2\\1\end{pmatrix}\right\rvert}{\sqrt{3^2+2^2+1^2}}$ |
| | $=\dfrac{\lvert-6\rvert}{\sqrt{14}}$<br>$=\dfrac{3\sqrt{14}}{7}$ |
| | $\Pi_1 :\ \mathbf{r}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=4$<br><br>$l_2 :\ \mathbf{r}=\begin{pmatrix}p\\2p+1\\-3\end{pmatrix}+t\begin{pmatrix}3q\\-3\\q\end{pmatrix}$<br><br>Let $D$ be a point on the plane $\Pi_1$.<br>Since $\begin{pmatrix}4\\0\\0\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=4$, $D$ is (4, 0, 0).<br>$\therefore\ \overrightarrow{DC}=\begin{pmatrix}p\\2p+1\\-3\end{pmatrix}-\begin{pmatrix}4\\0\\0\end{pmatrix}=\begin{pmatrix}p-4\\2p+1\\-3\end{pmatrix}$ |

| | |
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| | Perpendicular distance from $C$ to $\Pi_1$<br>= Length of projection of $\overrightarrow{DC}$ onto the normal of $\Pi_1$<br>$= \dfrac{\left\| \overrightarrow{DC} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} = \dfrac{\left\| \begin{pmatrix} p-4 \\ 2p+1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|}{\sqrt{6}} = \dfrac{15}{\sqrt{6}}$ |
| | $\left\|(p-4)-2(2p+1)-3\right\| = 15$<br>$\left\|-3p-9\right\| = 15$<br>$-3p-9=15$ or $-3p-9=-15$<br>$p=-8$ (rej $\because p>0$) $p=2$ |
| | Acute angle between $l_2$ and $\Pi_1$ $= \sin^{-1} \dfrac{\left\| \begin{pmatrix} 3q \\ -3 \\ q \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} 3q \\ -3 \\ q \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} = \sin^{-1}\left(\dfrac{2}{\sqrt{6}}\right)$ |
| | $\dfrac{3q+6+q}{\sqrt{9q^2+9+q^2}\sqrt{6}} = \dfrac{2}{\sqrt{6}}$ $\left(\left\|3q+6+q\right\| = 3q+6+q \text{ since } q>0\right)$ |
| | $4q+6 = 2\sqrt{10q^2+9}$<br>$(4q+6)^2 = 4\left(10q^2+9\right)$<br>$q^2-2q=0$<br>$q=2$ (since $q>0$) |

## Section B: Statistics [60 marks]

**5**

Systematic sampling of every 30$^{th}$ employee arranged in alphabetical order by name may result in too many employees chosen from a particular age group.

The method is stratified sampling.

To carry out a stratified sample, stratify the 1500 employees into the 3 age groups. Take a random sample from each stratum with sample size proportional to the relative size of the stratum:

| Age group | 21-40 | 41-60 | 60 and above |
|---|---|---|---|
| No. of employees | $35\% \times 50 = 17.5 \approx 18$ | $50\% \times 50 = 25$ | $15\% \times 50 = 7.5 \approx 7$ |

**6 (i)**

ENDANGERED: 3E, 2N, 2D, 1A, 1G, 1R

No of 4-letter code-words from E, N, D, A, G, R

$= {}^6C_4 \times 4!$

$= 360$

**(ii)**

Select 1 letter from N, D, A, G, R.

No of 4-letter code-words with the chosen letter and 3 "E"s

$= {}^5C_1 \times \dfrac{4!}{3!}$

$= 20$

**(iii)**

Case I: 3 same letters

No. of such code-words = 20

Case II: 1 pair of same letters

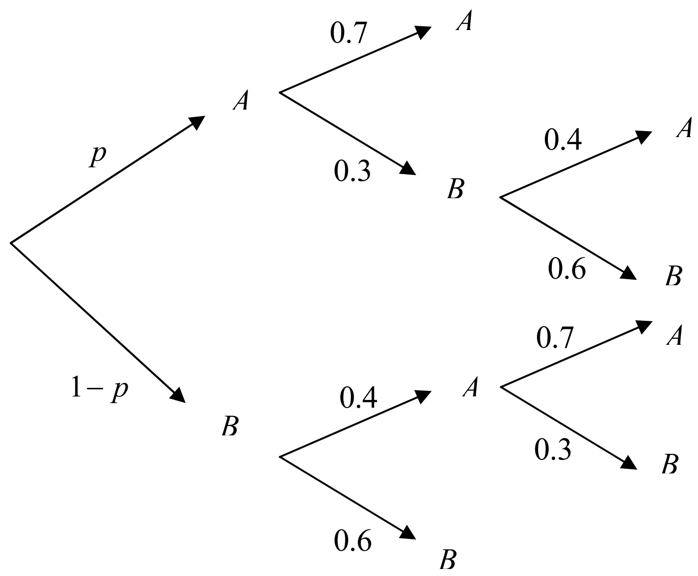
No. of such code-words $= {}^3C_1 \times {}^5C_2 \times \dfrac{4!}{2!} = 360$

Case III: 2 pairs of same letters

No. of such code-words $= {}^3C_2 \times \dfrac{4!}{2!2!} = 18$

No. of 4-letter code-words that contain at least 1 repeated letter

$= 20 + 360 + 18 = 398$

| 7 |  |
|---|---|
| **(i)** | For a 3 set match,<br>probability of *A* winning = probability of *B* winning<br>$p(0.3)(0.4)+(1-p)(0.4)(0.7)=p(0.3)(0.6)+(1-p)(0.4)(0.3)$ |
| | $0.12p+0.28-0.28p=0.18p+0.12-0.12p$<br>$0.22p=0.16$<br>$p=\frac{8}{11}$ |
| **(ii)** | $P(B \text{ wins the first set} \mid A \text{ wins the match})$<br>$=\frac{P(B \text{ wins the first set } \cap A \text{ wins the match})}{P(A \text{ wins the match})}$<br>$=\frac{\left(\frac{3}{11}\right)(0.4)(0.7)}{\left(\frac{8}{11}\right)(0.7)+\left(\frac{8}{11}\right)(0.3)(0.4)+\left(\frac{3}{11}\right)(0.4)(0.7)}$ |
| | $=0.11351\approx 0.114$ |

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| **8** | **(i)** | Let $A$ = weight of an apple in kg.<br>$A \sim \mathrm{N}\left(0.2, 0.05^2\right)$<br>Let $O$ = weight of an orange in kg.<br>$O \sim \mathrm{N}\left(0.3, 0.08^2\right)$<br><br>$A_1 + A_2 + A_3 - O \sim \mathrm{N}\left(3(0.2) - 0.3, 3\left(0.05^2\right) + 0.08^2\right)$<br>$= \mathrm{N}(0.3, 0.0139)$ |
| | | $\mathrm{P}\left(\lvert A_1 + A_2 + A_3 - O\rvert < 0.2\right)$<br>$= \mathrm{P}\left(-0.2 < A_1 + A_2 + A_3 - O < 0.2\right)$ |
| | | $= 0.19816 \approx 0.198$ |
| | **(ii)** | $T = 2\left(A_1 + A_2 + A_3 + A_4 + A_5\right) + 3\left(O_1 + O_2 + O_3 + O_4 + O_5\right)$<br>$T \sim \mathrm{N}\left(2(5)(0.2) + 3(5)(0.3), 2^2(5)\left(0.05^2\right) + 3^2(5)\left(0.08^2\right)\right)$<br>$= \mathrm{N}(6.5, 0.338)$ |
| | | $\mathrm{P}(T > 5) = 0.99506 \approx 0.995$ |
| | **(iii)** | $\overline{A} \sim \mathrm{N}\left(0.2, \frac{0.05^2}{n}\right)$ |
| | | $\mathrm{P}\left(\overline{A} > 0.21\right) < 0.3$<br>$\mathrm{P}\left(Z > \frac{0.21 - 0.2}{\sqrt{0.05^2/n}}\right) < 0.3$ |
| | | $1 - \mathrm{P}\left(Z \le \frac{\sqrt{n}}{5}\right) < 0.3$<br>$\mathrm{P}\left(Z \le \frac{\sqrt{n}}{5}\right) > 0.7$<br>$\frac{\sqrt{n}}{5} > 0.52440 \Rightarrow n > 6.8749$<br>$\therefore$ Least $n = 7$. |

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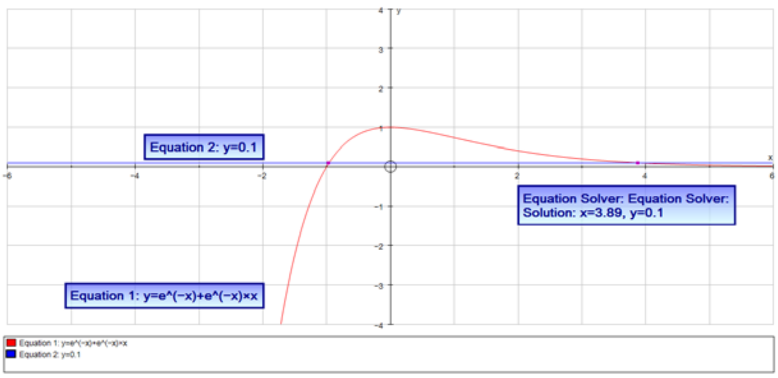
| **9** | **(i)** | 1. The defects in a reference book are independent of each other. |
| | | 2. The average number of defects in a reference book is constant. |
| | **(ii)** | $R \sim \text{Po}(3)$<br>$\text{P}(R>7)=\text{P}(R\geq 8)$<br>$=1-\text{P}(R\leq 7)=0.011905\approx 0.0119$ |
| | **(iii)** | Let $X$ = no. of reference books donated away out of 1000.<br>$X \sim \text{B}(1000, 0.011905)$ |
| | | Since $n$ is large, $np$ = 11.905 > 5 and $nq$ = 988.095 > 5, $X \sim \text{N}(11.905, 11.763)$ approximately. |
| | | $\text{P}(10\leq X\leq 15)\xrightarrow{c.c.}\text{P}(9.5\leq X\leq 15.5)$<br>$=0.61114\approx 0.611$ |
| | **(iv)** | Let $C$ = no. of defects in a children's book.<br>$C \sim \text{Po}(5)$<br><br>Hence, $R+C \sim \text{Po}(3+5)=\text{Po}(8)$ |
| | | $P(R+C<10)=P(R+C\leq 9)=0.71662\approx 0.717$ |
| | **(v)** | Now, $C \sim \text{Po}(\lambda)$<br><br>$\text{P}(C\leq 1)=0.1$<br>$\text{P}(C=0)+\text{P}(C=1)=0.1$<br>$\frac{e^{-\lambda}\lambda^0}{0!}+\frac{e^{-\lambda}\lambda^1}{1!}=0.1$ |
| | | $e^{-\lambda}+e^{-\lambda}\lambda=0.1$ |
| | | Using GC,<br><br>$\lambda=3.8897\approx 3.89$ |

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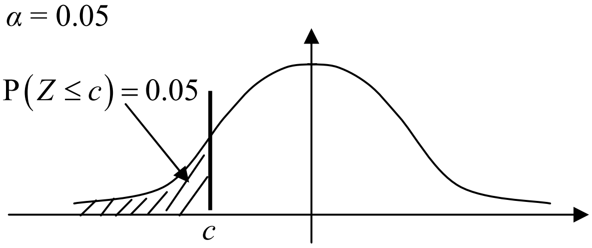
| **10** | **(i)** Let $X$ = payment made on a motor insurance claim.<br><br>Assumption: Payments made on motor insurance claims are normally distributed. |
| | $\bar{x} = \dfrac{\sum(x-10000)}{25} + 10000 = 9737.6$<br>$s^2 = \dfrac{1}{24}\left\{\sum(x-10000)^2 - \dfrac{\left[\sum(x-10000)\right]^2}{25}\right\} = 240506.5$ |
| | $H_0 : \mu = 10000$<br>$H_1 : \mu < 10000$ |
| | Under $H_0$, $\bar{X} \sim N\left(10000, \dfrac{\sigma^2}{25}\right)$.<br><br>Hence, test statistic $T = \dfrac{\bar{X}-10000}{\sqrt{S^2/25}} \sim t(24)$.<br><br>$\alpha = 0.01$<br><br>From GC, $t = \dfrac{9737.6-10000}{\sqrt{240506.5/25}} = -2.6753$. |
| | $p$-value = 0.00662<br><br>Since $p = 0.00662 < \alpha = 0.01$, we reject $H_0$ at 1% level of significance and conclude there is sufficient evidence that the mean payment made is less than \$10000. |
| | **(ii)** It means that there is a probability of 0.01 of concluding that the mean payment made is less than \$10000, given that the mean payment made is \$10000. |
| | **(iii)** $p$-value for the two-tailed test = 2 × 0.0066172<br>= 0.013234<br><br>Since $p$-value = 0.013234 > $\alpha = 0.01$, we do not reject $H_0$ at 1% level of significance, i.e. the conclusion in part **(i)** would not be the same. |
| | **(iv)** Let $\mu_0$ be the mean payment made that the insurance company should declare.<br><br>$H_0 : \mu = \mu_0$<br>$H_1 : \mu < \mu_0$<br>Under $H_0$, $\bar{X} \sim N\left(\mu_0, \dfrac{500^2}{25}\right)$.<br><br>Hence, test statistic $Z = \dfrac{\bar{X}-\mu_0}{\sqrt{500^2/25}} \sim N(0,1)$. |

$\alpha = 0.05$




Do not reject $H_0$ if $z > -1.6449$.

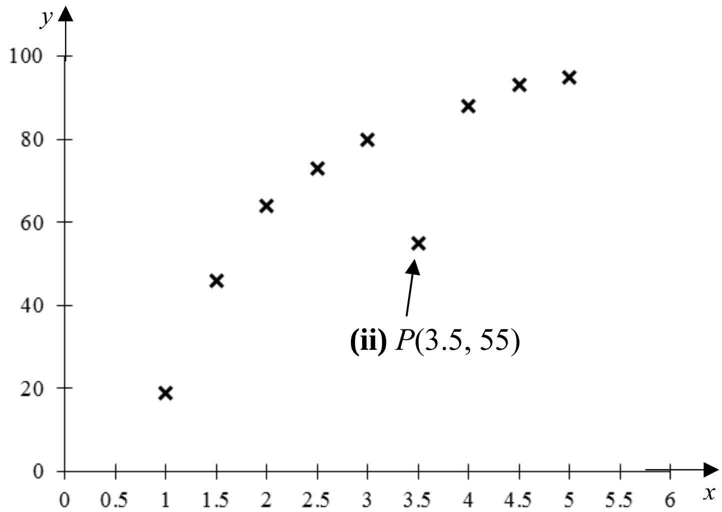
Since $z = \dfrac{9737.6 - \mu_0}{\sqrt{500^2/25}}$,

$\dfrac{9737.6 - \mu_0}{\sqrt{500^2/25}} > -1.6449$

$\mu_0 < 9902.09$

Largest $\mu_0$ = \$9902 (to nearest dollar)

**11** **(i)**




**(iii)** $y = 20.879 + 16.637x$

$y = 20.9 + 16.6x$

**(iv)** When $x = 5.0$, $y = 20.879 + 16.637(5.0) = 104.06 \approx 104$

The linear model is not suitable as contrast index $y$ cannot be greater than 100.

**(v)** A: $r = -0.99786 \approx -0.998$

B: $y = ax^b$

$\ln y = \ln\left(ax^b\right) = \ln a + \ln x^b$

$\ln y = \ln a + b \ln x$

$r = 0.92358 \approx 0.924$

Since $|r_A|$ is closer to 1 than $|r_B|$, model A is the better model.

| | |
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| | **(vi)** $y = 112.46 - \dfrac{95.651}{x}$ |
| | When $y = 75$, $75 = 112.46 - \dfrac{95.651}{x}$.<br>$x = 2.5534 \approx 2.55$ |
| | Since $y = 75$ is within the given range of data, this is an interpolation and the estimate is reliable. |